Determination of Non-Minimum Phase Systems by the Structural Approach

Andreas Daasch

Vollständiger Abdruck der von der Fakultät für Luft- und Raumfahrttechnik der Universität der Bundeswehr München zur Erlangung des akademischen Grades eines

Doktors der Ingenieurwissenschaften (Dr.-Ing.)

genehmigten Dissertation.

Gutachter:

1. Univ.-Prof. Dr.-Ing. Ferdinand Svaricek
2. Univ.-Prof. Dr.-Ing. habil. Klaus Röbenack

Die Dissertation wurde am 09.01.2019 bei der Universität der Bundeswehr München eingereicht und am 05.08.2019 durch die Fakultät für Luft- und Raumfahrttechnik angenommen. Die mündliche Prüfung fand am 10.12.2019 statt.

Danksagung

An erster Stelle möchte ich mich bei Prof. Ferdinand Svaricek für die angenehme Zusammenarbeit und die fundierte fachliche Betreuung bedanken. Des Weiteren möchte ich mich bei Prof. Klaus Röbenack für die Übernahme des Zweitgutachtens bedanken. Besonderer Dank gilt Matthias Schultalbers dafür, dass er den Rahmen geschaffen hat, in dem diese Arbeit entstanden ist.

Ohne die beiden folgenden Personen hätte meine Arbeit vermutlich nie ihre jetzige Form angenommen. Dieter Schwarzmann half mir vor allem dadurch, dass er mir den Weg aufgezeigt und freigemacht hat, meine Promotion ins Ziel zu führen. Ohne die wertvollen fachlichen Diskussionen mit meinem geschätzten Kollegen Alexander Weber hätte ich mir deutlich schwerer bei der Ausformulierung meiner Ergebnisse getan.

Nicht zu vergessen sind Maiko Garwon und Frank Wobbe, welche mir Starthilfe für meine Promotion gaben. In jeden Fall möchte ich mich auch bei den zahlreichen Korrekturlesern bedanken.

Contents

Deutsche Kurzfassung

Viele dynamische Prozesse und Systeme, die uns täglich begegnen, zeichnen sich durch die systemtheoretische Eigenschaft der Nichtminimalphasigkeit aus. Beispiele sind das Rückwärts-Einparken eines Autos mit Anhänger oder das Lenken eines Fahrrads. Für viele nichtminimalphasige Systeme ist charakteristisch, dass die zu steuernde Größe zuerst in die Gegenrichtung ausschlägt, in die sie gesteuert wird. Das bereits erwähnte Beispiel des Rückwärts-Einparkens mit Anhänger demonstriert dieses Verhalten sehr anschaulich. Systemtheoretisch spricht man von einem nichtminimalphasigen System, falls es eine instabile Nulldynamik besitzt oder - im Linearen - Nullstellen in der offen rechten komplexen Halbebene aufweist. Das Problem an nichtminimalphasigen Systemen ist, dass sie im Allgemeinen anspruchsvoll zu regeln sind. Der geschlossene Regelkreis neigt zur Instabilität, was dazu führt, dass nur geringe Reglerverstärkungen gewählt werden können. Außerdem sind nichtminimalphasige Systeme nicht stabil invertierbar, sodass insbesondere viele nichtlineare Methoden des Reglerentwurfs nicht anwendbar sind. Beide Eigenschaften führen zu einem schlechten Trajektorienfolgeverhalten. Aus diesem Grund ist es notwendig, Systeme vor dem Reglerentwurf auf ihre Nichtminimalphasigkeit hin zu überprüfen.

Wird eine Strecke modelliert, gibt es üblicherweise zwei Vorgehensweisen. Zum einen kann ein dynamisches Modell aus den physikalischen Zusammenhängen abgeleitet werden. Zum anderen können Messungen der Ein- und Ausgänge an der Strecke zur Systemidentifikation genutzt werden. Im ersten Fall bekommt man ein System mit zu bestimmenden Parametern, im zweiten Fall ein Satz numerischer Systeme. Bei vielen technischen Systemen entstehen dabei sehr umfangreiche Modelle. Dennoch sind viele Parameter nur ungenau bekannt. Das führt dazu, dass die Anwendung der herkömmlichen numerischen Methoden zur Analyse systemtheoretischer Eigenschaften aufwendig ist und trotzdem keine exakten Aussagen über die Strecke liefern kann. In diesen Fällen kann die strukturelle Analyse hilfreich sein. Diese betrachtet alleine die Abhängigkeiten im System, also nur, ob gewisse Systemzustände, Eingänge oder Ausgänge voneinander abhängen oder nicht. Das hat den Vorteil, dass schon frühzeitig Aussagen über das untersuchte System getroffen werden können ohne die genauen numerischen Werte der Parameter zu kennen. Außerdem gelten strukturelle Eigenschaften, die für ein System gefunden wurden, für alle Systeme gleicher Struktur.

In dieser Arbeit wird daher untersucht, ob allein anhand der Struktur eines Systems bestimmt werden kann, ob es nichtminimalphasig ist. Dazu wird wie folgt vorgegangen. Zunächst werden in Kapitel 2 die nötigen systemtheoretischen und mathematischen Grundlagen, wie die Repräsentation eines dynamischen Systems als Graph, grundlegende Eigenschaften wie Steuerbarkeit, Beobachtbarkeit, Invertierbarkeit und Rang, Stabilität und Nullstellen, sowie Nulldynamik und der Begriff der (Nicht-)minimalphasigkeit eingeführt. Bode führte 1940 die Bezeichnung nichtminimalphasig für stabile lineare Systeme mit mindestens einer Nullstelle in der rechten komplexen Halbebene ein. In den Achtzigern wurde dieser Begriff von Isidori für Systeme mit nicht asymptotisch stabiler Nulldynamik wiederverwendet. In Kapitel 3 wird daher untersucht, wie diese beiden Verwendungen dieses Begriffs vereint werden können. Hierzu wird die Byrnes-Isidori-Normalform, welche zur Bestimmung der Nulldynamik bei nichtlinearen quadratischen Systemen verwendet wird, auf allgemeine lineare Systeme erweitert. Das Resultat ist, dass im Wesentlichen die Definition der Nichtminimalphasigkeit nach Isidori auch auf Systeme, welche nach Bode nichtminimalphasig sind, zutrifft. Aus diesem Grund wird im Weiteren die Definition nach Isidori für nichtminimalphasige Systeme genutzt, das heißt, für Systeme mit mindestens einer Nullstelle in der offenen rechten komplexen Halbebene. Die in Kapitel 2 eingeführte Darstellung dynamischer Systeme als Graph wird in Kapitel 4 genutzt, um Nullstellen graphtheoretisch zu bestimmen. Dazu werden bekannte Methoden auf den allgemeinen Fall nichtquadratischer und auch degenerierter MIMO-Systeme erweitert. Diese erweiterten Methoden werden folglich in Kapitel 5 auf strukturelle Systeme angewendet. Hierfür werden zuerst die Begriffe strukturelles System und strukturelle Eigenschaft definiert und der Bezug zur Darstellung als Graph hergestellt. Strukturelle Eigenschaften sind dabei Eigenschaften, die im numerischen Sinn für fast alle Systeme gleicher Struktur gelten. Durch die Nichtexistenz bestimmter Subgraphen, den Feedback-Zyklusfamilien, im Graph eines Systems kann dann ein hinreichendes Kriterium für die Nichtminimalphasigkeit eines strukturellen Systems angegeben werden. Das heißt, fast alle numerischen Realisierungen eines strukturell nichtminimalphasigen Systems sind dann auch nichtminimalphasig im numerischen Sinne. Das wirft zugleich die Frage auf, ob es eine strukturelle Nichtminimalphasigkeit gibt, die für *alle* numerischen Realisierungen gilt. Dies führt zu dem bekannten Begriff der streng strukturellen Eigenschaften, die eben für alle numerischen Realisierungen eines strukturellen Systems gelten. In diesem Kontext werden die streng strukturell nichtminimalphasigen Systeme definiert. Abschließend werden drei Erweiterungen vorgestellt. Zuerst werden die eingeführten Methoden für die Analyse der Stabilität von strukturellen Systemen angewendet. Dies führt zu streng strukturell nicht asymptotisch stabilen Systemen, deren numerische Realisierungen unter keiner Wahl von numerischen von Null verschiedenen Parametern asymptotisch stabil sein können. Die nächste Erweiterung ist die Eigenschaft der Vorzeichen-Nichtminimalphasigkeit. Oft ist neben der Existenz einer Abhängigkeit im System auch die Richtung, d.h. das Vorzeichen, der Abhängigkeit bekannt. Es wird

gezeigt, dass für Systeme bei denen strukturell keine Nichtminimalphasigkeit ermittelt werden kann, durch Betrachtung der Vorzeichen der Abhängigkeiten eine Nichtminimalphasigkeit nachgewiesen werden kann. Schließlich wird die Anwendung der Methoden auf nichtlineare Systeme diskutiert. Es stellt sich heraus, dass strukturelle Nichtminimalphasigkeit bei nichtlinearen Systemen im Allgemeinen nicht existiert. Es lässt sich jedoch ein strukturelles Kriterium für nicht asymptotisch stabile nichtlineare Systeme finden.

Abstract

Many dynamic processes and systems that we encounter regularly are characterized by the system-theoretical property "non-minimum phase". A system is called non-minimum phase if it has unstable zero dynamics or, in the linear case, zeros with non-negative real parts. Non-minimum phase systems are generally challenging to control. The closed control loop tends to become unstable, which results in the fact that only small control gains can be chosen. Furthermore, non-minimum phase systems do not have a stable inverse, so that in particular many nonlinear methods of controller design are not applicable. Both properties lead to a poor trajectory tracking behavior. For this reason, it is necessary to check a priori if a system is non-minimum phase in order to select a suitable control method.

There are usually two approaches to model a plant. A dynamic model can be derived from the physical relations in the system or measurements of the inputs and outputs can be used for system identification methods. In the first case, a system with parameters, which have to be determined, is obtained and, in the second case, a set of numerical systems is generated. For technical systems, the obtained models are usually very complex. Nevertheless, many parameters are known only imprecisely. This leads to the fact that the application of common numerical methods for the analysis of system-theoretical properties is difficult. In these cases, the structural analysis may be helpful. The structural approach only considers the dependencies in the system, which means, whether certain system states, inputs or outputs depend on one another.

In this thesis, it is investigated whether it is possible to determine only by the structure of a system whether it is non-minimum phase. Therefore, the following approach is taken. The term (non-)minimum phase was used in 1940 by Bode for stable linear systems with at least one zero with positive real part and in the eighties by Isidori for systems with unstable zero dynamics. It is examined how these two concepts can be unified. The result is that essentially the definition of non-minimum phase systems according to Isidori applies also to systems that are non-minimum phase according to Bode. Further, the representation of dynamical systems as a graph is used to determine the zeros by the graph-theoretic approach. For this purpose, known methods are extended to the general case of non-square and degenerated MIMO systems. These extended methods are then applied to structural systems. Structural properties apply to almost all systems of the same structure in the numerical sense. By the non-existence of certain subgraphs - the feedback cycle families - in the graph of a system, a sufficient criterion for the non-minimum phase property of a numerical system can be given. This means that almost all numerical realizations of a structurally non-minimum phase system are also non-minimum phase in the numerical sense. This also raises the question whether there exists a structural non-minimum phase property that is valid numerically for *all* realizations. This leads to the well-known concept of strong-structural properties that hold numerically for all realizations of a structural system. In this context, strong-structurally non-minimum phase systems are defined. Finally, three extensions are presented. First, the developed methods are applied to analyze the stability of structural systems. This leads to strong-structurally not asymptotically stable systems, whose numerical realizations cannot be asymptotically stable under any choice of numerical non-zero parameters. Often, in addition to the existence of a dependency in the system, the direction, i. e. the sign, of the dependency is also known. It is shown that for systems, for which no non-minimum phase property can be determined structurally, a non-minimum phase property can be found by considering the signs of the dependencies in the system. Finally, the application of the methods to nonlinear systems is discussed. It turns out that in general the structurally non-minimum phase property does not exist for nonlinear systems. However, a structural criterion for not asymptotically stable nonlinear systems can be found.

Notation

A set of numbers or field is denoted by $\mathbb{F}$. More specifically, $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the field of complex numbers.

Vectors are given by lowercase bold letters $\boldsymbol{v}$. Matrices are represented by bold capital letters $\boldsymbol{M}$. Three types of matrices are considered. Polynomial matrices $\boldsymbol{P}(s) \in \mathbb{R}[s]^{n_1 \times n_2}$ or rational matrices $\boldsymbol{R}(s) \in \mathbb{R}(s)^{n_1 \times n_2}$ are matrices whose elements are polynomials or rational functions in a variable s. A parametrized matrix $\boldsymbol{M}[\boldsymbol{\mu}] \in \mathbb{F}^{n_1 \times n_2}$ is a matrix that has d parameters $\begin{bmatrix}\mu_1 & \mu_2 & \dots & \mu_d\end{bmatrix} =: \boldsymbol{\mu} \in \mathbb{F}^d$ as elements.

The identity matrix is denoted by $\boldsymbol{I}$. The notation $\operatorname{diag}(\boldsymbol{e})$ represents a diagonal matrix whose entries are given by the vector $\boldsymbol{e}$.

The matrix $\boldsymbol{M}^+$ is the pseudo inverse of the matrix $\boldsymbol{M}$, i. e. some matrix that fulfills the equation

$$\boldsymbol{M}\boldsymbol{M}^+ = \boldsymbol{I}\ . \tag{0.1}$$

The kernel of a matrix $\boldsymbol{M} \in \mathbb{F}^{n_1 \times n_2}$ is denoted by $\ker \boldsymbol{M} := \{\boldsymbol{v} \in \mathbb{F}^{n_2} | \boldsymbol{M}\boldsymbol{v} = \boldsymbol{0}\}$ and the image of the matrix is denoted by $\operatorname{im} \boldsymbol{M} := \{\boldsymbol{M}\boldsymbol{v} | \boldsymbol{v} \in \mathbb{F}^{n_2}\}$.

In some cases submatrices are considered, which are represented by

$$\boldsymbol{M}^{k_1,k_2,\dots,k_i}_{l_1,l_2,\dots,l_j} \tag{0.2}$$

where $k_1, k_2, \dots, k_i$ is a list of i rows and $l_1, l_2, \dots, l_j$ is a list of j columns of the matrix $\boldsymbol{M}$ of which the submatrix is constructed.

The $i \times j$ minor of $\boldsymbol{M}$, is given by

$$\boldsymbol{M}^{\{k_1,k_2,\dots,k_i\}}_{\{l_1,l_2,\dots,l_j\}} := \det \boldsymbol{M}^{k_1,k_2,\dots,k_i}_{l_1,l_2,\dots,l_j}\ . \tag{0.3}$$

In some cases if it is clear the explicit time dependency $\boldsymbol{x} := \boldsymbol{x}(t)$ of some variables is omitted for better readability. The derivative by time t is marked by a dot, i. e. $\dot{\boldsymbol{x}}(t) := \frac{d}{dt}\boldsymbol{x}(t)$.

Graphs

The basic structure of a graph is explained. For a complete description, see e. g. [Rei88, Appendix A1] or [Die10].

A *graph* $\mathcal{G}(\mathcal{V}, \mathcal{E})$ consists of a set of *vertices* $\mathcal{V}$ and a set of *edges* $\mathcal{E}$. The graph exposes a structure by connecting two vertices, $v_i \in \mathcal{V}$ and $v_j \in \mathcal{V}$, with an edge $e_{i,j} \in \mathcal{E}$. If both end-vertices of an edge coincide, the edge is called a *self-loop*. In a *directed* graph, the edges also assign a direction between the two connected vertices. This can be extended to a *directed weighted* graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$, which additionally contains a set of *weights* $\mathcal{W}$. In the weighted graph a value $w_{i,j} \in \mathcal{W}$ is assigned to every edge $e_{i,j}$. Graphically, vertices are represented by circles ◯ and edges by arrows $\rightarrow$ connecting the circles and revealing the direction of connection.

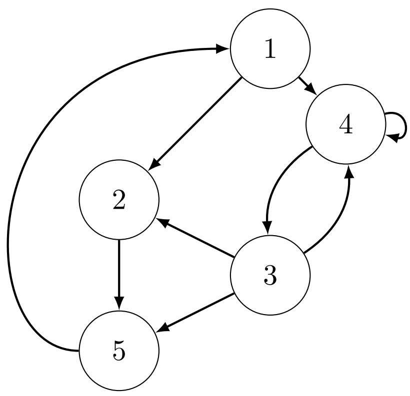

Figure 0.1: Example of a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

Some subgraphs can be identified in a graph. A *simple path* in the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a sequence of edges $\{e_{i,j}, e_{j,k}, \ldots\}$ connecting the vertices $\{v_i, v_j, v_k, \ldots\}$ in forward direction, wherein every vertex is visited only once. If the first and the last vertex of a simple path are identical, the sequence is called a *cycle*. A *cycle family* $\mathcal{C}$ is the set of disjoint cycles in the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, i. e. cycles that do not share vertices.

An example of a graph with cycles is drawn in Figure 0.1. This example will be used for the following explanation of notation. In the text self-loops are described by the contained vertex, e. g. 4. Cycles that contain two vertices are described by e. g. $4 \leftrightarrows 3$. Larger cycles are described by e. g. $1 \underleftarrow{\rightarrow 2 \rightarrow} 5$. A cycle family is marked by parentheses, e. g. $(1 \underleftarrow{\rightarrow 2 \rightarrow} 5, 4 \leftrightarrows 3)$.

An *adjacency matrix* $\mathcal{A}$ of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ represents the relation between the vertices $v_i \in \mathcal{V}$ in matrix form. The matrix elements $\alpha_{i,j} \in \mathcal{A}$ are determined by the edges $e_{i,j} \in \mathcal{E}$ in the following way [TS11]:

$$\alpha_{i,j} = \begin{cases} 0 & \text{if no directed edge between } v_i \text{ and } v_j \text{ exists} \\ w_{i,j} & \text{if a directed edge from } v_i \text{ to } v_j \text{ exists.} \end{cases} \tag{0.4}$$

This is an extension to the classical definition, where the entries of the adjacency matrix are only given by zeros or ones, depending on the existence of the considered edge.

1 Introduction

The terms “minimum phase” and “non-minimum phase” originate from communications engineering for a certain behavior of dynamical systems and are assumed to be first used by H. W. Bode. With a slightly different terminology, he described in [Bod40] the fact that there exist systems that have a minimal phase shift in their frequency response compared to other systems with the identical gain over frequency. This is, however, an abstract definition, since it involves the system description in the frequency domain. In state space, e. g. physical coordinates, systems that are non-minimum phase often expose a “wrong way behavior”, sometimes also called “inverse response” or “undershoot”. This means that the step response of such systems tends first in the opposite direction of where it will end up in steady state. This behavior is counterintuitive but very common for many dynamical systems. Examples are the problem of parking a car backwards, craning building materials to a specified position on a construction plant or steering a bicycle [ÅKL05]. There are many more examples, not only in the domain of mechanical systems. For instance the vertical dynamics of planes [SL91], vertical taking-off and landing aircrafts [MDP94], loading bridges, the inverse pendulum, electronic circuits with all-pass elements, water turbines and modern gasoline engines [DSS16b] can be non-minimum phase depending on which part of the system is actuated and which variables are measured. It cannot be said that a technical system is (non-)minimum phase by nature since the property depends on the selection of system inputs and outputs.

1.1 Motivation

When designing a controller for a system, in many cases, it is necessary to check whether the considered system is non-minimum phase, since systems with this property impose limitations on control. For instance, in closed loop they tend to get unstable allowing only a small feedback gain. Furthermore, they do not have a stable inverse, which obstructs the application of common control schemes such as Input-Output-Linearization [Isi95], Backstepping [KKK95] and Sliding Mode Control [SL91].

In order to determine whether a system is non-minimum phase, two approaches can be followed depending on the nature of the system description. Linear systems can be analyzed by the calculation of their zeros. Roughly speaking, the zeros are frequencies with which the inputs can be excited resulting in the outputs staying at zero. By the values of the zeros, it can be determined whether the considered system is non-minimum phase. A similar concept like the zeros exists for nonlinear systems called zero dynamics. The zero dynamics are the dynamics that describe the internal behavior once a feedback and initial conditions are chosen such that the output stays at zero. Then by investigating the stability of the zero dynamics, it is possible to tell whether the considered system is non-minimum phase.

The determination of zeros or the zero dynamics, or in general the application of control schemes, is often challenging for practical problems. In many cases an accurate description of the plant or the identification of physical systems, which is mandatory for proper application of control methods, leads to large complex models. Nonetheless, uncertainties in the model parameters may be present. In this case, structural analysis is a helpful tool because neither accurate numerical parameters of the model nor exact functional relations between its components are prerequisites. This has several advantages. First, a structural analysis can be done in a very early stage in the control design process without much information about the considered system. This enables early design decisions, e. g. which actuators or sensors have to be used for control and which control scheme is applicable. Furthermore, since only the structure of a system is considered, the results obtained are also applicable to all systems with the same structure in contrast to results gained by numerical methods, which are only valid for the investigated system. In addition, there are already various systems described in a structural form e. g. many types of communication networks, power grids and production processes for which structural methods can be applied directly. Another advantage is the possibility to represent structural systems graphically. This is often very insightful and many system theoretic properties can be checked manually without using numerical methods.

1.2 Previous Work

The non-minimum phase property is discussed in many standard textbooks about control engineering but the structural approach has been considered less often. Early investigations of structural systems was conducted by [Lin74; GS76] considering structural controllability and by [ITY71] considering structural “solvability”. The results for structural controllability and observability, finite and infinite zeros and poles in the case of linear systems were summarized in the book [Rei88] and in the survey [DCW03]. In the book [Mur09] the topic of structural properties is discussed rigorously. Structural properties like differential rank, infinite zeros and invertibility for nonlinear systems are described in the book [Wey02].

Recently, structural methods have been extensively applied in the complex networks domain, where, among others, biological, technological, social and economic networks are investigated. Examples are information transport in the brain, distribution of goods and news spreading on social media. In [LSB11] the structural controllability is applied to identify the set of nodes in a complex network that can control the whole network. Analogously, in [LSB13] the structural observability is used to find the set of sensors in a complex network to reconstruct the entire state of the network. Structural methods are used for fault detection in [PBB12] in order to ensure trustworthy computation in linear consensus networks. Furthermore, the structural approach is exploited in order to check for additional system theoretical properties like stability. For example in [Bel13] criteria are given to decide whether networks are stabilizable.

Further, subject specific references are given in the chapters where these topics are discussed in detail.

1.3 Contributions and Structure

The contribution of this work to the topic of structural systems analysis is the investigation of the stability of zeros or zero dynamics of systems with graph-theoretic methods. Therefore, sufficient criteria are developed to determine if a system is structurally non-minimum phase, meaning the considered system will be non-minimum phase for (almost) all numerical values for its parameters. Since non-minimum phase systems are challenging to control, this property is useful to select proper inputs and outputs in order to try to avoid non-minimum phase behavior or to select the appropriate control scheme, without knowing the exact numerical values a priori.

In addition to the already published work about criterions for structurally non-minimum phase systems [DSS16a] and strong-structurally non-minimum phase systems [DSS16c] this work generalizes these criterions for arbitrary linear systems without feed-through. That implies also a definition of zero dynamics for these systems, which before was only possible for square non-degenerated systems [Daa16].

This work is structured as follows. In the following chapter some mathematical and system theoretic background is presented. This implies the common mathematical representation of dynamical systems, their representation as graph, some basic properties of linear systems, such as controllability and observability, invertibility and rank, stability and zeros. Further, the zero dynamics of nonlinear systems are introduced and the term non-minimum phase is specified. In Chapter 3, the relation between the concepts zeros and zero dynamics is investigated for arbitrary linear systems. This includes the definition of the relative degree for non-square systems, whose existence is a precondition to determine the Byrnes-Isidori normal form, a normal form to determine the zero dynamics of a dynamical system. With the linear version of the Byrnes-Isidori normal form, the relationship between the zeros of a system and its zero dynamics is investigated in the general case. In Chapter 4, the graph-theoretic approach for the determination of the polynomials used for the analysis of the stability of the zeros and hence of the non-minimum phase property of linear systems is presented. Known results for square systems with one input and one output are generalized for arbitrary linear systems. In Chapter 5, this generalization is used to find sufficient criteria to decide whether a system is non-minimum phase only by its structure. These criteria will hold in almost all cases. Nevertheless, a concept is developed in order to obtain sufficient criteria for a structurally non-minimum phase property that will hold in all cases. In the end of this chapter, some extensions are presented. The strong-structural method is used to analyze the stability of a system leading to the definition of strong-structurally not asymptotically stable systems. Further, the extension of the structurally non-minimum phase property to systems, where besides the structure also the signs of the dependencies in the system are known and the extension of the structural approach to nonlinear systems is discussed. Moreover, the complexity of determining these structural properties is discussed. In the conclusion, a summary of the contributions of this work is given and links to further research are pointed out.

2 Preliminaries

In this chapter some fundamental properties of systems and concepts used in control engineering are introduced, which are used throughout this work. First, the representation of dynamical systems is presented for linear systems and for nonlinear systems. Subsequently, these representations are related to the graph-theoretic representation. Further, the basic properties of linear systems are summarized. This includes the observability and controllability, the invertibility and the rank, the stability and the zeros. Then, a normal form and the determination of zero dynamics are presented for nonlinear systems. Finally, the term “non-minimum phase” is defined and its usage in literature is discussed briefly.

2.1 Representations of Dynamical Systems

For reference, first the classical representations of dynamical systems as ordinary differential equations is given. After that, it is described, how to obtain a graph from the classical representation.

2.1.1 Classical Representation

Linear dynamical systems are commonly represented by the ordinary differential equations

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\ , \end{aligned} \tag{2.1}$$

where the system's state variables are denoted by $\boldsymbol{x}(t) \in \mathbb{R}^n$, the input vector by $\boldsymbol{u}(t) \in \mathbb{R}^m$ and the output vector by $\boldsymbol{y}(t) \in \mathbb{R}^p$. The order, and hence the number of state variables, is given by n, the number of inputs by m and the number of outputs by p. The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are real, constant and have appropriate dimensions. This representation is typically called *state space representation.* The linear space of the states is denoted by X, the space of inputs by U and the space of outputs by Y, i. e. $\boldsymbol{x}(t) \in \mathsf{X}$, $\boldsymbol{u}(t) \in \mathsf{U}$ and $\boldsymbol{y}(t) \in \mathsf{Y}$.

Since all practical systems are strictly proper [SP05, §1.3], there is no direct feedthrough and $\boldsymbol{D} = 0$ is assumed throughout this work. [1] Thus, usually the linear system

$$\Sigma_{LS}: \quad \begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \quad . \tag{2.2}$$

is regarded in this work. A system is called a *single-input-single-output (SISO)* system if $m = p = 1$ and a *multiple-input-multiple-output (MIMO)* system if $m > 1$ and $p > 1$. If $m = p$ holds, the system is called *square.*

The system Σ_{LS} can be Laplace transformed to

$$\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{Y}(s) \end{bmatrix} = \underbrace{\begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}}_{=:\boldsymbol{P}(s)} \begin{bmatrix} \boldsymbol{X}(s) \\ \boldsymbol{U}(s) \end{bmatrix} \tag{2.3}$$

with $s \in \mathbb{C}$ and $\boldsymbol{x}(0)$ some initial state, where $\boldsymbol{P}(s) \in \mathbb{R}[s]^{(n+p)\times(n+m)}$ is called the (Rosenbrock's) *system matrix.*

From (2.3) it is furthermore possible to determine the *transfer function matrix* $\boldsymbol{G}(s) \in \mathbb{R}(s)^{p\times m}$ from input to output

$$\boldsymbol{Y}(s) = \underbrace{\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}}_{=:\boldsymbol{G}(s)}\boldsymbol{U}(s) + \boldsymbol{C}\boldsymbol{x}(0) \; . \tag{2.4}$$

Occasionally nonlinear systems are considered in this work. The linear systems specified before can be seen as a special case of these systems. Nonlinear systems without direct feedthrough are represented by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t))\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{h}(\boldsymbol{x}(t)) \end{aligned} \quad , \tag{2.5}$$

where the continuous differentiable vector fields $\boldsymbol{f}(\boldsymbol{x}(t)), \boldsymbol{g}(\boldsymbol{x}(t)) \in \mathbb{R}^n$ and $\boldsymbol{h}(\boldsymbol{x}(t)) \in \mathbb{R}^p$.

[1] Sometimes high frequency behavior is modeled by a non-zero $\boldsymbol{D}$.

2.1.2 Graph-Theoretic Representation

Especially for the investigation of structural properties, the graph-theoretic representation of dynamical systems is beneficial, because the graph visualizes the mutual dependencies of the state variable and the inputs and outputs. Therefore, this method is used by many authors, e. g. [Rei88; DCW03; Che76; Wey02; LSB11].

The system Σ_{LS} can be represented as a graph using the following definition.

Definition 2.1.1 (System Graph). The (weighted) *system graph* $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ ($\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$) of the system (2.2) consists of m input vertices $u_1, \ldots, u_m \in \mathcal{U} \subset \mathcal{V}$, n states vertices $x_1, \ldots, x_n \in \mathcal{X} \subset \mathcal{V}$ and p output vertices $y_1, \ldots, y_p \in \mathcal{Y} \subset \mathcal{V}$. The vertices are connected by directed (and weighted) edges, generated by the following rules:

1. There exists a directed edge $e_{b_{i,j}} \in \mathcal{E}$ from input vertex u_j to state vertex x_i if in $\boldsymbol{B}$ the element $b_{i,j}$ in the i-th row and j-th column is nonzero. (The resulting weight $w_{b_{i,j}} \in \mathcal{W}$ of this edge is given by $b_{i,j}$.)

2. There exists a directed edge $e_{a_{i,j}} \in \mathcal{E}$ from state vertex x_j to state vertex x_i if in $\boldsymbol{A}$ the element $a_{i,j}$ in the i-th row and j-th column is nonzero. (The resulting weight $w_{a_{i,j}} \in \mathcal{W}$ of this edge is given by $a_{i,j}$.)

3. There exists a directed edge $e_{c_{i,j}} \in \mathcal{E}$ from state vertex x_j to output vertex y_i if in $\boldsymbol{C}$ the element $c_{i,j}$ in the i-th row and j-th column is nonzero. (The resulting weight $w_{c_{i,j}} \in \mathcal{W}$ of this edge is given by $c_{i,j}$.)

If a system Σ_{LS} is represented as a graph as previously defined, the state vertices have a special function. They symbolize integrators that integrate the values (multiplied by the weights) of the state vertices to which they are connected as end-vertices of an edge. Then they share their values to the vertices to which they are connected as start-vertices of an edge. This property of state vertices is fundamental to this work.

Often vertices are connected in a cyclic manner, as described previously in the Notation section. The number of state vertices in cycle families is of major interest, so a property for this value is defined as follows.

Definition 2.1.2 (Width of a Cycle Family). The *width* of a cycle family is the number of state vertices it touches.

The translation from the classical to the graph-theoretic representation is demonstrated by the next example.

Example 2.1.1. Consider the generic second order system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix} u, \quad y = \begin{bmatrix} c_{1,1} & c_{1,2} \end{bmatrix} \boldsymbol{x} \,. \tag{2.6}$$

Applying Definition 2.1.1 yields its graph-theoretic representation depicted in Figure 2.1. Note, since the width of a cycle family is important in this work, state

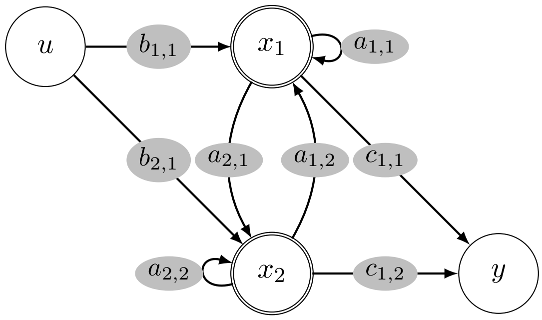

Figure 2.1: $\mathcal{G}_{sys}$ of (2.6).

vertices are marked by double borders for better visibility.

As it is possible to transfer the classical representation of a dynamical system to its graph-theoretic representation, the reverse is also possible. The adjacency matrix of a system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ is given by

$$\mathcal{A}_{sys} = \begin{bmatrix} 0 & \boldsymbol{C} & 0 \\ 0 & \boldsymbol{A} & \boldsymbol{B} \\ 0 & 0 & 0 \end{bmatrix} \,. \tag{2.7}$$

2.2 Some Fundamental Properties of Linear Systems

In this section, the familiar concepts of controllability and observability, invertibility and rank, stability and zeros of linear systems are introduced for reference.

2.2.1 Controllability and Observability

Controllability and observability are central properties of linear systems. The controllability of a system indicates whether all state variables of a dynamical system can be driven by the input.

Definition 2.2.1 (**Controllability** [ZDG96]). A system (2.2) is called *controllable* if there exists an input $\boldsymbol{u}(t)$, $t \in [0, T]$, which transfers the state $\boldsymbol{x}(t)$ from an initial value $\boldsymbol{x}_1$ at $t = 0$ to a final value $\boldsymbol{x}_2$ in some finite time T. Otherwise, the system is said to be *uncontrollable*.

The controllability can be checked by the following criteria, see e. g. [AM07].

Theorem 2.2.1. A system is controllable if the *controllability matrix*

$$\boldsymbol{Q}_c = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \tag{2.8}$$

has full rank, i. e.

$$\operatorname{rank} \boldsymbol{Q}_c = n \, . \tag{2.9}$$

This is equivalent to the condition

$$\operatorname{rank} \begin{bmatrix} \lambda_i \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = n \tag{2.10}$$

for all $\lambda_i \in \mathbb{C}$ that are eigenvalues of $\boldsymbol{A}$.

By (2.10) it is possible to determine which eigenvalues are controllable and which are not.

Definition 2.2.2 (**Controllable Eigenvalue**). An eigenvalue λ_i of $\boldsymbol{A}$ is called controllable if it fulfills (2.10). Otherwise, it is called uncontrollable.

The dual concept of controllability is observability. It indicates whether the state variables can be calculated from the measurement of the outputs.

Definition 2.2.3 (**Observability** [ZDG96]). The system (2.2) is said to be *observable* if, for any $T > 0$, the initial state $\boldsymbol{x}_1$ at $t = 0$ can be determined from the time history of the input $\boldsymbol{u}(t)$ and the output $\boldsymbol{y}(t)$ in the interval of $[0, T]$. Otherwise, the system is said to be *unobservable*.

The observability of a system can be determined by the following theorem.

Theorem 2.2.2. A system is observable if the *observability matrix*

$$\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix} \tag{2.11}$$

has full rank, i. e.

$$\operatorname{rank} \boldsymbol{Q}_o = n \; . \tag{2.12}$$

This is equivalent to the condition

$$\operatorname{rank} \begin{bmatrix} \lambda_i \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} = n \tag{2.13}$$

for all $\lambda_i \in \mathbb{C}$ that are eigenvalues of $\boldsymbol{A}$.

By (2.13) it is possible to determine which eigenvalues are observable and which are not.

Definition 2.2.4 (**Observable Eigenvalue** [TSH01])**.** An eigenvalue λ_i of $\boldsymbol{A}$ is called observable if it fulfills (2.13). Otherwise, it is called unobservable.

Observability and controllability are dual properties. The dual system of a system (2.2) defined by the triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is constructed by

$$\bar{\boldsymbol{A}} := \boldsymbol{A}^T, \; \bar{\boldsymbol{B}} := \boldsymbol{C}^T \text{ and } \bar{\boldsymbol{C}} := \boldsymbol{B}^T \; . \tag{2.14}$$

Now, if the original system is observable, its dual system is controllable and vice versa. The same holds for their controllable and observable eigenvalues.

2.2.2 Invertibility and Rank

The systems invertibilities are crucial for many problems considered in control engineering [Est+07]. For instance, left invertibility is a precondition for failure detection and isolation, and right invertibility is a precondition for reference tracking and disturbance rejection.

Common definitions of the invertibilities of dynamical systems are given by Moylan [Moy77].

Definition 2.2.5 (**Right Invertibility**)**.** A dynamical system is *right invertible* if, for any [sufficiently smooth] $\boldsymbol{y}_{ref}(t)$ defined on $[0, \infty[$, there exists an $\boldsymbol{u}(t)$ and a choice of $\boldsymbol{x}(0)$ such that $\boldsymbol{y}(t) = \boldsymbol{y}_{ref}(t)$ for all $t \in [0, \infty[$.

Definition 2.2.6 (**Left Invertibility**)**.** Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be any two inputs of a dynamical system and let $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ be the corresponding outputs for the same initial condition $\boldsymbol{x}(0)$. The system is said to be *left invertible* if $\boldsymbol{y}_1(t) = \boldsymbol{y}_2(t)$ for all $t > 0$ implies that $\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$ for all $t > 0$.

More formally, this means that if a system is right invertible, the mapping between the input space and the output space is surjective. Analogously, if a system is left invertible, the mapping between the input space and the output space is injective.

This leads to the equivalent definition:

Definition 2.2.7 (**Right Invertible, Left Invertible, Degenerated,** Definition 3.5.1 [CLS12])**.** The system (2.2) is called *right invertible* if there exists a rational matrix function $\boldsymbol{R}(s)$, such that

$$\boldsymbol{G}(s)\boldsymbol{R}(s) = \boldsymbol{I}_p \ , \tag{2.15}$$

with $\boldsymbol{I}_p$, the $p \times p$ identity matrix. The system (2.2) is called *left invertible* if there exists a rational matrix function $\boldsymbol{L}(s)$, such that

$$\boldsymbol{L}(s)\boldsymbol{G}(s) = \boldsymbol{I}_m \ , \tag{2.16}$$

with $\boldsymbol{I}_m$, the $m \times m$ identity matrix. If Σ_{LS} is both left and right invertible, it is said to be *invertible*. If it is neither left nor right invertible, it is said to be *degenerated*.

If a system is right invertible, it is possible to calculate an input $\boldsymbol{u}(t)$ and an initial condition $\boldsymbol{x}(0)$ such that the system follows a given reference trajectory $\boldsymbol{y}_{ref}(t)$. For left invertible systems, the input $\boldsymbol{u}(t)$ can be determined by the signal of the output $\boldsymbol{y}(t)$ and the initial condition $\boldsymbol{x}(0)$.

Obviously, for a right invertible system, the number of inputs has to be greater or equal to the number of outputs, i. e. $m \geq p$. In the same way for a left invertible system, the number of inputs has to be less or equal to the number of outputs, i. e. $m \leq p$. For invertible systems the same number of inputs and outputs is necessary, i. e. $m = p$. Such systems are called *square*.

Invertibility is a dual property as explained next. The system matrix of the dual system is given by[2]

$$\bar{\boldsymbol{P}}(s) = \boldsymbol{P}(s)^T \tag{2.17}$$

and its transfer function by

$$\bar{\boldsymbol{G}}(s) = \boldsymbol{G}(s)^T \ . \tag{2.18}$$

From linear algebra it is known, that if a matrix $\boldsymbol{M}$ is right invertible, its transposed $\boldsymbol{M}^T$ is left invertible and vice versa. Thus, left and right invertibility is also a dual property. The duality between left and right invertible systems is discussed in detail by [Est+07].

The invertibility of a system can be checked by a rank criterion. Therefore, the rank of a polynomial matrix $\boldsymbol{M}(s)$ is briefly discussed.

Rank of Parameterized Matrices

In linear algebra text books, the rank of a matrix is often defined in the following way:

> **Definition 2.2.8** (**Rank**, Definition 5.2.2. [Mir55])**.** The *rank* of a matrix $\boldsymbol{M}$ is the maximum size k of a $k \times k$ submatrix of $\boldsymbol{M}$ for that the determinant does not vanish.

The determinant $\det \boldsymbol{M}$ of an $n \times n$ matrix $\boldsymbol{M}$ with the elements $m_{i,j}$ can be calculated by

$$\det \boldsymbol{M} := \sum_{(\blacklozenge)} \prod_{i=1}^{n} m_{i,t_i} - \sum_{(\lozenge)} \prod_{i=1}^{n} m_{i,t_i} \tag{2.19}$$

where $\{t_1, t_2, \ldots, t_n\}$ is a permutation of $\{1, 2, \ldots, n\}$ and $(\blacklozenge)$ is the sum of even permutations and $(\lozenge)$ the sum of odd permutations [Rei88].

In literature there are further concepts of the rank of a matrix. In addition to Definition 2.2.8 the following terms are considered: “Generic rank”, “normal rank”, “maximal rank”, “full rank” and “term rank”. Note, to better distinguish between the enumerated terms and Definition 2.2.8, it will sometimes be referred to this rank as “numerical rank”. First, the generic rank, refer to [Mur09; Wou91b], is defined.

[2]This equality holds if the system matrix is defined by $\boldsymbol{P}(s) := \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}$, which is also very common.

Definition 2.2.9 (Generic Rank). The *generic rank* of a parameterized matrix $\boldsymbol{M}[\boldsymbol{\mu}] \in \mathbb{F}^{n \times m}$ with $\begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_d \end{bmatrix} = \boldsymbol{\mu} \in \mathbb{F}^d$ is given by

$$\text{g-rank}\, \boldsymbol{M}[\boldsymbol{\mu}] := \max_{\boldsymbol{\mu}} \text{rank}\, \boldsymbol{M}[\boldsymbol{\mu}] \; . \tag{2.20}$$

Note, generally it is not easy to evaluate the generic rank since the parameters in $\boldsymbol{\mu}$ may be dependent on each other. Therefore, it is not possible to find the generic rank by maximizing the rank for every parameter μ_i independently.

A special case of parameterized matrices are polynomial and rational matrices, where the entries are functions of a complex variable s. In the case of polynomial matrices, these functions are polynomials in s. The elements of rational matrices are rational functions in s.

Since these types of matrices are often considered in control engineering, e. g. $\boldsymbol{G}(s)$ and $\boldsymbol{P}(s)$, their generic rank is called “normal rank” [Kai80; CLS12; Rei88; TSH01] or “maximal rank” [Lun13].

Definition 2.2.10 (Normal Rank). The *normal rank*, denoted by norm-rank, of a polynomial matrix $\boldsymbol{M}(s) \in \mathbb{R}[s]^{n_1 \times n_2}$ or rational matrix [3] $\mathbb{R}(s)^{n_1 \times n_2}$ with $s \in \mathbb{C}$ is given by

$$\text{norm-rank}\, \boldsymbol{M}(s) := \max_{s} \text{rank}\, \boldsymbol{M}(s) \; . \tag{2.21}$$

A polynomial or rational matrix has *full rank* if $\text{norm-rank}\, \boldsymbol{M}(s) = \min(n_1, n_2)$ holds.

Later on, also parameterized polynomial or rational matrices are considered. Since the normal rank is a special case of the generic rank the following equation holds by redefining the parameters:

$$\text{g-rank}\, \boldsymbol{M}[\boldsymbol{\nu}](s) = \text{g-rank}\, \boldsymbol{M}[\boldsymbol{\mu}] \; , \tag{2.22}$$

with $\boldsymbol{M}[\boldsymbol{\nu}](s)$ as a matrix with elements that are polynomial or rational functions in $\boldsymbol{\nu} \in \mathbb{R}^d$ and $s \in \mathbb{C}$ and $\boldsymbol{M}[\boldsymbol{\mu}]$ as the same matrix with redefined parameters $\boldsymbol{\mu} := \boldsymbol{f}(\boldsymbol{\nu}, s)$. Obviously, for fixed parameters (except s)

$$\text{g-rank}\, \boldsymbol{M}(s) = \text{norm-rank}\, \boldsymbol{M}(s) \tag{2.23}$$

holds.

[3] In the case of rational matrices, elements may diverge for values of s that are poles of the rational function the element consists of. Since the definition of rank of a matrix with infinite elements is commonly not given, these values of s are excluded.

In order to illustrate the application of the generic rank, the following example is given.

Example 2.2.1. Consider the parameterized system defined by the triple

$$\boldsymbol{A} = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} , \; \boldsymbol{B} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix} , \boldsymbol{C} = \begin{bmatrix} \text{-}\beta & \gamma \end{bmatrix} . \tag{2.24}$$

Often the norm rank of the system matrix $\boldsymbol{P}(s)$ is investigated. The system matrix of the considered system is given by

$$\boldsymbol{P}(s) = \begin{bmatrix} s & \text{-}\alpha & \text{-}\beta \\ \text{-}\alpha & s & \text{-}\gamma \\ \text{-}\beta & \gamma & 0 \end{bmatrix} . \tag{2.25}$$

As described by Definition 2.2.8 the rank of a matrix is defined by the maximum size of a submatrix that has a non-vanishing determinant. Now consider the largest submatrix, the matrix itself, i. e.

$$\det \boldsymbol{P}(s) = (\gamma^2 - \beta^2)s \neq 0 . \tag{2.26}$$

Since the normal rank Definition 2.2.10 is determined by maximizing only over s, it is not applicable without knowing the numerical values of β and γ. So in this case the generic rank of $\boldsymbol{P}(s)$ with the parameters $\boldsymbol{\mu} := [\alpha, \beta, \gamma, s]$ has to be considered. Now the generic rank depends on the relations between these parameters. For instance if they are all independent, maximizing over $\boldsymbol{\mu}$ leads to a non-vanishing determinant, i. e. $\det \boldsymbol{P}(s) \neq 0$ and thus g-rank $\boldsymbol{P}(s) = 3$. However, in the case that γ and β are depending on each other, e. g. $\beta = -\gamma$ will yield g-rank $\boldsymbol{P}(s) < 3$. This demonstrates that the evaluation of a rank of a parameterized matrix, like $\boldsymbol{P}(s)$ or $\boldsymbol{G}(s)$, must be carried out with care.

In contrast to the definitions of rank described above, the term rank has a combinatorial meaning.

Definition 2.2.11 (Term Rank, Section 2.1.3 [Mur09]**).** The *term rank* of a matrix $\boldsymbol{M}$ is the maximum size k of a $k \times k$ submatrix of $\boldsymbol{M}$ for that at least one term $m_{1,t_1} m_{2,t_2} \dots m_{k,t_k}$ of its determinant (2.19) does not vanish, i. e. there occurs no multiplication by zero in $m_{1,t_1} m_{2,t_2} \dots m_{k,t_k}$.

The term rank will play an important role later when considering the graph-theoretic approach. It is possible to calculate the term rank with the `sprank` function of MatLab.

The relation between the different types of rank are given by the next lemma.

Lemma 2.2.1. It holds that

1. $\text{rank}\, \boldsymbol{M}(s) \leq \text{norm-rank}\, \boldsymbol{M}(s)$,
2. $\text{rank}\, \boldsymbol{M}(s) \leq \text{term-rank}\, \boldsymbol{M}(s)$ and
3. $\text{norm-rank}\, \boldsymbol{M}(s) \leq \text{term-rank}\, \boldsymbol{M}(s)$.

Proof. The first relation is a direct consequence of Definition 2.2.10. The second and third relation hold because if $\text{rank}\, \boldsymbol{M}(s) = k$ or $\text{norm-rank}\, \boldsymbol{M}(s) = k$ at least some minor of $\boldsymbol{M}(s)$ is not zero therefore at least one term $m_{1,t_1} m_{2,t_2} \ldots m_{k,t_k}$ in (2.19) must not vanish. □

To illustrate the differences between these concepts of rank an example is given.

Example 2.2.2. In this example four cases, collected in Table 2.1, are discussed where the three concepts of rank differ.

Case	$\boldsymbol{M}_{i)}(s)$	$\det \boldsymbol{M}_{i)}(s)$	rank	norm-rank	g-rank	term-rank
a)	$\begin{bmatrix} s\text{-}a & e & d \\ 0 & s\text{-}b & 0 \\ 0 & c & 0 \end{bmatrix}$	0	2	2	2	2
b)	$\begin{bmatrix} s\text{-}a & 0 & e \\ 0 & s\text{-}a & \text{-}d \\ d & e & 0 \end{bmatrix}$	$(s-a)ed$ $-d(s-a)e$	2	2	2	3
c)	$\begin{bmatrix} s\text{-}a & 0 & e \\ 0 & s\text{-}a & d \\ d & e & 0 \end{bmatrix}$	$-(s-a)ed$ $-d(s-a)e$	2	3	3	3
d)	$\begin{bmatrix} s\text{-}a & c & 0 \\ 0 & s\text{-}b & d \\ e & 0 & 0 \end{bmatrix}$	ecd	3	3	3	3

Table 2.1: Evaluation of the distinct types of rank for polynomial matrices.

Consider the parameters a to e are non-zero, algebraically independent and fixed. This leads to equivalence of the generic rank and normal rank in the given examples. The (numerical) rank will be evaluated at $s = a$.

Considering case a) all terms of $\det \boldsymbol{M}_{a)}(s)$ vanish, so that $\boldsymbol{M}_{a)}(s)$ has rank deficiency for all four types of rank.

In case b) some terms of the determinant do not vanish. However, if $\det \boldsymbol{M}_{b)}(s)$ is evaluated, it becomes zero. Therefore, case b) has rank deficiency for the numerical rank and the generic rank but not for the term rank. Note that in this case the rank deficiency is independent of the actual values of a, d, e and s.

Case c) is very similar to case b). However, the determinant only vanishes for $s = a$, so a numeric rank deficiency is present.

Comparing case b) and case c) reveals that the generic rank of a matrix may change by numerical cancellations in contrast to the term rank.

In the last case d) the determinant is independent of s, so $\boldsymbol{M}_{d)}(s)$ has full rank for all three types of rank.

Rank of a Linear System

Typically, the *rank of a system* Σ_{LS} is referred as

$$r := \text{norm-rank}\, \boldsymbol{G}(s) \; . \tag{2.27}$$

The system is right-invertible if $r = p$ and left-invertible if $r = m$ [TSH01, Sec. 8.2]. The system is said to be of *full rank* or *non-degenerated* if $r = \min(m, p)$. Otherwise the system is called *degenerated* [SS87, Def. 2.3].

The ranks of the transfer function and the system matrix are related as given by the next lemma.

Lemma 2.2.2 (Lemma 8.9 and proof [TSH01])**.**

$$\text{norm-rank}\, \boldsymbol{P}(s) = \text{norm-rank}\, \boldsymbol{G}(s) + n \; . \tag{2.28}$$

Clearly, for non-degenerated systems the conditions

$$\begin{aligned} \operatorname{rank} \boldsymbol{B} &= m \text{ and} \\ \operatorname{rank} \boldsymbol{C} &= p \end{aligned} \tag{2.29}$$

must be met, meaning that there is no direct redundant actuation or measuring of the plant. Throughout this work it is assumed that (2.29) is always met. However, these conditions are not sufficient for non-degenerated systems, as explained in the following example.

Example 2.2.3. Consider the system defined by the matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{2.30}$$

with $m = p = 2$ inputs and outputs. Although the input matrix $\boldsymbol{B}$ and the output matrix $\boldsymbol{C}$ has rank 2, the system matrix does not have maximal rank.

$$\text{norm-rank} \begin{bmatrix} s-1 & 0 & -1 & 0 & 0 \\ -1 & s-1 & -1 & 0 & -1 \\ -1 & -1 & s-1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = 4\,. \tag{2.31}$$

Hence, the example system is degenerated.

Therefore, in addition to the conditions (2.29), the matrix $\boldsymbol{A}$ has to be considered to determine the non-degeneracy of a system.

2.2.3 Stability

Stability is of major interest in control theory. A brief summery is given here. Stability for dynamical systems was defined by Lyapunov [Lya92] in the following way [Kha02, Def. 4.1]:

Definition 2.2.12 (Asymptotic Stability). An equilibrium point $\bar{\boldsymbol{x}}$, $\boldsymbol{0} = \boldsymbol{A}\bar{\boldsymbol{x}}$ or $\boldsymbol{0} = \boldsymbol{f}(\bar{\boldsymbol{x}})$, of system (2.1) , is called *stable*, if for any $\Omega > 0$ there exists an $\epsilon > 0$ such that

$$\|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| < \epsilon \Rightarrow \|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}\| < \Omega \text{ for all } t > 0\,. \tag{2.32}$$

If additionally $\lim_{t\to\infty} \boldsymbol{x}(t) = \bar{\boldsymbol{x}}$ holds, $\bar{\boldsymbol{x}}$ is called *asymptotically stable*. Otherwise, the equilibrium point is called *unstable*.

Since the stability of linear systems is uniquely determined by the eigenvalues of $\boldsymbol{A}$, stability can be considered as a property of the system. So for linear systems the following theorem has been formulated, see e. g. [Gan59, V.§6 Theorem 3] or [Kha02, Theorem 4.5].

Theorem 2.2.3. A linear system Σ_{LS} is stable if the eigenvalues of the matrix $\boldsymbol{A}$ have non-positive real parts and for every eigenvalue λ_i with zero real part and algebraic multiplicity $q_i \geq 2$, $\operatorname{rank}(\boldsymbol{A} - \lambda_i \boldsymbol{I}) = n - q_i$ holds. If it has only negative real parts, the system is asymptotically stable.

The eigenvalues of $\boldsymbol{A}$ are also called the *(system) poles* of Σ_{LS}. They are determined by the roots of the *characteristic polynomial*

$$\chi(\lambda) := \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n \,. \tag{2.33}$$

Some of the system poles are the poles of $\boldsymbol{G}(s)$, also called *transmission poles.* The transmission poles of a system are those system poles that are simultaneously observable and controllable.

It is immediately possible to give the number of roots in 0 of a polynomial as given by (2.33). The lowest order of λ for that a coefficient α_k exists, equals the number of roots in 0. This means a polynomial can be always factorized so that it contains no roots in 0.

For the determination of stability, the position of the (remaining) roots of (2.33) is of interest. Usually this is done by the Routh-Hurwitz criterion [LT85; Rou77; Hur95], which consists of a necessary and a sufficient condition. The necessary condition is the well-known Theorem of Viète on polynomial roots [Gir29] given below. [4]

Lemma 2.2.3. A necessary condition that the roots σ_{0_i} of the polynomial equation

$$c_0\sigma^\delta + c_1\sigma^{\delta-1} + \ldots + c_{\delta-1}\sigma + c_\delta = 0 \tag{2.34}$$

of order δ have *strictly negative real parts* is, that all coefficients c_i are nonzero and of equal sign, i. e.

$$(c_i > 0 \;\forall i) \vee (c_i < 0 \;\forall i) \quad i = 0, 1, 2, \ldots, \delta \tag{2.35}$$

holds.

Since the sufficient condition of the Routh-Hurwitz criterion is of no further interest in this work, the reader is referred to any textbook about linear control systems.

[4] This condition is also known as Stodola condition.

2.2.4 Zeros

Zeros of a dynamical system are the origin of the phenomenon in which outputs are identically zero although inputs and state variables are subject to motion. They often occur due to competing effects in the system that cancel out each other.

In literature, the term "zeros" of a system is not always used in a consistent way [Sva13]. Hence, a short summary of the topic is given. The classifications of [MK76] are used, since these are widely used in literature. According to these classifications, the zeros of a linear system can be described by the four sets: *transmission zeros (TZ)*, *decoupling zeros(DZ)*, *invariant zeros (IZ)* and *system zeros (SZ)*.

In order to classify the zeros of Σ_{LS} (2.2) a canonical form of a rational matrix, the *Smith-McMillan-Form* [Kai80], is used. The Smith-McMillan-Form is an extension of the *Smith-Form*, a canonical form of the polynomial matrix $\boldsymbol{N}(s) \in \mathbb{R}[s]^{n_1 \times n_2}$ with normal rank $\rho \leq \min(n_1, n_2)$. The Smith-Form $\boldsymbol{S}_N(s)$ of $\boldsymbol{N}(s)$ is obtained by elementary matrix operations

$$\boldsymbol{V}(s)\boldsymbol{N}(s)\boldsymbol{W}(s) = \boldsymbol{S}_N(s) = \begin{bmatrix} \mathrm{diag}(\epsilon_1(s), \epsilon_2(s), \dots, \epsilon_\rho(s)) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad (2.36)$$

where the transformation matrices $\boldsymbol{V}(s)$ and $\boldsymbol{W}(s)$ are unimodular. Herein $\epsilon_i(s)$ are unique monic polynomials where $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$ for $i = 1, \dots, \rho - 1$. The polynomials $\epsilon_i(s)$ are called the *invariant polynomials* of $\boldsymbol{N}(s)$.

Consider the rational matrix $\boldsymbol{M}(s) \in \mathbb{R}[s]^{n_1 \times n_2}$ with normal rank $\rho \leq \min(n_1, n_2)$. It can be rewritten as

$$\boldsymbol{M}(s) = \frac{\boldsymbol{N}(s)}{d(s)} \qquad (2.37)$$

and transformed to

$$\frac{\boldsymbol{S}_N(s)}{d(s)} \qquad (2.38)$$

(with $\boldsymbol{N}(s)$ being the appropriate polynomial matrix, $\boldsymbol{S}_N(s)$, its Smith-Form and $d(s) \in \mathbb{R}[s]$), the monic least common denominator of the elements of $\boldsymbol{M}(s)$.

Reducing the elements of $\frac{\boldsymbol{S}_N(s)}{d(s)}$ to the lowest terms yields the Smith-McMillan-Form

$$\boldsymbol{S}_M(s) = \begin{bmatrix} \mathrm{diag}(\frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, \dots, \frac{\epsilon_\rho(s)}{\psi_\rho(s)}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \qquad (2.39)$$

of $\boldsymbol{M}(s)$. Herein $\epsilon_i(s)$ and $\psi_i(s)$ are coprime monic polynomials where $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$ and $\psi_{i+1}(s)$ divides $\psi_i(s)$ for $i = 1, \ldots, \rho - 1$ and $\psi_1(s) = d(s)$.

With the help of the Smith-Form and the Smith-McMillan-Form, the various sets of zeros of Σ_{LS} can be defined as follows.

Definition 2.2.13 (Zeros).

- The transmission zeros are the zeros of the set of the polynomials $\epsilon_i(s)$ of the Smith-McMillan-Form (2.39) of the transfer function $\boldsymbol{G}(s)$.

- The input decoupling zeros (IDZ), identical with the uncontrollable eigenvalues, are the zeros of the invariant polynomials of

$$\boldsymbol{P}_I(s) := \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} . \tag{2.40}$$

- The output decoupling zeros (ODZ), identical with the unobservable eigenvalues, are the zeros of the invariant polynomials of

$$\boldsymbol{P}_O(s) := \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} . \tag{2.41}$$

- The input output decoupling zeros ($IODZ$), identical with the unobservable and uncontrollable eigenvalues, are those output decoupling zeros that disappear when the input decoupling zeros are eliminated.

- The decoupling zeros are that list of values that fulfill the criterion for the IDZ or for the ODZ, reduced by the list of values that fulfill the criterion for the IODZ, i. e.

$$DZ = [IDZ] + [ODZ] - [IODZ] . \tag{2.42}$$

- The system zeros are the union of the transmission zeros and the decoupling zeros, i. e. $SZ = TZ \cup DZ$.

- The invariant zeros are the zeros of the invariant polynomials of the system matrix $\boldsymbol{P}(s)$.

Since the transformation to the Smith-Form is unimodular, which leads to the rank of $\boldsymbol{N}(s)$ and $\boldsymbol{S}_N(s)$ being identical for all s, the invariant zeros can also be determined by

$$IZ = \{s_0 \in \mathbb{C}\,|\text{rank}\,\boldsymbol{P}(s_0) < \text{norm-rank}\,\boldsymbol{P}(s)\} \tag{2.43}$$

and the decoupling zeros by

$$IDZ = \{s_0 \in \mathbb{C}\,|\text{rank}\,\boldsymbol{P}_I(s_0) < \text{norm-rank}\,\boldsymbol{P}_I(s)\} \tag{2.44}$$

and

$$ODZ = \{s_0 \in \mathbb{C}\,|\text{rank}\,\boldsymbol{P}_O(s_0) < \text{norm-rank}\,\boldsymbol{P}_O(s)\}\ . \tag{2.45}$$

However, in these cases the multiplicity of the zeros cannot be determined.

Furthermore, the system zeros can be directly calculated by the following lemma of [Ros74].

Lemma 2.2.4. Let r be the rank of Σ_{LS}. Consider the minors of $\boldsymbol{P}(s)$

$$\boldsymbol{P}(s)^{\{1,2,\ldots,n,n+i_1,n+i_2,\ldots,n+i_r\}}_{\{1,2,\ldots,n,n+j_1,n+j_2,\ldots,n+j_r\}} \tag{2.46}$$

formed by the rows $\{1,2,\ldots,n,n+i_1,n+i_2,\ldots,n+i_r\}$ and columns $\{1,2,\ldots,n,n+j_1,n+j_2,\ldots,n+j_r\}$, where $n+i_1,n+i_2,\ldots,n+i_r$ and $n+j_1,n+j_2,\ldots,n+j_r$ are subsets of size r. The system zeros may now be determined by the roots of the monic greatest common divisor $p_{SZ}(s)$ of all these minors that do not vanish.

The various sets of zeros are visualized in Figure 2.2.

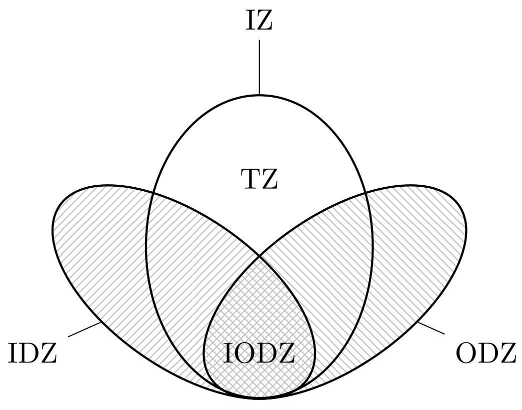

Figure 2.2: Diagram of the described sets the system zeros consist of. The labels of the borders refer to the sets enclosed by the border. The labels of the areas refer to the set delimited by the border of the labeled area.

If the system (2.2) is non-degenerated and square, the invariant zeros are the roots of the polynomial resulting from the determinant of the system matrix (2.3),

$$IZ = \{s \in \mathbb{C} | \det \boldsymbol{P}(s) = 0\} \ . \tag{2.47}$$

In this case, the set of invariant zeros consists of the non-intersecting sets of the transmission zeros and decoupling zeros, i. e. $IZ = TZ \cup DZ$.

Moreover, if the system Σ_{LS} is also minimal, there are no decoupling zeros and hence $SZ = IZ = TZ$.

The zeros of $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ and the dual counterpart $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{C}})$ are related. Considering transmission zeros the Smith-McMillan-Form (2.39) of $\bar{\boldsymbol{G}}(s)$ and of $\boldsymbol{G}(s)$ is identical since it is diagonal. Hence, the transmission zeros of a system and its dual system are identical. In the same manner, this holds true for the invariant zeros by considering the Smith-Form of $\bar{\boldsymbol{P}}(s)$ and of $\boldsymbol{P}(s)$. For the decoupling zeros the case is different since these are related to observability and controllability. Therefore, it holds that $\bar{\boldsymbol{P}}_I(s) = \boldsymbol{P}_O^T(s)$ and $\bar{\boldsymbol{P}}_O(s) = \boldsymbol{P}_I^T(s)$. Considering the Smith-Form of these matrices leads to the conclusion that the input decoupling zeros of a system are the output decoupling zeros of its dual system and vice versa.

The topic of system zeros is comprehensively covered in [Sma06].

2.3 Zero Dynamics

Many linear systems are obtained by linearization of nonlinear equations resulting from physical modelling. These linear systems may contain zeros as described in the previous section. This raises the question whether nonlinear systems contain something similar to “nonlinear zeros”. Therefore, in [BI84] the concept of zero dynamics was introduced, which is a generalization of the concept of zeros of a linear system to nonlinear square systems. According to Isidori, refer to [Isi13], the zero dynamics can be defined as follows:

> **Definition 2.3.1** (**Zero Dynamics**)**.** The *zero dynamics* of a dynamical system are identical with the dynamical system that characterizes the internal behavior of the system once initial conditions and inputs are chosen in such a way as to constrain the output to be identically zero.

These dynamics can be isolated from the nonlinear system (2.5) by a special transformation.

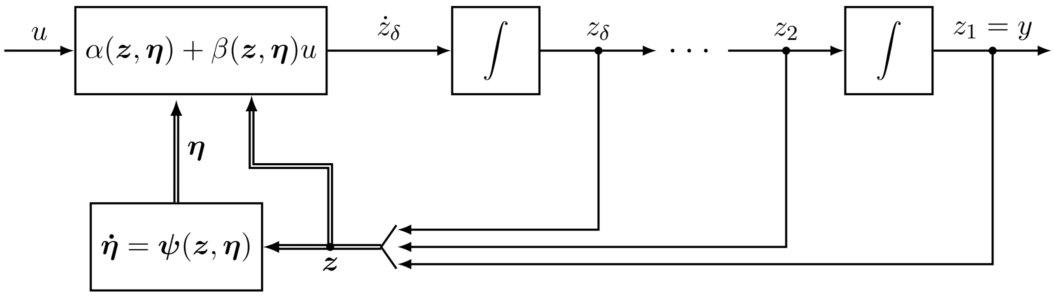

Figure 2.3: Scheme of a nonlinear SISO system in Byrnes-Isidori normal form.

This transformation splits up the system locally into external dynamics and internal dynamics. The result of this transformation is called the *Byrnes-Isidori normal form*. The procedure is strongly related to the control approach called Input-Output-Linearization [Isi95].

The Byrnes-Isidori normal form for a nonlinear SISO system is sketched in Figure 2.3. The top row indicates the external dynamics containing a nonlinear algebraic equation of the input and a chain of simple integrators leading to the output. In the bottom row the internal dynamics are depicted, which consist of a nonlinear dynamical system. In order to obtain the Byrnes-Isidori normal form, some preconditions must be met.

Suppose the system (2.5) is square, i. e. $m = p$, and has an equilibrium $\boldsymbol{x}_0$ at $\boldsymbol{x} = \mathbf{0}$ and Ω is a neighbourhood around $\boldsymbol{x}_0$. Let $\delta_1, \delta_2, \ldots, \delta_p$ be integers with $\delta_i > 0$ so that $\sum_{i=1}^{p} \delta_i \leq n$ holds. These are determined subsequently.

The system (2.5) can be locally transformed to

$$\dot{\boldsymbol{z}} = \begin{bmatrix} z_{2,1} \\ z_{3,1} \\ \vdots \\ z_{\delta_1,1} \\ \alpha_1(\boldsymbol{z},\boldsymbol{\eta}) + \boldsymbol{\beta}_1(\boldsymbol{z},\boldsymbol{\eta})\boldsymbol{u} \\ \vdots \\ z_{2,p} \\ z_{3,p} \\ \vdots \\ z_{\delta_p,p} \\ \alpha_p(\boldsymbol{z},\boldsymbol{\eta}) + \boldsymbol{\beta}_p(\boldsymbol{z},\boldsymbol{\eta})\boldsymbol{u} \end{bmatrix} \qquad (2.48) \qquad \text{and} \qquad \dot{\boldsymbol{\eta}} = \boldsymbol{\psi}(\boldsymbol{z},\boldsymbol{\eta}) \qquad (2.49)$$

with $\boldsymbol{z} \in \mathbb{R}^{\delta}$ and $\boldsymbol{\eta} \in \mathbb{R}^{n-\delta}$ if there exists a transformation

$$\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\eta} \end{bmatrix} = \boldsymbol{\Phi}(\boldsymbol{x}) \tag{2.50}$$

which is diffeomorph in Ω. The Byrnes-Isidori normal form splits the transformed system up into the internal dynamics (2.49) and the external dynamics (2.48).

In order to determine the transformation (2.50), each output y_i for $i = 1, 2, \ldots, p$ is derived. For compact description, the Lie derivative L is used here, defined as

$$\dot{y}_i = \frac{\partial}{\partial \boldsymbol{x}} h_i(\boldsymbol{x})^T \dot{\boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} h_i(\boldsymbol{x})^T (\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}) = L_{\boldsymbol{f}} h_i(\boldsymbol{x}) + L_{\boldsymbol{g}} h_i(\boldsymbol{x})\boldsymbol{u} \,. \tag{2.51}$$

The output is derived until the first time an input appears, i. e.

$$\begin{aligned} \dot{y}_i &= L_{\boldsymbol{f}} h_i(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{g}} h_i(\boldsymbol{x})}_{=\boldsymbol{0}} \boldsymbol{u} \\ \ddot{y}_i &= L_{\boldsymbol{f}}^2 h_i(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{f}} L_{\boldsymbol{g}} h_i(\boldsymbol{x})}_{=\boldsymbol{0}} \boldsymbol{u} \\ &\vdots \\ \overset{(\delta_i-1)}{y_i} &= L_{\boldsymbol{f}}^{\delta_i-1} h_i(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{f}}^{\delta_i-2} L_{\boldsymbol{g}} h_i(\boldsymbol{x})}_{=\boldsymbol{0}} \boldsymbol{u} \\ \overset{(\delta_i)}{y_i} &= L_{\boldsymbol{f}}^{\delta_i} h_i(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{f}}^{\delta_i-1} L_{\boldsymbol{g}} h_i(\boldsymbol{x})}_{\neq\boldsymbol{0}} \boldsymbol{u} \,. \end{aligned} \tag{2.52}$$

The number of differentiations needed for this is denoted by δ_i. This means that the Lie derivative $L_{\boldsymbol{f}}^j L_{\boldsymbol{g}} h_i(\boldsymbol{x}) = 0$ for $j = 0, 1, \ldots, \delta_i - 1$.

With

$$\boldsymbol{a}(\boldsymbol{x}) = \begin{bmatrix} L_{\boldsymbol{f}}^{\delta_1} h_1(\boldsymbol{x}) \\ L_{\boldsymbol{f}}^{\delta_2} h_2(\boldsymbol{x}) \\ \vdots \\ L_{\boldsymbol{f}}^{\delta_p} h_p(\boldsymbol{x}) \end{bmatrix} \tag{2.53}$$

and

$$\boldsymbol{D}(\boldsymbol{x}) = \begin{bmatrix} L_{\boldsymbol{f}}^{\delta_1-1} L_{\boldsymbol{g}} h_1(\boldsymbol{x}) \\ L_{\boldsymbol{f}}^{\delta_2-1} L_{\boldsymbol{g}} h_2(\boldsymbol{x}) \\ \vdots \\ L_{\boldsymbol{f}}^{\delta_p-1} L_{\boldsymbol{g}} h_p(\boldsymbol{x}) \end{bmatrix} \tag{2.54}$$

the derivatives $\overset{(\boldsymbol{\delta})}{\boldsymbol{y}} = \begin{bmatrix} \overset{(\delta_1)}{y_1} & \dots & \overset{(\delta_p)}{y_p} \end{bmatrix}^T$ can be combined to

$$\overset{(\boldsymbol{r})}{\boldsymbol{y}} = \boldsymbol{a}(\boldsymbol{x}) + \boldsymbol{D}(\boldsymbol{x})\boldsymbol{u} \; . \tag{2.55}$$

As described by (2.49) and (2.48) the integer δ gives the dimensions of the internal and external dynamics of a system. Since this is a fundamental property of dynamical systems, δ was named "relative degree" and the following definition, see [Kha02, §13.2] or [Isi95, §5.2], was given.

Definition 2.3.2 (**Relative Degree**)**.** *In Ω the relative degree δ_i of an output y_i, $j = 1, 2, \dots, p$ is the number of differentiations by time of this output that have to be carried out until the first time any input u_j, $j = 1, 2, \dots, m$ will appear. If furthermore the matrix $\boldsymbol{D}(\boldsymbol{x})$ is non-singular in Ω, the vector $\boldsymbol{\delta} := \begin{bmatrix} \delta_1 & \delta_2 & \dots & \delta_p \end{bmatrix}^T$ is called the vector relative degree* [5] *of the nonlinear system (2.5) and the overall relative degree is given by the sum $\delta := \sum_{i=1}^{p} \delta_i \leq n$.*

Remark 2.3.1. It may happen that a system will not have a proper relative degree. If for instance for some output y_i there is no relation to any input u_j, the differentiation (2.52) may be carried out infinite times without occurrence of an input in the equations. If the system has a relative degree δ it is limited by the number of states n, see [Isi95, Proposition 5.1.2], and thus for each output y_i by $\delta_i \leq n - p + 1$. Exceptions are systems with feed-through. There, the relative degree associated with an output may have the value 0 and hence the maximum relative degree of an output is independent of the number of outputs and thus $\delta_i \leq n$.

With (2.52) the first δ rows of the transformation (2.50) are given by

$$\boldsymbol{z} = \begin{bmatrix} z_{1,1} \\ z_{2,1} \\ \vdots \\ z_{\delta_1,1} \\ \vdots \\ z_{1,p} \\ z_{2,p} \\ \vdots \\ z_{\delta_p,p} \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ \vdots \\ y_1^{(\delta_1-1)} \\ \vdots \\ y_p \\ \dot{y}_p \\ \vdots \\ y_p^{(\delta_p-1)} \end{bmatrix} . \tag{2.56}$$

[5] In [Isi85] this property was also called *well-defined (vector) relative degree.*

In order to complete the transformation (2.50) further $n - \delta$ coordinates have to be found that fulfill the diffeomorphism in Ω. This is done by searching $n - \delta$ linear independent coordinates which are orthogonal on $\boldsymbol{z}$. Therefore, the last $n - \delta$ rows $\Phi_i(\boldsymbol{x})$, $i \in [\delta + 1, n]$, of (2.50) have to satisfy the partial differential equations

$$L_{\boldsymbol{g}}\Phi_i(\boldsymbol{x}) = \boldsymbol{0}\,. \tag{2.57}$$

A solution to this system of partial differential equations is not always possible to find due to difficulties in the required symbolic computations [Jag95; Röb17]. A systematic approach to find the transformation (2.50) has been published in [Röb17] for SISO systems.

By Definition 2.3.1 the zero dynamics can be isolated by setting all outputs y_i and their δ_i derivatives for $i = 1, 2, \ldots, p$ to zero, i. e. $\boldsymbol{z} = \boldsymbol{0}$ and $\boldsymbol{u} = \text{-}\boldsymbol{D}^{-1}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x})$. This yields the zero dynamics

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\psi}(\boldsymbol{0}, \boldsymbol{\eta})\,. \tag{2.58}$$

2.4 Non-Minimum Phase Systems

Originally the term “minimum phase” was introduced by H. W. Bode: “Poles of the transfer immittance may occasionally be found in the right half-plane, even for passive networks. Transfer immittances having no poles in the right half-plane, however, have the special property of being ’minimum phase shift’ functions.” [Bod45, p. 121]. In modern terminology, this equals to transfer functions of linear systems that have no zeros in the right complex half plane. He called the opposite “non-minimum phase” functions, meaning transfer functions that have zeros in the right complex half-plane. For both cases, he assumed that the poles of the transfer function are in the left complex half plane.

As mentioned in the introduction, non-minimum phase systems impose several limitations on control. Control limitations for linear non-minimum phase systems, i. e. systems with right half plane zeros, are described in [SP05]. First, such systems may show inverse response behavior and in many other control applications this behavior is disapproved. Another negative property of linear non-minimum phase systems is high gain instability. This means, by increasing the feedback gain for achieving faster response behavior, the closed loop system tends to get unstable since the poles are moving in the direction of the (right half plane) zeros. Related to this are the bandwidth limitations for the achievable control performance. The frequency where tight control is possible are approximately either half of the frequency of the slowest zero or double of the frequency of the fastest zero. All this leads to poor trajectory tracking behavior.

The definition of Bode was later extended by [BI84] and related to the zero dynamics of nonlinear systems: “In analogy with the case of linear systems, which are traditionally said to be ’minimum phase’ when all their transmission zeros have negative real part, nonlinear systems whose zero dynamics have a globally stable equilibrium at z = 0 are also called minimum phase systems.” [Isi95, p.436]. Since this definition is more general, and in absence of a concise term for the stability of zero dynamics, the following definition by Isidori is used.

Definition 2.4.1 (**(Non-)Minimum Phase**). A system that has none or asymptotically stable zero dynamics is called *minimum phase*. Otherwise, the system is called *non-minimum phase*.

As will be shown in Chapter 3, generally all zeros of a linear system are included in its zero dynamics. The result is that if a linear system has non-negative zeros, it will be non-minimum phase according to Definition 2.4.1. It should be mentioned, that although the use of the term non-minimum phase in this case is common, it is controversial [Zei14]. In contrast to the original definition by Bode, this definition is independent of the position of the poles when considering linear systems. It makes also a precise distinction between zeros on the imaginary axis and asymptotic stable zeros. Both extensions are very reasonable if you regard minimum phase systems as systems that do not impose limitations on control due to the location of their zeros. The position of the poles, as long as they are not decoupling zeros, is no obstacle for control, since they can be arbitrarily placed by feedback. That zeros in 0 may be problematic is demonstrated by the following example.

Example 2.4.1. Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \boldsymbol{x} + d\,, \tag{2.59}$$

where d is an output disturbance. This system has 3 eigenvalues in 0. One is a transmission pole and the remaining two are decoupling zeros. Both of the zeros are IDZ, one of them is also ODZ, this means it is an IODZ. Now only the input-output behavior is regarded, which is described by

$$\boldsymbol{G}(s) = \frac{1}{s}\,. \tag{2.60}$$

For any initial state $x_1(0) \neq 0$ the output will diverge. To "stabilize" the output, one may conclude to use the controller

$$u = -y \; . \tag{2.61}$$

As shown in the simulation depicted in Figure 2.4 this controller will succeed in stabilizing the output. However, the internal states x_2 and x_3 will diverge, which

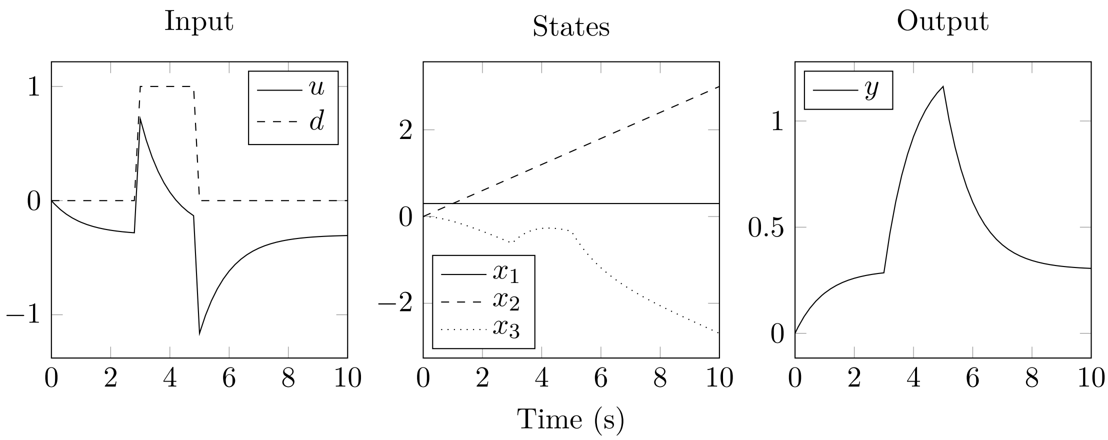

Figure 2.4: Simulation of system (2.59) with controller (2.61), output disturbance d and initial states $\boldsymbol{x} = [0.3\ 0\ 0]^T$.

certainly is a non-desirable behavior for any control loop. The reason for this are the decoupling zeros in 0.

3 Zero Dynamics of Linear Systems

The following chapter will cover two closely related concepts in control theory: the zeros of linear systems and the zero dynamics of dynamical systems. Originally, the zero dynamics were introduced to obtain a concept similar to the zeros of linear systems for nonlinear systems as described in Section 2.3. In the following sections, it will be investigated to which extend the concept of zero dynamics is applicable for linear systems and how it can be related to the zeros.

Since nonlinear systems (2.5) are a generalization of linear systems (2.2), the concept of zero dynamics should be transferable to linear systems. It is known that the linear approximation of the zero dynamics coincides with the zero dynamics of the linear approximation around an equilibrium of a nonlinear system [Isi95, § 4.3]. In the following introductory example the Byrnes-Isidori normal form (BIN), refer to Section 2.3, will be applied to a square non-degenerated linear system. It will be investigated how invariant zeros and the zero dynamics of this system are related.

Example 3.0.1. Consider the linear system given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}. \tag{3.1}$$

Calculating the invariant zeros, as described in Subsection 2.2.4, yields one transmission zero at 2, an input decoupling zero at -4 and one output decoupling zero at -1. In order to get the BIN of the system, the relative degree of it has to be determined. This is done by deriving each output until any input occurs for the first time in the equation. Deriving the output vector once yields

$$\dot{\boldsymbol{y}}(t) = \begin{bmatrix} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & -4 & 0 & 3 \end{bmatrix} \boldsymbol{x}(t) + \underbrace{\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}}_{=\boldsymbol{D}} \boldsymbol{u}(t) \ . \tag{3.2}$$

This means that each output has relative degree 1. Since the matrix $\boldsymbol{D}$ is invertible, the overall relative degree is 2. Therefore, the order of the external dynamics is 2 and the order of the internal dynamics has to be 3 since the system order is 5. The size of the internal dynamics corresponds to the number of the invariant zeros. As defined in Section 2.3, the transformation matrix will be constructed by $\boldsymbol{C}$. In order to complete this matrix, 3 additional linearly independent rows have to be found. Since the internal dynamics should be independent of the input, they can be constructed from the kernel of $\boldsymbol{B}^T$, i. e. $\boldsymbol{v}_i \in \ker \boldsymbol{B}^T \mid \boldsymbol{v}_i \notin \operatorname{im} \boldsymbol{C}$. This yields

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} . \tag{3.3}$$

With this matrix the system can be transformed by $\hat{\boldsymbol{A}} = \boldsymbol{\Phi}\boldsymbol{A}\boldsymbol{\Phi}^{-1}$, $\hat{\boldsymbol{B}} = \boldsymbol{\Phi}\boldsymbol{B}$ and $\hat{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{\Phi}^{-1}$ resulting in

$$\begin{aligned} \dot{\begin{bmatrix} z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 1 & 2 & 1 & 0 & -7 \\ 1 & 1 & 2 & 0 & -2 \\ 2 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}}_{=\hat{\boldsymbol{A}}} \begin{bmatrix} z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 \\ -2 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{=\hat{\boldsymbol{B}}} \boldsymbol{u} \ , \\ \boldsymbol{y} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{=\hat{\boldsymbol{C}}} \begin{bmatrix} z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \ . \end{aligned} \tag{3.4}$$

Herein are z_i the coordinates of the external dynamics and η_i the coordinates of the internal dynamics. Finally, the internal dynamics will be separated to gain the zero dynamics. This is done by solving the "zeroing the output problem". The aim is to find a control law that guarantees $\boldsymbol{y} = \mathbf{0}$ for all times. The relation between input and output is given by

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 1 & 2 & 1 & 0 & -7 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix} \boldsymbol{u} \,. \tag{3.5}$$

The requirement $\dot{\boldsymbol{y}} = \mathbf{0}$ leads to the control law

$$\boldsymbol{u} = -\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 1 & 2 & 1 & 0 & -7 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \,. \tag{3.6}$$

Insertion in the transformed system leads to the autonomous system

$$\dot{\begin{bmatrix} z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & -2 \\ 2 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{3.7}$$

where the external dynamics are now independent of the internal dynamics and hence the zero dynamics are isolated. The zero dynamics of the system are identified by the marked submatrix. This submatrix describes the dynamics of the system once all outputs are 0. As expected, the eigenvalues of it are the invariant zeros of the system. Note, this part of the matrix is unchanged by the feedback law and thus the zero dynamics can be determined directly from the BIN (3.4). Since the transmission zero is positive, the system is non-minimum phase. In the considered case, the problem of controlling such a system is apparent by the dependence of the internal coordinates on the external coordinates. This means, even if the zero dynamics are in the equilibrium, they may be excited and thus diverge.

The relation between the zero dynamics and the invariant zeros of a system was already investigated for certain systems. Square minimal and non-degenerated SISO systems were considered in the state space by [Isi95] and in the frequency domain in [IW13]. MIMO systems were considered in [HL12] and in the framework of the “special coordinate basis” in [GS10]. However, the relation between the zeros and zero dynamics has not yet been investigated for arbitrary linear systems. In this chapter, the application of the BIN to non-square and non-minimal linear systems will be investigated as an extension of [Daa16], where the square non-degenerated case was treated. This is mainly motivated by a presentation of Isidori where he claims zero dynamics were not yet investigated for systems with more inputs than outputs [Isi11]. Some approaches for the non-square case were already considered by [Pol03], mainly in the differential algebraic context.

Apart from the investigation of zero dynamics of linear systems by the Byrnes-Isidori normal form, other normal forms were introduced, e. g. for linear square systems by [Mül09; IK15], for linear square time-varying systems in [BI10] and for non-square linear systems in [Kho16].

In this chapter, the zero dynamics of general linear systems will be determined by a linear version of the Byrnes-Isidori normal form. Therefore, first the relative degree of general linear systems will be investigated. Concluding, the relation to the zeros of the system will be shown.

3.1 Relative Degree of Linear Systems

As already described, a precondition to obtain the Byrnes-Isidori normal form is the existence of a relative degree of the considered system. A closer look will reveal that the method used to determine a relative degree is strongly related to the problem of decoupling linear systems considered by [FW67] and [Gil69]. The steps of determining the relative degree and the decoupling matrix are identical. These approaches for decoupling control have been transferred to non-linear systems by [Por70] and [Fre75]. These publications were possibly the foundation for the Byrnes-Isidori normal form. For decoupling control as well as for the determination of the zero dynamics by e. g. the BIN some kind of system inverse has to exist, namely $\boldsymbol{D}^{-1}$. Thus, invertibility of a system is relevant for this kind of transformations. Consider a plant that has several outputs. In order to steer each output independently, e. g. to zero, at least as many inputs as outputs are required. If this is possible, the system is right invertible as explained in the next section. Decoupling control is always feasible for right invertible systems as demonstrated by [HH83] in frequency space and in the state space by considering a special coordinate basis in [SS87]. Left inverse systems can be partially decoupled as shown in [Dwo11]. For the general case of linear systems, decoupling control was unified in [WA86]. In the context of decoupling control the zeroing the output problem was already considered in [SP71].

By the similarity of the methods, it is apparent that, if decoupling control is possible, also the transformation to BIN should be possible. One major difference between decoupling control and the transformation to Byrnes-Isidori normal form is, that for decoupling a stable controller is desirable, whereas the transformation to BIN is mainly a tool for analysis. For decoupling control, this requires that the considered system has to be minimum phase. For analysis this is obviously not a requirement. Decoupling control is an extensive issue with its own terminology which will not be introduced here. Only the basic relations to the BIN will be referenced in this section.

As already mentioned, a requirement to transform a system to Byrnes-Isidori normal form is that it possesses a (vector) relative degree as defined by Definition 2.3.2. The relative degree is not a term that was introduced in the nonlinear control domain. For linear SISO systems it is used to indicate the difference between the largest order of s in the nominator and the largest order of s in the denominator of the transfer function $\boldsymbol{G}(s)$, also called "relative order" [SP05]. Generally, the relative degree defines the order of the external dynamics. The external dynamics characterize the system part that is fully controllable and observable. Hence, the eigenvalues associated with it are simultaneously controllable and observable [Mit77]. Since these are relevant properties for linear systems and it is mandatory for the Byrnes-Isidori normal form, the relative degree for arbitrary linear systems will be discussed in this section.

In the next subsection, it is shown how the relative degree can be obtained for right invertible systems. This result will be transferred to left invertible systems in order to define the relative degree for general linear systems. Not all systems have innately a relative degree. Therefore, a method is given on how to extend these systems to obtain a relative degree for them. Since linear systems are considered, it will be shown, how to determine the relative degree in the frequency domain. With the relative degree, it is then possible to state the linear form of the BIN and finally some illustrating examples are given.

3.1.1 Right Invertible Systems

In order to transfer a system to BIN, its relative degree, see Definition 2.3.2, has to be known. For right invertible systems, the mapping from the inputs u_j to the y_i is surjective. This means that any output has to be influenced by at least one input. Therefore, identically to the nonlinear approach (2.52), for each output y_i an integer $1 \leq \delta_i \leq n$ can be found by deriving the output δ_i times

$$\overset{(\delta_i)}{y_i} = \boldsymbol{c}_i \boldsymbol{A}^{\delta_i} \boldsymbol{x} + \boldsymbol{c}_i \boldsymbol{A}^{\delta_i - 1} \boldsymbol{B} \boldsymbol{u} \tag{3.8}$$

with $\boldsymbol{c}_i \boldsymbol{A}^{\delta_i - 1} \boldsymbol{B} \neq \boldsymbol{0}$ and $\boldsymbol{c}_i \boldsymbol{A}^{k} \boldsymbol{B} = \boldsymbol{0}$ for $k < \delta_i - 1$.

The method of determination of the relative degree (3.8) shows another interpretation of the relative degree. For every output, it defines the minimal number of integration steps from the input. This means the (vector) relative degree equals the minimal number of integrations between the inputs and outputs. This will be illustrated by the following example:

Example 3.1.1. Consider the right invertible system defined by the triple

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.9}$$

Deriving the output vector yields

$$\begin{aligned} \dot{\boldsymbol{y}}(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{u}(t) \\ \ddot{\boldsymbol{y}}(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=\boldsymbol{D}} \boldsymbol{u}(t) . \end{aligned} \tag{3.10}$$

According to (3.8), the relative degree $\delta_{1,2}$ for the first and second output are identically 2. By the original definition of the relative degree, Definition 2.3.2, the matrix $\boldsymbol{D}$ has to be non-singular. This is only possible for square matrices. For right invertible systems with $m > p$ $\boldsymbol{D}$ is never square. That means, in the general case this matrix has to be of maximal rank, as required for decoupling [SS87]. Hence, the relative degree of the considered system is $\delta = 4$. This is also visible from the graph $\mathcal{G}_{sys}$ of the system depicted in Figure 3.1. Herein the relative degree

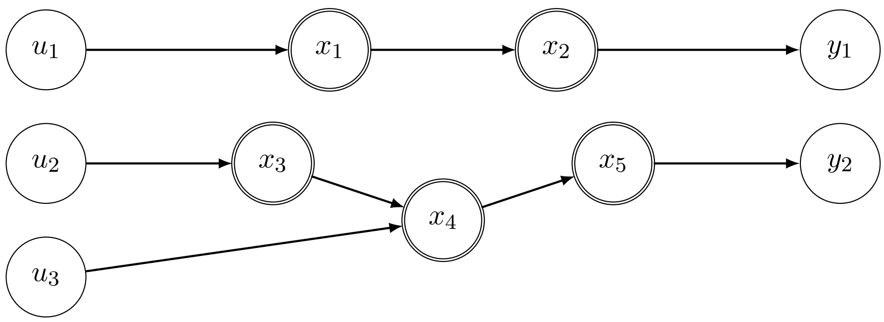

Figure 3.1: $\mathcal{G}_{sys}$ of the system (3.9).

is directly visible from the minimum count of state vertices between the outputs and their nearest input.

3.1.2 Left Invertible and Degenerated Systems

For left invertible systems, the issue of determining the relative degree is not as straight forward as for right invertible systems. In general, left invertible but not right invertible systems contain more outputs than inputs. These systems are injective, which is a problem if a relative degree has to be determined. The injectivity involves that in general there is no unique relationship between the outputs and the inputs. In some cases, even an output may not have any relation to an input. This is, however, a precondition for the determination of the relative degree by its definition for square systems. Hence, in general it is not possible to determine the (vector) relative degree of a left inverse system in the way described before.

The problem of determination of the relative degree is demonstrated by the next example.

Example 3.1.2. The dual system of Example 3.1.1 is given by ($\bar{\boldsymbol{A}} = \boldsymbol{A}^T, \bar{\boldsymbol{B}} = \boldsymbol{B}^T, \bar{\boldsymbol{C}} = \boldsymbol{C}^T$). Its graph is shown in Figure 3.2. Deriving the outputs yields

$$\begin{aligned}
\dot{\boldsymbol{y}}(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}(t) \\
\ddot{\boldsymbol{y}}(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}(t) \\
\dddot{y}_2(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{u}(t) \; .
\end{aligned} \tag{3.11}$$

This leads to $\delta_{1,3} = 2$ for $y_{1,3}(t)$ and to $\delta_2 = 3$ for $y_2(t)$. The decoupling matrix is then given by

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \; . \tag{3.12}$$

It has maximal rank, however, the summation of $\delta_1, \delta_2, \delta_3$ would lead to a relative degree of $\delta = 7$. This is not possible since $r \leq \delta \leq n$ and the order of the system n is only 5.

In order to solve this problem, consider again the duality to Example 3.1.1. There, the overall relative degree equals 4, since the minimal number of integrators is given by two paths from the two outputs to two inputs, omitting one of the inputs, see Figure 3.1. However, in the actual example there are three paths as seen by Figure 3.2. The difference is that in Example 3.1.1 the

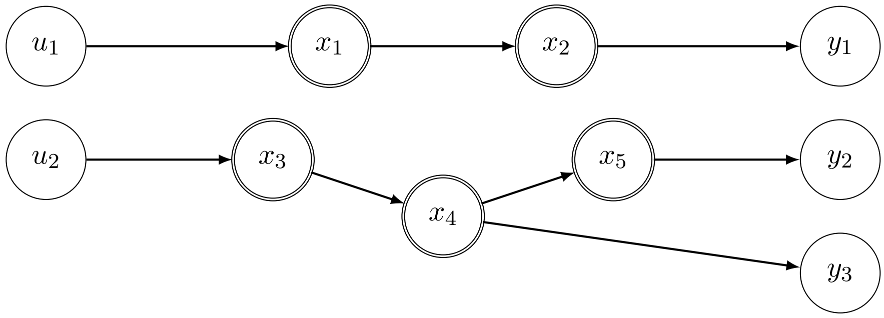

Figure 3.2: $\mathcal{G}_{sys}$ of the dual system of (3.9).

paths do not share any vertices, i. e. they are vertex-disjoint. As will be shown in Section 4.2, the minimal number of vertex disjoint paths from the inputs to the outputs is directly related to the rank of a system. In order to solve the problem of relative degree for left invertible systems, the rank of the system can be considered. For left or right invertible systems the rank equals the minimum count of inputs or outputs, i. e. $r = \min(m, p)$ (2.27). In general, the rank of a system determines the maximum number of outputs that can be decoupled, see e. g. [SS87]. As it holds for the right inverse systems, where the relative degree equals the minimal number of integrators between inputs and outputs, a similar approach is possible for left inverse systems considering only the r outputs with the minimal relative degree δ_i.

With this in mind, it is possible to determine a relative degree for general linear systems. As by (3.8) for each output y_i a δ_i is determined by

$$\overset{(k_i)}{y_i} = \boldsymbol{c}_i \boldsymbol{A}^{k_i} \boldsymbol{x} + \boldsymbol{c}_i \boldsymbol{A}^{k_i - 1} \boldsymbol{B} \boldsymbol{u} \tag{3.13}$$

with $\boldsymbol{c}_i \boldsymbol{A}^{k_i-1} \boldsymbol{B} = \boldsymbol{0}$ for $k_i < \delta_i \leq n$ and $\boldsymbol{c}_i \boldsymbol{A}^{\delta_i - 1} \boldsymbol{B} \neq \boldsymbol{0}$. k_i and hence δ_i is limited by the systems order n because of the Cayley-Hamilton Theorem, see e. g. [Ros70]. This means that if $\boldsymbol{c}_i \boldsymbol{A}^{k_i-1} \boldsymbol{B} = \boldsymbol{0}$ for $k_i = n$, also all higher derivatives $k_i > n$ of the considered output will not be influenced by any input. These special outputs do not have a relative degree δ_i. Let $\tilde{p}$ be the number of outputs with relative degrees $\delta_1, \delta_2, \ldots, \delta_{\tilde{p}}$. By reordering the outputs, the input-output relation can be written as

$$\begin{bmatrix} \overset{(\delta_1)}{y_1} \\ \vdots \\ \overset{(\delta_{\tilde{p}})}{y_{\tilde{p}}} \\ y_{\tilde{p}+1} \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \boldsymbol{A}^{\delta_1} \\ \vdots \\ \boldsymbol{c}_{\tilde{p}} \boldsymbol{A}^{\delta_{\tilde{p}}} \\ \boldsymbol{c}_{\tilde{p}+1} \\ \vdots \\ \boldsymbol{c}_p \end{bmatrix} \boldsymbol{x} + \underbrace{\begin{bmatrix} \boldsymbol{c}_1 \boldsymbol{A}^{\delta_1 - 1} \\ \vdots \\ \boldsymbol{c}_{\tilde{p}} \boldsymbol{A}^{\delta_{\tilde{p}}-1} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{B}}_{=\boldsymbol{D}} \boldsymbol{u} \, . \tag{3.14}$$

The system rank r will be used for the determination of the relative degree as argued before. Hence, the second criterion for the relative degree of a system, that the matrix $\boldsymbol{D}$ has to be of full rank, can be changed to the precondition, that the rank of the decoupling matrix has to be identical with the system rank. This extension of the relative degree is summarized by the following definition.

Definition 3.1.1 (**Relative Degree**)**.** A linear system Σ_{LS} of rank r has relative degree δ if there exists a minimal sum of the relatives degrees δ_i (3.13) of r outputs such that the matrix $\tilde{\boldsymbol{D}}$, constructed by the lines of $\boldsymbol{D}$ (3.14) that correspond to the r selected outputs, has rank r, i. e.

$$\delta := \min_{\{\delta_1, \delta_2, \ldots, \delta_{\tilde{p}}\}} \sum_j \delta_j \mid j \in J \subseteq \{1, 2, \ldots, \tilde{p}\} \wedge |J| = r \wedge \operatorname{rank} \tilde{\boldsymbol{D}} = r \, . \tag{3.15}$$

Herein, the reduced decoupling matrix $\tilde{\boldsymbol{D}}$ is given by

$$\tilde{\boldsymbol{D}} := \begin{bmatrix} \boldsymbol{c}_{j_1} \boldsymbol{A}^{\delta_{j_1} - 1} \\ \vdots \\ \boldsymbol{c}_{j_r} \boldsymbol{A}^{\delta_{j_r} - 1} \end{bmatrix} \boldsymbol{B} \, . \tag{3.16}$$

The vector relative degree $\boldsymbol{\delta}$ of the system is given by the vector of the relative degrees of the r selected outputs $\{y_{j_1}, y_{j_2}, \ldots, y_{j_r}\}$.

If a relative degree according to this definition exists, the system can be split up in external and internal dynamics similarly to nonlinear systems as described in Section 2.3. The external dynamics can be decoupled by the feedback controller

$$\boldsymbol{u} = -\tilde{\boldsymbol{D}}^{+} \left(\begin{bmatrix} \boldsymbol{c}_{j_1} \boldsymbol{A}^{\delta_{j_1}} \\ \vdots \\ \boldsymbol{c}_{j_r} \boldsymbol{A}^{\delta_{j_r}} \end{bmatrix} \boldsymbol{x} - \boldsymbol{\vartheta} \right) \tag{3.17}$$

with $\tilde{\boldsymbol{D}}^{+}$ the pseudo inverse of $\tilde{\boldsymbol{D}}$ and a new input $\boldsymbol{\vartheta}$. With this feedback controller, r outputs can be driven independently.

Remark 3.1.1. It may happen that there is more than one possible choice of r outputs that lead to a minimal relative degree. Since the selection of the outputs that form the relative degree determines the external dynamics and internal dynamics, the transformation is not unique.

Considering again Example 3.1.2, the relative degree of this system can now be determined. The rank of the system is 2. This means, two of three outputs have to be selected. It is obvious from the decoupling matrix that the first output has to be used. In order to minimize the overall relative degree, the third output is also selected since $\delta_2 > \delta_3$. This selection leads to a reduced decoupling matrix of rank 2 and, thus, to a relative degree $\delta = \delta_1 + \delta_3 = 4$ of the example system. Note, this is actually the same relative degree as the dual right invertible system possesses.

Remark 3.1.2. In general, it would be also possible to define the relative degree such that the external dynamics are as large as possible by finding the maximum sum of the relative degrees of r outputs under the conditions of Definition 3.1.1. However, this would lead to different values of the relative degree for a system and its dual counterpart; compare the examples. By choosing always the minimum sum, the relative degree is constant regarding duality.

Definition 3.1.1 is also valid for degenerated systems as demonstrated next.

Example 3.1.3. Consider the degenerated system (2.30) from the preliminaries. Derivation of the outputs yields

$$\begin{aligned} \dot{\boldsymbol{y}}(t) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}(t) \\ \ddot{y}_2(t) &= \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{u}(t) \end{aligned} \tag{3.18}$$

This means the relative degrees for the first and second output are $\delta_1 = 1$ and $\delta_2 = 2$. Considering the decoupling matrix

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.19}$$

this leads to an overall relative degree of $\delta = \delta_1 = 1$ by Definition 3.1.1.

3.1.3 Systems without Relative Degree

Not every linear system has (vector) relative degree since its decoupling matrix may not have the same rank as the system itself. In the decoupling control domain, such systems are called to have “weak inherent coupling” [Gil69]. As it is possible to decouple these systems by dynamic feedback, it is possible to extend a system such that it gets a relative degree. This is done by *dynamic extension* [Isi95, Sec. 5.4]. In the general nonlinear case described there, this may fail if the decoupling matrix $\boldsymbol{D}(\boldsymbol{x})$ (2.54) does not have constant rank in the neighbourhood of the considered equilibrium. In the linear case, this is no issue since the rank of $\boldsymbol{D}$ is always constant. Although this method is stated for square systems, only small modifications are necessary to adapt it for the case of Definition 3.1.1. This is possible since only r inputs and r outputs have to be considered, i. e. a square subsystem, in order to obtain a reduced decoupling matrix of rank r. By dynamic extension, the static feedback law (3.17) is extended to a dynamical feedback by introducing additional integrators on the inputs of the plant. This leads to additional state variables in the system description. The aim is to adjust the individual relative degrees of the outputs such that the (reduced) decoupling matrix gets the desired rank. This can be done by the *dynamic extension algorithm* [Isi95, Sec. 5.4], which is directly applicable for linear systems.

Algorithm 3.1.1. Suppose a linear system Σ_{LS} of rank r has a decoupling matrix $\boldsymbol{D}$ with $\operatorname{rank} \boldsymbol{D} < r$. The associated outputs have relative degrees $\delta_1 \dots \delta_{\tilde{p}}$. Let, after possibly reordering of the rows and hence the outputs, $\boldsymbol{d}_i$ be the i-th row of $\boldsymbol{D}$, $\boldsymbol{b}^j$ the j-th column of $\boldsymbol{B}$ and k_i some constants.

1. Find two integers i_0 and j_0 such that

   a) for some integer $1 < l \leq \tilde{p}$

   $$\boldsymbol{d}_l = \sum_{i=1}^{l-1} k_i \boldsymbol{d}_i \tag{3.20}$$

   holds with $k_{i_0} \neq 0$ and

   b)

   $$d_{i_0, j_0} = \boldsymbol{c}_{i_0} \boldsymbol{A}^{\delta_{i_0}-1} \boldsymbol{b}^{j_0} \neq 0 \; , \tag{3.21}$$

   i. e. a nonzero element in the i_0-th row of $\boldsymbol{D}$.

Then a new system can be constructed by

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}} + \frac{1}{d_{i_0,j_0}}\boldsymbol{b}^{j_0}\left(\xi - \sum_{\substack{j=1\\ j\neq j_0}}^{m} d_{i_0,j}u_j\right)\\ \dot{\xi} &= \vartheta \ ,\end{aligned} \tag{3.22}$$

where $\tilde{\boldsymbol{B}}$ is $\boldsymbol{B}$ without the j_0-th column, $\tilde{\boldsymbol{u}}$ is $\boldsymbol{u}$ without the j_0-th entry and ϑ is a new input.

2. Determine the decoupling matrix for the new system. If it is now possible to find a $\tilde{\boldsymbol{D}}$ with rank $\tilde{\boldsymbol{D}} = r$ according to Definition 3.1.1, you are done.

3. Else, repeat with the new system until the following stopping condition, [Isi95, Prop. 5.4.3], is met. Provided that $r_0 = \operatorname{rank} \boldsymbol{D} < r$ and $\delta_1, \ldots, \delta_{r_0}$ are the relative degrees that belong to r_0 independent rows of $\boldsymbol{D}$. Let $\delta_* := \min\{\delta_j : r_0 < j \leq \tilde{p}\}$. If possible, at most $(n - \delta_1 - \ldots - \delta_{r_0} - \delta_*)r_0$ iterations of the algorithm have to be carried out to increment the rank of $\boldsymbol{D}$. Else the algorithm failed and it is not possible to find a relative degree for the system by dynamic extension.[1]

Another algorithm for linear systems is given in [WA86].

The application of Algorithm 3.1.1 is demonstrated by the next example.

Example 3.1.4. Consider the system defined by the triple

$$\begin{aligned}\boldsymbol{A} &= \begin{bmatrix} 0 & 1 & 0 & 0\\ -2 & -3 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -2 & -3\end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix},\\ \boldsymbol{C} &= \begin{bmatrix} 1 & 3 & 1 & 0\\ 1 & 3 & 2 & 0\\ 1 & 3 & 3 & 0\end{bmatrix}.\end{aligned} \tag{3.23}$$

[1]Note, as described in e. g. [WA86], it is always possible to find a dynamic extension for right invertible systems, i. e. if $r = p$.

This system has rank $r = 2$. Deriving the output vector yields

$$\dot{\boldsymbol{y}}(t) = \begin{bmatrix} -6 & -8 & 0 & 1 \\ -6 & -8 & 0 & 2 \\ -6 & -8 & 0 & 3 \end{bmatrix} \boldsymbol{x}(t) + \underbrace{\begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 3 & 0 \end{bmatrix}}_{=\boldsymbol{D}} \boldsymbol{u}(t) \ . \tag{3.24}$$

Obviously, the decoupling matrix $\boldsymbol{D}$ is not of rank 2. Applying one iteration of Algorithm 3.1.1 with $i_0 = 1$ and $j_0 = 1$ yields the new system

$$\hat{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \ \hat{\boldsymbol{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} , \tag{3.25}$$

with $\hat{\boldsymbol{C}} = \boldsymbol{C}$. Deriving the output vector now yields

$$\begin{aligned} \dot{\boldsymbol{y}}(t) &= \begin{bmatrix} -6 & -8 & 0 & 1 & 1 \\ -6 & -8 & 0 & 2 & 1 \\ -6 & -8 & 0 & 3 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}(t) \\ \ddot{\boldsymbol{y}}(t) &= \begin{bmatrix} 16 & 18 & -2 & -3 & -\frac{8}{3} \\ 16 & 18 & -4 & -6 & -\frac{8}{3} \\ 16 & 18 & -6 & -9 & -\frac{8}{3} \end{bmatrix} \boldsymbol{x}(t) + \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{=\boldsymbol{D}} \boldsymbol{u}(t) \ . \end{aligned} \tag{3.26}$$

Herein, the decoupling matrix $\boldsymbol{D}$ is of rank 2, and the system has relative degree $\delta = 4$ with one additional state variable.

3.1.4 Frequency Domain Approach

The relative degrees of the outputs can alternatively be determined directly from $\boldsymbol{G}(s)$. Consider $\boldsymbol{g}_i(s)$ the i-th row of $\boldsymbol{G}(s)$ belonging to the output y_i. Every row is given by a vector of rational functions $\boldsymbol{g}_i(s) = \begin{bmatrix} \frac{d_{i,1}(s)}{n_{i,1}(s)} & \frac{d_{i,2}(s)}{n_{i,2}(s)} & \dots & \frac{d_{i,m}(s)}{n_{i,m}(s)} \end{bmatrix}$ where $d_{i,j}(s)$ are the denominators and $n_{i,j}(s)$ are the nominators. They are polynomials in s. The formalism $\deg d_{i,j}(s)$ or $\deg n_{i,j}(s)$ gives the degree, i. e. the highest order of s, of these polynomials. Now the relative degree of each output y_i can be determined by

$$\delta_i = \min_{1 \leq j \leq m} \deg d_{i,j}(s) - \deg n_{i,j}(s) \ , \tag{3.27}$$

i. e. the relative degree for an output is given by the minimal difference of the denominator and nominator degree in the entries of the corresponding row in $\boldsymbol{G}(s)$. If a row has only zero entries, the associated output has no relative degree. This method is equivalent to (3.13).

With possibly reordering, the relative degrees of all outputs can be combined to a relative degree matrix

$$\boldsymbol{S}_\delta = \mathrm{diag}(s^{\delta_1}, \dots, s^{\delta_{\bar{p}}}, 1, \dots, 1) \ , \tag{3.28}$$

which is a diagonal matrix of s to the power of the relative degree of each output and 1 for the outputs that do not have a relative degree.

Now the decoupling matrix $\boldsymbol{D}$ can be directly calculated from the transfer function matrix $\boldsymbol{G}(s)$.

Theorem 3.1.1. For the decoupling matrix of a linear system

$$\boldsymbol{D} = \lim_{s \to \infty} \boldsymbol{S}_\delta \boldsymbol{G}(s) \tag{3.29}$$

holds.

This is a well-known result from decoupling control, compare e. g. [WA86], however, by considering the state space approach an alternative proof can be derived.

Proof. In order to prove this theorem, first the relation between the frequency domain representation, $\boldsymbol{G}(s)$, and the state space relation (3.14) has to be established. Each line of (3.14)

$$\overset{(\delta_i)}{y_i} = \boldsymbol{c}_i \boldsymbol{A}^{\delta_i} \boldsymbol{x} + \boldsymbol{c}_i \boldsymbol{A}^{\delta_i - 1} \boldsymbol{B} \boldsymbol{u} \tag{3.30}$$

with $\boldsymbol{c}_i \boldsymbol{A}^k \boldsymbol{B} = \boldsymbol{0}$ for $k < \delta_i - 1$ that corresponds to an output that has a relative degree is Laplace-transformed to

$$s^{\delta_i} Y_i(s) = \boldsymbol{c}_i \boldsymbol{A}^{\delta_i} \boldsymbol{X}(s) + \boldsymbol{c}_i \boldsymbol{A}^{\delta_i - 1} \boldsymbol{B} \boldsymbol{U}(s) \ . \tag{3.31}$$

The Laplace state variable vector $\boldsymbol{X}(s)$ resolves to $\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{B} \boldsymbol{U}(s)$ and thus can be replaced. Now the following conversions can be made:

$$\begin{aligned}
s^{\delta_i} Y_i(s) &= \boldsymbol{c}_i \boldsymbol{A}^{\delta_i}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) + \boldsymbol{c}_i \boldsymbol{A}^{\delta_i-1}\boldsymbol{B}\boldsymbol{U}(s) \\
&= \boldsymbol{c}_i \left(\boldsymbol{A}^{\delta_i}(s\boldsymbol{I}-\boldsymbol{A})^{-1} + \boldsymbol{A}^{\delta_i-1}\right)\boldsymbol{B}\boldsymbol{U}(s) \\
&= \boldsymbol{c}_i \left(\boldsymbol{A}^{\delta_i} + \boldsymbol{A}^{\delta_i-1}(s\boldsymbol{I}-\boldsymbol{A})\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) \\
&= \boldsymbol{c}_i \left(s\boldsymbol{A}^{\delta_i-1}\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) \qquad (3.32) \\
&= \boldsymbol{c}_i \left(s\boldsymbol{A}^{\delta_i-1} - s^{\delta_i}\boldsymbol{I} + s^{\delta_i}\boldsymbol{I}\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) \\
&= \underbrace{\boldsymbol{c}_i \left(s\boldsymbol{A}^{\delta_i-1} - s^{\delta_i}\boldsymbol{I}\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}}_{\overset{(a)}{=}\boldsymbol{0}}\boldsymbol{U}(s) + \underbrace{s^{\delta_i}\boldsymbol{c}_i(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}}_{\overset{(b)}{=}s^{\delta_i}\boldsymbol{G}_i(s)}\boldsymbol{U}(s) \qquad (3.33)
\end{aligned}$$

The equality (a) holds because it can be reformulated to

$$\begin{aligned}
&\boldsymbol{c}_i \left(s\boldsymbol{A}^{\delta_i-1} - s^{\delta_i}\boldsymbol{I}\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} = \\
&s\boldsymbol{c}_i \left(\boldsymbol{A}^{\delta_i-1} - s^{\delta_i-1}\boldsymbol{I}\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} = \\
&s\boldsymbol{c}_i (\boldsymbol{A}-s\boldsymbol{I})\left(\boldsymbol{A}^{\delta_i-2} + s\boldsymbol{A}^{\delta_i-3} + \ldots + s^{\delta_i-3}\boldsymbol{A} + s^{\delta_i-2}\boldsymbol{I}\right)(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\ .
\end{aligned} \qquad (3.34)$$

In the last row $\left(\boldsymbol{A}^{\delta_i-1} - s^{\delta_i-1}\boldsymbol{I}\right)$ is factorized. Since $\boldsymbol{A}$ commutates with itself and $\boldsymbol{I}$ it follows:

$$\begin{aligned}
&s\boldsymbol{c}_i \left(\boldsymbol{A}^{\delta_i-2} + s\boldsymbol{A}^{\delta_i-3} + \ldots + s^{\delta_i-3}\boldsymbol{A} + s^{\delta_i-2}\boldsymbol{I}\right)(\boldsymbol{A}-s\boldsymbol{I})(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} = \\
&s\boldsymbol{c}_i\boldsymbol{A}^{\delta_i-2}\boldsymbol{B} + s^2\boldsymbol{c}_i\boldsymbol{A}^{\delta_i-3}\boldsymbol{B} + \ldots + s^{\delta_i-2}\boldsymbol{c}_i\boldsymbol{A}\boldsymbol{B} + s^{\delta_i-1}\boldsymbol{c}_i\boldsymbol{B} = \boldsymbol{0}
\end{aligned} \qquad (3.35)$$

since $\boldsymbol{c}_i\boldsymbol{A}^k\boldsymbol{B} = \boldsymbol{0}$ for $k < \delta_i - 1$.

The second equality (b) holds because

$$Y_i(s) = \boldsymbol{G}_i(s)\boldsymbol{U}(s) = \boldsymbol{c}_i(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)\ . \qquad (3.36)$$

By combining the rows of (3.33) for all outputs, the following relation for (3.14) is given:

$$\begin{aligned}\boldsymbol{S}_\delta \boldsymbol{Y}(s) &= \begin{bmatrix} \boldsymbol{c}_1 \boldsymbol{A}^{\delta_1} \\ \vdots \\ \boldsymbol{c}_{\tilde{p}} \boldsymbol{A}^{\delta_{\tilde{p}}} \\ \boldsymbol{c}_{\tilde{p}+1} \\ \vdots \\ \boldsymbol{c}_p \end{bmatrix} \boldsymbol{X}(s) + \underbrace{\begin{bmatrix} \boldsymbol{c}_1 \boldsymbol{A}^{\delta_1 - 1} \\ \vdots \\ \boldsymbol{c}_{\tilde{p}} \boldsymbol{A}^{\delta_{\tilde{p}}-1} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{B}}_{=\boldsymbol{D}} \boldsymbol{U}(s) \\ &= \underbrace{\begin{bmatrix} \boldsymbol{c}_1 \boldsymbol{A}^{\delta_1 - 1} \\ \vdots \\ \boldsymbol{c}_{\tilde{p}} \boldsymbol{A}^{\delta_{\tilde{p}}-1} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} s\boldsymbol{I}(s\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{B}}_{=\boldsymbol{\Gamma}(s)} \boldsymbol{U}(s) = \boldsymbol{S}_\delta \boldsymbol{G}(s) \boldsymbol{U}(s)\end{aligned} \tag{3.37}$$

with $\boldsymbol{\Gamma}(s)$ from (3.32) and proper reordering the outputs. Finally

$$\lim_{s \to \infty} \boldsymbol{\Gamma}(s) = \boldsymbol{D} \tag{3.38}$$

since $\lim_{s\to\infty} s\boldsymbol{I}(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{I}$. □

3.1.5 Byrnes-Isidori Normal Form for Linear Systems

By Definition 3.1.1, if present, a relative degree for general linear systems can be determined. With this, it is possible to state a linear version of the Byrnes-Isidori normal form, refer to

Section 2.3. This yields the external dynamics of the form

$$\dot{\boldsymbol{z}} = \begin{bmatrix} z_{2,1} \\ z_{3,1} \\ \vdots \\ z_{\delta_{j_1},1} \\ \boldsymbol{\alpha}_1 \boldsymbol{z} + \boldsymbol{\kappa}_1 \boldsymbol{\eta} + \boldsymbol{\beta}_1 \boldsymbol{u} \\ \vdots \\ z_{2,r} \\ z_{3,r} \\ \vdots \\ z_{\delta_{j_r},r} \\ \boldsymbol{\alpha}_r \boldsymbol{z} + \boldsymbol{\kappa}_r \boldsymbol{\eta} + \boldsymbol{\beta}_r \boldsymbol{u} \end{bmatrix} \tag{3.39}$$

and the internal dynamics of the form

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\Psi} \boldsymbol{z} + \boldsymbol{\Lambda} \boldsymbol{\eta} (+\tilde{\boldsymbol{\Theta}} \boldsymbol{u}) \ . \tag{3.40}$$

The vectors $\boldsymbol{\alpha}_i$, $\boldsymbol{\kappa}_i$ and $\boldsymbol{\beta}_i$ are of size $1 \times \delta$, $1 \times n - \delta$ and $1 \times m$ respectively, and the matrices $\boldsymbol{\Psi}$, $\boldsymbol{\Lambda}$ and $\tilde{\boldsymbol{\Theta}}$ are respectively of dimensions $n - \delta \times \delta$, $n - \delta \times n - \delta$ and $n - \delta \times m$. The term in parentheses, $(+\tilde{\boldsymbol{\Theta}} \boldsymbol{u})$, is sometimes present when there exists no transformation, or no such transformation is chosen, that makes the internal dynamics independent of the input as described next.

The transformation to the linear Byrnes-Isidori normal form is achieved by the linear form of the transformation (2.50)

$$\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{\eta} \end{bmatrix} = \boldsymbol{\Phi} \boldsymbol{x} \ . \tag{3.41}$$

The first δ rows of the transformation matrix $\boldsymbol{\Phi}$ are constructed by

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} \boldsymbol{c}_{j_1} \\ \boldsymbol{c}_{j_1} \boldsymbol{A} \\ \vdots \\ \boldsymbol{c}_{j_1} \boldsymbol{A}^{\delta_{j_1}-1} \\ \vdots \\ \boldsymbol{c}_{j_r} \\ \boldsymbol{c}_{j_r} \boldsymbol{A} \\ \vdots \\ \boldsymbol{c}_{j_r} \boldsymbol{A}^{\delta_{j_r}-1} \end{bmatrix} . \tag{3.42}$$

The remaining $n-\delta$ rows of $\mathbf{\Phi}$ have to be chosen such that the coordinate transformation is complete.

Remark 3.1.3. In order to make the internal dynamics (3.40) independent of the inputs, the remaining $n-\delta$ rows of $\mathbf{\Phi}$ should be constructed by further $n-\delta$ linearly independent row vectors that are part of kernel of $\boldsymbol{B}^T$, i. e.

$$\mathbf{\Phi}_2 = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_{n-\delta} \end{bmatrix} \text{ with } \boldsymbol{v}_i \in \ker \boldsymbol{B}^T \mid \boldsymbol{v}_i \notin \operatorname{im} \mathbf{\Phi}_1 \, , \tag{3.43}$$

since then $\mathbf{\Phi}_2 \boldsymbol{B} = \mathbf{0}$. However, this is not always possible because $\mathbf{\Phi}_1$ is of dimension $\delta \times n$ and $\delta \geq r$. By construction $\delta - r$ rows of $\mathbf{\Phi}_1$ are element of $\ker \boldsymbol{B}^T$. It is $\dim \ker \boldsymbol{B}^T = n - m$ since $\operatorname{rank} \boldsymbol{B} = m$ was given as a precondition. That means, for systems with $r < m$, there are $m - r$ rows of the internal dynamics that are dependent on $\boldsymbol{u}$. This is always true for non-square right inverse systems since $r = p < m$. Hence, if necessary to complete the transformation, the third part of $\mathbf{\Phi}$ is given by $m - r$ linearly independent row vectors

$$\mathbf{\Phi}_3 = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{w}_{m-r} \end{bmatrix} \text{ with } \boldsymbol{w}_i \in \ker \begin{bmatrix} \mathbf{\Phi}_1^T & \mathbf{\Phi}_2^T \end{bmatrix} \tag{3.44}$$

and finally

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1 \\ (\mathbf{\Phi}_3) \\ \mathbf{\Phi}_2 \end{bmatrix} . \tag{3.45}$$

This order is chosen such that the lower part is independent of the inputs.

With

$$\mathbf{\Xi} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_r \end{bmatrix} , \; \boldsymbol{K} = \begin{bmatrix} \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \vdots \\ \boldsymbol{\kappa}_r \end{bmatrix} \text{ and } \mathbf{\Theta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_r \end{bmatrix} \tag{3.46}$$

the derivatives $\overset{(\boldsymbol{\delta})}{\boldsymbol{y}} = \begin{bmatrix} \overset{(\delta_{j_1})}{y_{j_1}} & \dots & \overset{(\delta_{j_r})}{y_{j_r}} \end{bmatrix}^T$ can be combined to

$$\overset{(\boldsymbol{\delta})}{\boldsymbol{y}} = \mathbf{\Xi} \boldsymbol{z} + \boldsymbol{K} \boldsymbol{\eta} + \mathbf{\Theta} \boldsymbol{u} \, . \tag{3.47}$$

Finally, by setting $\boldsymbol{z} = \mathbf{0}$ the linear zero dynamics is given by

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\Lambda}\boldsymbol{\eta} \; . \tag{3.48}$$

In the case where it is not possible to make the internal dynamics independent of the inputs, the zero dynamics is given by

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\Lambda}\boldsymbol{\eta} - \tilde{\boldsymbol{\Theta}}\boldsymbol{\Theta}^{+}\boldsymbol{K}\boldsymbol{\eta} = \tilde{\boldsymbol{\Lambda}}\boldsymbol{\eta} \; , \tag{3.49}$$

by inserting the feedback law

$$\boldsymbol{u}_z = -\boldsymbol{\Theta}^{+}\boldsymbol{K}\boldsymbol{\eta} \; . \tag{3.50}$$

where $\boldsymbol{\Theta}^{+}$ is a pseudo inverse of $\boldsymbol{\Theta}$.

Remark 3.1.4. What actually happens by the state feedback in transformed coordinates (3.50) or in original coordinates (3.17), is that as many poles as possible are moved to the positions of the zeros of the system. Hence, the system is made maximal unobservable [Sas99, p.398][Isi95, p.166].

However, the zero dynamics of a right inverse system and the dual zero dynamics of a left inverse system do not have the same properties. Definition 2.3.1 says that initial conditions and inputs have to be determined such that the output is identically zero in order to obtain the zero dynamics. In general, this is not possible for left inverse systems that have more outputs than inputs. Nevertheless, it is possible to minimize the dimensions of the output space, i. e. under suitable control $\dim \mathsf{Y} = p - m$ holds. This is obvious from the linear algebra since for left inverse systems $\dim \operatorname{im} \boldsymbol{G}(s) = m$ applies. So only m directions of the outputs $\boldsymbol{y}(t)$ can be influenced by the inputs $\boldsymbol{u}(t)$ leaving $p - m$ dimensions uncontrolled. This is analogously valid for degenerated systems.

A definition that incorporates this issue is the also common definition by Schwarz.

Definition 3.1.2 (**Zero Dynamics** [Sch91])**.** The dynamics of a system (2.5) that (additionally) can be made unobservable by state feedback are called zero dynamics.

In the case of right inverse systems, this definition is equivalent to Definition 2.3.1.

3.1.6 Examples

In order to illustrate the BIN for linear systems some examples are given next.

Example 3.1.5. Consider the left invertible system

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \tag{3.51}$$
$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The derivation of the outputs

$$\dot{\boldsymbol{y}} = \boldsymbol{CAx} + \boldsymbol{CBu} = \begin{bmatrix} 1 & 0 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & -1 \\ -2 & -1 \\ 0 & -2 \end{bmatrix} \boldsymbol{u} \tag{3.52}$$

yields a relative degree of $\delta = 2$ considering the first two outputs. The feedback law to decouple the system is given by

$$\boldsymbol{u}_z = -\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 3 \end{bmatrix} \boldsymbol{x} \tag{3.53}$$

and the transformation to BIN is

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3.54}$$

This leads to internal dynamics

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 3 & 0 & 0 & 10 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \boldsymbol{\eta}, \tag{3.55}$$

which are independent of $\boldsymbol{u}$ and thus are identical with the zero dynamics of the system.

In order to show the differences in the zero dynamics the dual system of this example is considered.

Example 3.1.6. The dual system of Example 3.1.5 is given by the triple ($\bar{\boldsymbol{A}} = \boldsymbol{A}^T, \bar{\boldsymbol{B}} = \boldsymbol{C}^T, \bar{\boldsymbol{C}} = \boldsymbol{B}^T$). Hence, the system is right invertible. The derivation of the outputs

$$\dot{\boldsymbol{y}} = \bar{\boldsymbol{C}}\bar{\boldsymbol{A}}\boldsymbol{x} + \bar{\boldsymbol{C}}\bar{\boldsymbol{B}}\boldsymbol{u} = \begin{bmatrix} 0 & -1 & 3 & 0 & 0 & -3 \\ -1 & 0 & -3 & 0 & -1 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & -2 & 0 \\ -1 & -1 & -2 \end{bmatrix} \boldsymbol{u} \,. \tag{3.56}$$

Here, all outputs are used to determine the relative degree. It is identical to that of the right invertible system, i. e. $\delta = 2$. The feedback that keeps the outputs at zero is

$$\boldsymbol{u}_z = -\begin{bmatrix} \frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{2} & 0 \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 & -1 & 3 & 0 & 0 & -3 \\ -1 & 0 & -3 & 0 & -1 & -3 \end{bmatrix} \boldsymbol{x} \,. \tag{3.57}$$

The transformation to BIN is given by

$$\boldsymbol{\Phi} = \begin{bmatrix} \bar{\boldsymbol{C}} \\ \boldsymbol{w}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \,. \tag{3.58}$$

Note, in this case it is not possible to make the internal dynamics independent of the input because $n - \delta = 4$ but $\dim\ker \boldsymbol{B}^T = 3$. Therefore, the third row of $\boldsymbol{\Phi}$ is given by $\boldsymbol{w}_1$ and thus the first coordinate of the internal dynamics is influenced by $\boldsymbol{u}$. Hence for the isolation of the zero dynamics

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} \frac{7}{5} & \frac{1}{10} & -\frac{4}{5} & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -5 & \frac{5}{2} & 0 & -4 \end{bmatrix} \boldsymbol{\eta} \tag{3.59}$$

the feedback law $\boldsymbol{u}_z$ has to be inserted.

Finally, an example for a degenerated system is given.

Example 3.1.7. Consider the system given by the triple

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (3.60)$$
$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Derivation of the outputs yields

$$\begin{aligned} \dot{\boldsymbol{y}}(t) &= \begin{bmatrix} 1 & 1 & 1 & 0 & 8 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}(t) \\ \ddot{y}_2(t) &= \begin{bmatrix} 2 & 1 & 2 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{u}(t)\,. \end{aligned} \qquad (3.61)$$

The system has rank $r = 1$. Thus the output with the least relative degree is selected, i. e. the first output. This leads to the system relative degree of $\delta = \delta_1 = 1$. The feedback law is given by

$$\boldsymbol{u}_z = -\begin{bmatrix} 1 & 1 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} \qquad (3.62)$$

and the transformation to BIN is

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{w}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (3.63)$$

This leads to internal dynamics

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 \\ -1 & -6 & 7 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \boldsymbol{\eta}\,, \qquad (3.64)$$

which are independent of $\boldsymbol{u}$ since the second line of $\boldsymbol{u}_z$ vanishes and thus the internal dynamics are identical with the zero dynamics of the system.

3.2 Zeros and Zero dynamics

A property of (invariant) zeros is that if s_0 is a zero of a linear system, the input

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 e^{s_0 t} \tag{3.65}$$

for certain initial conditions, $\boldsymbol{u}_0$ and $\boldsymbol{x}_0$, will not produce any output [MK76]. This is very similar to the definition of the zero dynamics (Definition 2.3.1), where a feedback and initial conditions have to be found such that the output stays at zero. The question of how these two issues are related will be answered for arbitrary linear systems in this section. First the relation to the invariant zeros is studied. Then, the relation to the decoupling zeros is considered. Finally, the results are combined to state the relation to the system zeros.

3.2.1 Relation to Invariant Zeros

The relation to the zero dynamics for invariant zeros of square non-degenerated systems is shown by [HL12]. In this subsection, the method used there will be generalized to non-square and degenerated systems. This is done by proving the following lemma.

Lemma 3.2.1. The invariant zeros of a linear system with relative degree are contained in the set of eigenvalues of its zero dynamics.

Proof. Suppose, a system (2.2) is of vector relative degree $\delta < n$ and has invariant zeros, which are, according to (2.43), the solutions to s for

$$\text{rank} \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} < \text{norm-rank}\, \boldsymbol{P}(s) \leq n + r\ . \tag{3.66}$$

With the BIN (3.39),(3.40) and (3.46), a feedback law

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{x} = \boldsymbol{\Theta}^{+}(-\boldsymbol{\Xi}\boldsymbol{z} - \boldsymbol{K}\boldsymbol{\eta}) \tag{3.67}$$

is defined. Next, each component of the system is transformed by (3.41) to

$$\hat{\boldsymbol{A}} = \boldsymbol{\Phi}\boldsymbol{A}\boldsymbol{\Phi}^{-1},\ \hat{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{\Phi}^{-1},\ \hat{\boldsymbol{B}} = \boldsymbol{\Phi}\boldsymbol{B} \text{ and } \hat{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{\Phi}^{-1} \tag{3.68}$$

with

$$\hat{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix} \tag{3.69}$$

and

$$\hat{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix} . \tag{3.70}$$

For square invertible systems

$$\boldsymbol{B}_2 = \boldsymbol{0},\ \boldsymbol{C}_2 = \boldsymbol{0} \tag{3.71}$$

holds. For non-square systems right invertible systems

$$\boldsymbol{B}_2 = \begin{bmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{\Theta}} \end{bmatrix},\ \boldsymbol{C}_2 = \boldsymbol{0} , \tag{3.72}$$

and for non-square systems left invertible systems

$$\boldsymbol{B}_2 = \boldsymbol{0},\ \boldsymbol{C}_2 = \begin{bmatrix} \tilde{\boldsymbol{C}} & \boldsymbol{0} \end{bmatrix},\ \tilde{\boldsymbol{C}} \in \mathbb{R}^{p \times p-m} \tag{3.73}$$

holds, see Remark 3.1.3.

Inserting the feedback law (3.67) yields the transformed system

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{0} \\ \tilde{\boldsymbol{\Psi}} & \tilde{\boldsymbol{\Lambda}} \end{bmatrix} = \hat{\boldsymbol{A}} + \hat{\boldsymbol{B}}\hat{\boldsymbol{F}} . \tag{3.74}$$

with $\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi}$ in the left invertible case and $\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi} - \tilde{\boldsymbol{\Theta}}\boldsymbol{\Theta}^{+}\boldsymbol{\Xi}$ in the general case. Note that the triple $(\boldsymbol{B}_1, \boldsymbol{H}, \boldsymbol{C}_1)$ is observable as well as controllable, since it represents the external dynamics. Due to the structure of the transformed system, the sole solutions to s for

$$\operatorname{rank} \begin{bmatrix} s\boldsymbol{I} - \hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{F}} & \text{-}\hat{\boldsymbol{B}} \\ \hat{\boldsymbol{C}} & \boldsymbol{0} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{H} & \boldsymbol{0} & \text{-}\boldsymbol{B}_1 \\ -\tilde{\boldsymbol{\Psi}} & s\boldsymbol{I} - \tilde{\boldsymbol{\Lambda}} & \text{-}\boldsymbol{B}_2 \\ \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{0} \end{bmatrix} < n + r \tag{3.75}$$

are in a subset of the eigenvalues of $\tilde{\boldsymbol{\Lambda}}$ which are the eigenvalues of the zero dynamics of the considered system. By the equivalence

$$\begin{bmatrix} s\boldsymbol{I} - \hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{F}} & \text{-}\hat{\boldsymbol{B}} \\ \hat{\boldsymbol{C}} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} s\boldsymbol{I} - \boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{F} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{3.76}$$

these solutions for s coincide with the invariant zeros of the system (3.66). □

In the case of non-square or degenerated systems, the number of eigenvalues of $\tilde{\boldsymbol{\Lambda}}$ is generally larger than the number of zeros. The reason is that the number of inputs or outputs is larger than the system rank. The eigenvalues of $\tilde{\boldsymbol{\Lambda}}$ that are not invariant zeros are those values for s where

$$\text{rank}\begin{bmatrix}-\tilde{\boldsymbol{\Psi}} & s\boldsymbol{I}-\tilde{\boldsymbol{\Lambda}} & \text{-}\boldsymbol{B}_2\end{bmatrix} = \text{norm-rank}\begin{bmatrix}-\tilde{\boldsymbol{\Psi}} & s\boldsymbol{I}-\tilde{\boldsymbol{\Lambda}} & \text{-}\boldsymbol{B}_2\end{bmatrix} \tag{3.77}$$

or

$$\text{rank}\begin{bmatrix}s\boldsymbol{I}-\tilde{\boldsymbol{\Lambda}} \\ \boldsymbol{C}_2\end{bmatrix} = \text{norm-rank}\begin{bmatrix}s\boldsymbol{I}-\tilde{\boldsymbol{\Lambda}} \\ \boldsymbol{C}_2\end{bmatrix} . \tag{3.78}$$

The lemma is demonstrated by the next example.

Example 3.2.1. Consider again Example 3.1.5. This system has one transmission zero at 2, one input decoupling zero at -4, and one output decoupling zero at -1. The zeros at 2 and -1 are also invariant zeros. The eigenvalues of its zero dynamics are $s_1 = 3$, $s_2 = 2$, $s_3 = -1$ and $s_4 = -4$. Hence, the invariant zeros are included. The eigenvalues s_1 and s_4 are not in the set of invariant zeros because they do not cause a drop in the column rank of (3.75), i. e. they are no solution of

$$\text{rank}\begin{bmatrix}s\boldsymbol{I}-\tilde{\boldsymbol{\Lambda}} \\ \boldsymbol{C}_2\end{bmatrix} = \text{rank}\begin{bmatrix} s-3 & 0 & 0 & -10 \\ 0 & s-2 & 0 & 2 \\ 0 & 0 & s+1 & 2 \\ 0 & 0 & 0 & s+4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} < 4 \tag{3.79}$$

with

$$\boldsymbol{C}\boldsymbol{\Phi}^{-1} = \begin{bmatrix}\boldsymbol{C}_1 & \boldsymbol{C}_2\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} . \tag{3.80}$$

In the square non-degenerated case, the set of system zeros consists of the invariant zeros. In the non-square case the set of system zeros is larger than the set of invariant zeros because there are additional decoupling zeros [MK76]. The here discussed method can be easily extended to show that input decoupling zeros are always part of the zero dynamics by considering the equivalence transformation

$$\begin{bmatrix}s\boldsymbol{I}-\hat{\boldsymbol{A}}-\hat{\boldsymbol{B}}\hat{\boldsymbol{F}} & \text{-}\hat{\boldsymbol{B}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Phi}\end{bmatrix}\begin{bmatrix}s\boldsymbol{I}-\boldsymbol{A} & \text{-}\boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{F} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} . \tag{3.81}$$

However, it is not possible to formulate a similar equivalence transformation for output decoupling zeros because, to find the zero dynamics, feedback through the input is necessary. A method that shows that the decoupling zeros are generally part of the zero dynamics is described in the next subsection.

3.2.2 Relation to Decoupling Zeros and System Zeros

In order to complete the set of system zeros, the coincidence of the decoupling zeros and the eigenvalues of the zero dynamics of systems with unobservable or uncontrollable eigenvalues is shown.

Lemma 3.2.2. The decoupling zeros of a linear system with relative degree are contained in the set of eigenvalues of its zero dynamics.

Proof. Assume system (2.2) is of order $n = n_{c\bar{o}} + n_{co} + n_{\bar{c}o} + n_{\bar{c}\bar{o}}$ and has only decoupling zeros. Applying the Kalman decomposition [Kal62] yields

$$\dot{\begin{bmatrix} \boldsymbol{x}_{c\bar{o}} \\ \boldsymbol{x}_{co} \\ \boldsymbol{x}_{\bar{c}o} \\ \boldsymbol{x}_{\bar{c}\bar{o}} \end{bmatrix}} = \begin{bmatrix} \boldsymbol{A}_{c\bar{o}} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \boldsymbol{A}_{14} \\ \boldsymbol{0} & \boldsymbol{A}_{co} & \boldsymbol{A}_{23} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{\bar{c}o} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_{43} & \boldsymbol{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{c\bar{o}} \\ \boldsymbol{x}_{co} \\ \boldsymbol{x}_{\bar{c}o} \\ \boldsymbol{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{c\bar{o}} \\ \boldsymbol{B}_{co} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u} \tag{3.82a}$$

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{C}_{co} & \boldsymbol{C}_{\bar{c}o} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{c\bar{o}} & \boldsymbol{x}_{co} & \boldsymbol{x}_{\bar{c}o} & \boldsymbol{x}_{\bar{c}\bar{o}} \end{bmatrix}^T \tag{3.82b}$$

with $\boldsymbol{x}_{c\bar{o}} \in \mathbb{R}^{n_{c\bar{o}}}$, $\boldsymbol{x}_{co} \in \mathbb{R}^{n_{co}}$, $\boldsymbol{x}_{\bar{c}o} \in \mathbb{R}^{n_{\bar{c}o}}$, $\boldsymbol{x}_{\bar{c}\bar{o}} \in \mathbb{R}^{n_{\bar{c}\bar{o}}}$ and $\boldsymbol{u} \in \mathbb{R}^m, \boldsymbol{y} \in \mathbb{R}^p$. The pair $(\boldsymbol{A}_{c\bar{o}}, \boldsymbol{B}_{c\bar{o}})$ is only controllable so the eigenvalues of $\boldsymbol{A}_{c\bar{o}}$ are ODZ. The pair $(\boldsymbol{A}_{\bar{c}o}, \boldsymbol{C}_{\bar{c}o})$ is only observable so the eigenvalues of $\boldsymbol{A}_{\bar{c}o}$ are IDZ. The subsystem $\boldsymbol{A}_{\bar{c}\bar{o}}$ is neither controllable nor observable so its eigenvalues are IODZ and the triple $(\boldsymbol{A}_{co}, \boldsymbol{B}_{co}, \boldsymbol{C}_{co})$ is both observable and controllable. Furthermore, since the triple $(\boldsymbol{A}_{co}, \boldsymbol{B}_{co}, \boldsymbol{C}_{co})$ has no transmission zeros its vector relative degree is $\delta_{co} = n_{co} = \delta$.

The differentiation of the outputs $\boldsymbol{y}$ until the first occurrence of any component of the input vector $\boldsymbol{u}$ yields

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{C}_{co}\boldsymbol{x}_{co} + \boldsymbol{C}_{\bar{c}o}\boldsymbol{x}_{\bar{c}o} \\
\overset{(\boldsymbol{1})}{\boldsymbol{y}} &= \boldsymbol{C}_{co}\boldsymbol{A}_{co}\boldsymbol{x}_{co} + \underbrace{\boldsymbol{C}_{co}\boldsymbol{B}_{co}}_{=\boldsymbol{0}}\boldsymbol{u} + \left(\boldsymbol{C}_{co}\underbrace{\boldsymbol{A}_{23}}_{=\boldsymbol{\Lambda}_{(\boldsymbol{1})}} + \boldsymbol{C}_{\bar{c}o}\boldsymbol{A}_{\bar{c}o}\right)\boldsymbol{x}_{\bar{c}o} \\
&\vdots \\
\overset{(\boldsymbol{\delta}-\boldsymbol{1})}{\boldsymbol{y}} &= \boldsymbol{C}_{co}\boldsymbol{A}_{co}^{[\boldsymbol{\delta}-\boldsymbol{1}]}\boldsymbol{x}_{co} + \underbrace{\boldsymbol{C}_{co}\boldsymbol{A}_{co}^{[\boldsymbol{\delta}-\boldsymbol{2}]}\boldsymbol{B}_{co}}_{=\boldsymbol{0}}\boldsymbol{u} + \left(\boldsymbol{C}_{co}\boldsymbol{\Lambda}_{(\boldsymbol{\delta}-\boldsymbol{1})} + \boldsymbol{C}_{\bar{c}o}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{\delta}-\boldsymbol{1}]}\right)\boldsymbol{x}_{\bar{c}o} \\
\overset{(\boldsymbol{\delta})}{\boldsymbol{y}} &= \boldsymbol{C}_{co}\boldsymbol{A}_{co}^{[\boldsymbol{\delta}]}\boldsymbol{x}_{co} + \underbrace{\boldsymbol{C}_{co}\boldsymbol{A}_{co}^{[\boldsymbol{\delta}-\boldsymbol{1}]}\boldsymbol{B}_{co}}_{\neq\boldsymbol{0}}\boldsymbol{u} + \left(\boldsymbol{C}_{co}\boldsymbol{\Lambda}_{(\boldsymbol{\delta})} + \boldsymbol{C}_{\bar{c}o}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{\delta}]}\right)\boldsymbol{x}_{\bar{c}o}
\end{aligned}
\tag{3.83}
$$

where

$$
\boldsymbol{\Lambda}_{(\boldsymbol{\delta})} = \left(\boldsymbol{A}_{co}^{[\boldsymbol{\delta}]}\boldsymbol{A}_{23} + \boldsymbol{A}_{co}^{[\boldsymbol{\delta}-\boldsymbol{1}]}\boldsymbol{A}_{23}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{1}]} + \ldots + \boldsymbol{A}_{co}^{[\boldsymbol{1}]}\boldsymbol{A}_{23}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{\delta}-\boldsymbol{1}]} + \boldsymbol{A}_{23}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{\delta}]}\right), \tag{3.84}
$$

$\overset{(\boldsymbol{\delta})}{\boldsymbol{y}} = \begin{bmatrix} \overset{(\delta_1)}{y_1} & \ldots & \overset{(\delta_r)}{y_r} \end{bmatrix}^T$ and the abbreviation $\boldsymbol{C}_*\boldsymbol{A}_*^{[\boldsymbol{\delta}]} = \begin{bmatrix} \boldsymbol{c}_{*,1}\boldsymbol{A}_*^{\delta_1} \\ \vdots \\ \boldsymbol{c}_{*,r}\boldsymbol{A}_*^{\delta_r} \end{bmatrix}$ is a collocation of the row vectors belonging to each output $y_1 \ldots y_r$. From (3.83) the first δ rows for the transformation matrix $\boldsymbol{\Phi}$ are determined and to complete the coordinate transformation the remaining $n-\delta$ rows are constructed as simple as possible, resulting in

$$
\boldsymbol{\Phi} = \begin{bmatrix}
\boldsymbol{0} & \boldsymbol{C}_{co} & \boldsymbol{C}_{\bar{c}o} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{C}_{co}\boldsymbol{A}_{co}^{[\boldsymbol{1}]} & \left(\boldsymbol{C}_{co}\boldsymbol{\Lambda}_{(\boldsymbol{1})} + \boldsymbol{C}_{\bar{c}o}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{1}]}\right) & \boldsymbol{0} \\
\vdots & \vdots & \vdots & \vdots \\
\boldsymbol{0} & \boldsymbol{C}_{co}\boldsymbol{A}_{co}^{[\boldsymbol{\delta}-\boldsymbol{1}]} & \left(\boldsymbol{C}_{co}\boldsymbol{\Lambda}_{(\boldsymbol{\delta}-\boldsymbol{1})} + \boldsymbol{C}_{\bar{c}o}\boldsymbol{A}_{\bar{c}o}^{[\boldsymbol{\delta}-\boldsymbol{1}]}\right) & \boldsymbol{0} \\
\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}
\end{bmatrix}. \tag{3.85}
$$

Transforming the system (3.82) with (3.85) and setting $\boldsymbol{z} = \boldsymbol{0}$ the internal dynamics (3.40) results in the zero dynamics

$$
\dot{\boldsymbol{\eta}} = \begin{bmatrix}
\boldsymbol{A}_{c\bar{o}} & \boldsymbol{A}_{\#} & \boldsymbol{A}_{14} \\
\boldsymbol{0} & \boldsymbol{A}_{\bar{c}o} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{A}_{43} & \boldsymbol{A}_{\bar{c}\bar{o}}
\end{bmatrix}\boldsymbol{\eta}, \tag{3.86}
$$

with $\boldsymbol{A}_{\#}$ being a mixed matrix, which has no influence on the eigenvalues of (3.86) due to the block matrix structure of (3.86). Comparing (3.86) with (3.82) shows that the eigenvalues of the zero dynamics are exactly the decoupling zeros. □

Finally, an example is given, which also incorporates transmission zeros.

Example 3.2.2. Consider the following system in Kalman normal form (3.82)

$$\boldsymbol{A} = \begin{bmatrix} -4 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (3.87)$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

with $p = 4$ outputs and $m = 3$ inputs. It possesses the unobservable eigenvalue -4, the uncontrollable eigenvalue -1 and the simultaneously unobservable and uncontrollable eigenvalue 7. Additionally, this system has one transmission zero at 2.

The transformation to BIN can be determined to

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (3.88)$$

Transforming the system leads to internal dynamics, which are independent of $\boldsymbol{u}$

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} 3 & 0 & 0 & -5 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ \frac{5}{2} & \frac{1}{2} & -4 & 5 & 6 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix} \boldsymbol{\eta}. \qquad (3.89)$$

Hence, these dynamics are identical with the zero dynamics. Due to its block triangular form, its eigenvalues can be read from the diagonal. It contains the

IODZ at 7, the IDZ at -1, the ODZ at -4 and the TZ at 2. Additionally, there is one eigenvalue at 3 due to the asymmetric number of inputs and outputs.

Remark 3.2.1. The additional eigenvalues, which appear in the zero dynamics for non-square systems, are well-known from decoupling control, refer to [SS87, §4], [KM76, §5] or [Var80]. For systems with different number of inputs and outputs, they appear in the "squaring up" or "squaring down" problem. These additional "zeros" are not invariant, since their value can be assigned by the feedback law.

As indicated by the example, both sets, the set of transmission zeros and the set of decoupling zeros, are contained in the zero dynamics. Since these sets are non-intersecting, it is possible to correlate the following from Lemma 3.2.1 and Lemma 3.2.2.

Corollary 3.2.1. The system zeros of a linear system with relative degree are contained in the set of eigenvalues of its zero dynamics.

Further, in the case of square non-degenerated systems with relative degree, the eigenvalues of the zero dynamics and invariant zeros are coincident as investigated in [Daa16].

Theorem 3.2.1. The invariant zeros of a linear square non-degenerated system (2.2) (with relative degree) coincide with the eigenvalues of its zero dynamics.

Considering Corollary 3.2.1 and Definition 2.4.1 the following definition seems reasonable.

Definition 3.2.1 (Non-Minimum Phase). A linear systems is non-minimum phase if it has at least one non-negative zero.

By the isolation of the zero dynamics, it is possible to determine all zeros of a square non-degenerated system. Thus, besides the Smith-McMillan-Form Definition 2.2.13, the rank criteria (2.43) or the characteristic polynomial of $\boldsymbol{P}(s)$ (2.47), a further method is available to determine the zeros of the system. All these methods are based on matrix calculations. In the next chapter a method for the graph-theoretic determination of zeros and poles of linear systems is introduced.

4 Graph-Theoretic Determination of Zeros and Poles

The poles and zeros of a linear system Σ_{LS} can be determined by certain polynomials as introduced in the preliminaries. The poles are given by the roots of the characteristic polynomial (2.33). Since square non-degenerated systems only contain invariant zeros, they can be determined by (2.47). In all other cases, Lemma 2.2.4 yields polynomials for the determination of the system zeros. By graph-theoretic tools, it is possible to obtain these polynomials as well.

4.1 Characteristic Polynomial

The characteristic polynomial of a matrix can be determined from its associated graph. This was first investigated by [Kön16] and much later reformulated by [Che67; Pon66]. Similar to [Che76, Theorem 3.21] a theorem for the determination of the characteristic polynomial of a linear system Σ_{LS} can be specified.

Consider a linear system Σ_{LS} and its corresponding system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$.

Theorem 4.1.1. The coefficient α_k of (2.33) is determined by the cycle families of width k within the weighted system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$.

1. If no such cycle family of width k exists, the coefficient α_k is equal to zero.

2. If only one cycle family of width k exists, α_k equals the product of the weights of their edges multiplied by $(-1)^d$, where d is the number of cycles the family consists of.

3. If there is more than one cycle family of width k, α_k equals the sum of the corresponding weight products of the cycle families.

In order to prove this the following two lemmas are necessary. The coefficients α_k of $\chi(\lambda)$ (2.33) can be calculated by the following lemma [Mir55, Theorem 7.1.2]:

Lemma 4.1.1. For $0 \leq k < n$, the coefficient α_k of $\chi(\lambda)$ (2.33) is equal to $(\text{-}1)^k$ times the sum of all k-rowed principal minors of $\boldsymbol{A}$, i. e.

$$\alpha_k = (\text{-}1)^k \sum_{|\ell|=k} \boldsymbol{A}_\ell^\ell \,, \tag{4.1}$$

where $\ell := \{i_1, i_2, \ldots, i_k\}$ are all subsets of the row or column indices $[1, n]$ of size $|\ell| = k$ and $\boldsymbol{A}_\ell^\ell$ is a (principal) minor of $\boldsymbol{A}$ considering its ℓ rows and columns.

By [Che76] it is possible to obtain the determinant of a square matrix (adjacency matrix) $\mathcal{A}$ by the cycle families of its related graph.

Lemma 4.1.2 (See Theorem A2.1 in [Rei88] or originally Theorem 3.1 in [Che76])**.** If $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is the adjacency matrix of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$, each summand $m_{1,t_1} m_{2,t_2} \ldots m_{n,t_n}$ of (2.19) corresponds to a cycle family touching all vertices of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$. The value of the summand is given by the product of the weights of the edges in the cycle family. If the cycle family consists of d disjoint cycles, the sign factor of the summand is given by $(\text{-}1)^{n-d}$.

Proof of Theorem 4.1.1. By Lemma 4.1.2, each determinant of a $k \times k$ submatrix of $\boldsymbol{M}$ having same row and column indices and containing the diagonal elements of $\boldsymbol{M}$, i. e. each principal minor, can be determined by cycle families of size k. By Lemma 4.1.1, the value of α_k is given by the sum of the principal minors. This means, no specific cycle families have to be considered but the sum of the product of the weights of cycle families of same size k. The sign factors of both lemmas combine to

$$(\text{-}1)^{k-d}(\text{-}1)^k = (\text{-}1)^d \,. \tag{4.2}$$

Since the input vertices $u_i \in \mathcal{U}$ are only start-vertices of edges and the output vertices $y_i \in \mathcal{Y}$ are only end-vertices of edges, they will not be contained in any cycle family of $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$. Thus, the input and output vertices can be removed from $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ yielding a graph whose adjacency matrix is $\boldsymbol{A}$. □

The application of this theorem will be demonstrated by an example.

Example 4.1.1. Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} a_{1,1} & 0 & a_{1,3} \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & 0 & a_{3,3} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} c_1 & 0 & c_3 \end{bmatrix} \boldsymbol{x} \tag{4.3}$$

and its corresponding system graph depicted in Figure 4.1. Applying Theorem 4.1.1

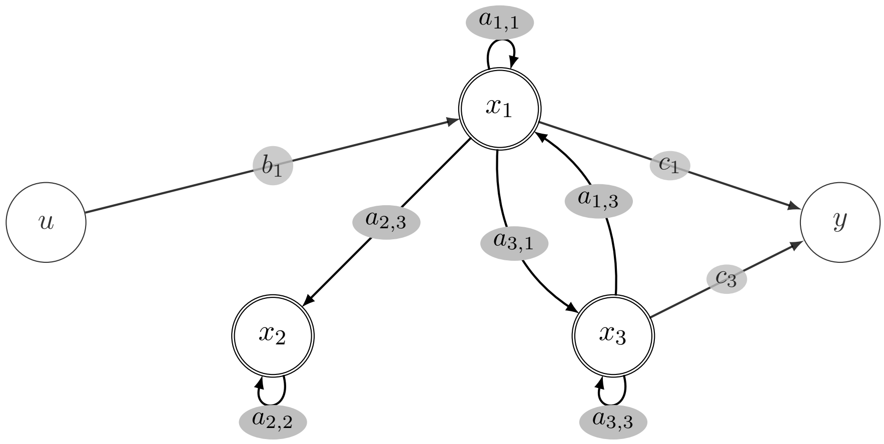

Figure 4.1: $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ of the system (4.3).

on the graph in Figure 4.1 yields Table 4.1. From this table the characteristic

	cycle families $\mathcal{C}_i$	$\sum \prod \mathcal{W} \in \mathcal{C}_i$
α_1	$(x_1), (x_2), (x_3)$	$-a_{1,1} - a_{2,2} - a_{3,3}$
α_2	$(x_1 \leftrightarrows x_3), (x_1, x_2)$	$-a_{1,3}a_{3,1} + a_{1,1}a_{2,2}$
	$(x_2, x_3), (x_3, x_1)$	$+a_{2,2}a_{3,3} + a_{3,3}a_{1,1}$
α_3	$(x_1 \leftrightarrows x_3, x_2), (x_1, x_2, x_3)$	$a_{1,3}a_{3,1}a_{2,2} - a_{1,1}a_{2,2}a_{3,3}$

Table 4.1: Cycle families and their corresponding values for the coefficients of (2.33).

polynomial of its A-matrix

$$\lambda^3 - (a_{1,1} + a_{2,2} + a_{3,3})\lambda^2 + (a_{1,1}a_{2,2} + a_{2,2}a_{3,3} + a_{3,3}a_{1,1} - a_{1,3}a_{3,1})\lambda + a_{1,3}a_{3,1}a_{2,2} - a_{1,1}a_{2,2}a_{3,3} = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) \tag{4.4}$$

is obtained.

4.2 Structural System Rank

For the graph-theoretic determination of the zeros, the first step is to decide whether a system is square and non-degenerated. In this case, the determination of the zeros is much simpler compared to the other cases as will be seen in the subsequent sections. The squareness of a system is obvious by the number of inputs and outputs, respectively, the number of input vertices and output vertices. In order to examine whether a system is degenerated, the rank criterion (2.27) has to be checked. Therefore, a property from the structural analysis is anticipated. There are some approaches to determine the rank of a system by graph-theoretic methods. By [Rei88, Definition 32.9] the "structural norm rank" is defined as follows.

Definition 4.2.1 (Structural Norm Rank). The *structural norm rank* of the transfer matrix $\boldsymbol{G}(s)$ for a class of systems given by the set of numerical realizations $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ of $\Sigma_{\circledast}$[1] is defined as

$$\max_{(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})\in\Sigma_{\circledast}} \max_{s\in\mathbb{C}} \boldsymbol{G}(s)\ . \tag{4.5}$$

This rank can be determined by [Rei88, Theorem 32.7] using a concept called "feedback families", which for this work will be defined more narrowly as described in the next section. In order to determine the rank of a system graph-theoretically without using feedback families or structural systems $\Sigma_{\circledast}$, a tailored version of the rank criterion in [Mur09, Remark 2.1.13] is given.

Definition 4.2.2 (Structural System Rank). The *structural system rank* of a linear system, denoted by s-rank Σ_{LS}, is given by the maximal number of vertex disjoint simple paths between the input vertices and output vertices, called *input-output paths*, in its system graph $\mathcal{G}_{sys}$.

The structural rank of a linear system is related to its rank (2.27) as given by the next theorem.

Theorem 4.2.1. The structural system rank of a system Σ_{LS} is an upper bound of the normal rank of its transfer matrix, i. e.

$$\text{norm-rank}\, \boldsymbol{G}(s) \leq \text{s-rank}\, \Sigma_{LS}\ . \tag{4.6}$$

[1]See Definition 5.1.1.

Proof. Consider the submatrix

$$\boldsymbol{P}^{\boldsymbol{i}_k}_{\boldsymbol{j}_k} := \boldsymbol{P}(s)^{1,2,\dots,n,n+i_1,n+i_2,\dots,n+i_k}_{1,2,\dots,n,n+j_1,n+j_2,\dots,n+j_k} \tag{4.7}$$

of $\boldsymbol{P}(s)$ where $\boldsymbol{i}_k := n+i_1, n+i_2, \dots, n+i_k$ and $\boldsymbol{j}_k := n+j_1, n+j_2, \dots, n+j_k$ are some subsets of size k of, respectively, the row and the column indices greater n of $\boldsymbol{P}(s)$. Let $\mathcal{G}_{\boldsymbol{P}^{\boldsymbol{i}_k}_{\boldsymbol{j}_k}}$ be the corresponding graph for which $\boldsymbol{P}^{\boldsymbol{i}_k}_{\boldsymbol{j}_k}$ is the adjacency matrix.

The equality term-rank $\boldsymbol{P}^{\boldsymbol{i}_0}_{\boldsymbol{j}_0} = n$ always holds. Therefore, with Lemma 4.1.2 the term rank of $\boldsymbol{P}(s)$ is given by the largest k, denoted by $\tilde{k}$, for that a cycle family in $\mathcal{G}_{\boldsymbol{P}^{\boldsymbol{i}_{\tilde{k}}}_{\boldsymbol{j}_{\tilde{k}}}}$ exists that touches all vertices plus n, i. e. term-rank $\boldsymbol{P}(s) = \tilde{k} + n$. This cycle family touches exactly $\tilde{k}$ vertices that each relates to a combination of an input u_i and an output y_j, to which are referred as *input-output vertices.*

The system graph $\mathcal{G}_{sys}$ can be transformed to the graph $\mathcal{G}_{\boldsymbol{P}^{\boldsymbol{i}_{\tilde{k}}}_{\boldsymbol{j}_{\tilde{k}}}}$ by merging the $i_1, i_2, \dots, i_{\tilde{k}}$ input vertices and $j_1, j_2, \dots, j_{\tilde{k}}$ output vertices, deleting the remaining input and output vertices and their connecting edges, changing the signs of the weights belonging to $\boldsymbol{A}$ and $\boldsymbol{B}$ and adding self-loops of weight s to the state vertices. Hence, the cycles which touch the input-output vertices in $\mathcal{G}_{\boldsymbol{P}^{\boldsymbol{i}_{\tilde{k}}}_{\boldsymbol{j}_{\tilde{k}}}}$ become in $\mathcal{G}_{sys}$ to $\tilde{k}$ vertex disjoint simple paths from the input vertices to the output vertices. Since $\boldsymbol{P}^{\boldsymbol{i}_{\tilde{k}}}_{\boldsymbol{j}_{\tilde{k}}}$ is the largest submatrix with a cycle family touching all vertices, there will be no further simple paths from the input vertices to the output vertices in $\mathcal{G}_{sys}$. Hence, the structural rank of Σ_{LS} equals $\tilde{k}$.

Since norm-rank $\boldsymbol{P}(s)$ = norm-rank $\boldsymbol{G}(s) + n$ and norm-rank $\boldsymbol{P}(s) \leq$ term-rank $\boldsymbol{P}(s)$ by Lemma 2.2.2 and Lemma 2.2.1, respectively, the proof is complete. □

Parts of this proof are taken from [CDP90; CDP91]. An alternative proof is provided in [Wey02, Bew. 5.5].

Remark 4.2.1. In cases where the structural system rank r_s of Σ_{LS} is greater than its normal rank r, numerical cancellations in some minors of size r_s of $\boldsymbol{P}(s)$ occur. The reason is that the rank of a system is given by norm-rank $\boldsymbol{G}(s)$ whereas the cycle families, as shown in the proof, are related to the term-rank $\boldsymbol{P}(s)$. However, as will be shown in Subsection 5.4.1, in almost all cases the structural system rank r_s equals the rank of the system r. The relationship between the structural system rank and the structural norm rank will also be explained there.

4.3 Invariant Zeros Polynomial

Consider a system Σ_{LS} with the restriction that the number of outputs equals the number of inputs and it has full rank, i. e. $m = p = r$. According to (2.47) the invariant zeros of this system can be determined by the roots of the *invariant zeros polynomial*

$$\begin{aligned} p_{IZ}(s) &:= \det \boldsymbol{P}(s) \\ &= p_m s^{n-m} + p_{m+1} s^{n-m-1} + \ldots + p_{n-1} s + p_n = \sum_{k=m}^{n} p_k s^{n-k} \ . \end{aligned} \tag{4.8}$$

In [Rei88, Chapter 31] a method is introduced, on how to obtain the coefficients of the polynomial (4.8) from the graph-theoretic representation of a SISO minimal system. This is done by introducing a feedback in the system graph $\mathcal{G}_{sys}$, which then yields the feedback graph of a system $\mathcal{G}_{fb}$. The following definition applies for (square non-degenerated) MIMO systems.

Definition 4.3.1 (**Feedback Graph, Feedback Cycle Family**). The *feedback graph* $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ is constructed by inserting *feedback edges* in the system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ of Σ_{LS}. The feedback edges connect the inputs u_i to the outputs y_i by the feedback law

$$\boldsymbol{u} = -\boldsymbol{I}\boldsymbol{y} \ , \tag{4.9}$$

with $\boldsymbol{I}$ an $r \times r$ identity matrix. A *feedback cycle family* is a cycle family that contains exactly r feedback edges.

Remark 4.3.1. Since invariant zeros, as the name indicates, are invariant in respect to feedback [MK76], the feedback law for the generation of a feedback graph may be chosen nearly arbitrary. The only condition that must be met for a feedback law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{y}$ is

$$\det \boldsymbol{K} \neq 0 \ , \tag{4.10}$$

i. e. $\boldsymbol{K}$ is non-singular. This guarantees that the system is not altered by information loss due to a non-injective mapping from $\boldsymbol{y}$ to $\boldsymbol{u}$.

With feedback cycle families, it is possible to determine the coefficients of (4.8) by the next theorem.

Theorem 4.3.1. The coefficient p_k of (4.8) is determined by the feedback cycle families of width k within the feedback graph $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E}, \mathcal{W})$.

1. If no such cycle family of width k exists, the coefficient p_k is equal to zero.

2. Otherwise, if only one cycle family of width k exists, p_k equals the product of the weights of their edges multiplied by (-1)^d, where d is the number of cycles the family consists of.

3. If there is more than one cycle family of width k, p_k equals the sum of the corresponding weight products of the cycle families.

Proof. Similar to Lemma 4.1.1 it is possible to expand the determinant of $\boldsymbol{P}(s)$ to

$$\det \begin{bmatrix} s\boldsymbol{I}-\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} =$$
$$s^{n-m} \sum_{i_m} \sum_{\dots<i_m} \dots \sum_{i_2<\dots} \sum_{i_1<i_2} \begin{bmatrix} \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}^{\{i_1,i_2,\dots,i_m n+1,n+2,\dots,n+m\}}_{\{i_1,i_2,\dots,i_m n+1,n+2,\dots,n+m\}} +$$
$$s^{n-m-1} \sum_{i_{m+1}} \sum_{i_m<i_{m+1}} \dots \sum_{i_2<\dots} \sum_{i_1<i_2} \begin{bmatrix} \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}^{\{i_1,i_2,\dots,i_m,i_{m+1},n+1,n+2,\dots,n+m\}}_{\{i_1,i_2,\dots,i_m,i_{m+1},n+1,n+2,\dots,n+m\}} +$$
$$\dots +$$
$$s \sum_{i_{n-1}} \sum_{\dots<i_{n-1}} \dots \sum_{i_2<\dots} \sum_{i_1<i_2} \begin{bmatrix} \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}^{\{i_1,i_2,\dots,i_m,\dots,i_{n-1},n+1,n+2,\dots,n+m\}}_{\{i_1,i_2,\dots,i_m,\dots,i_{n-1},n+1,n+2,\dots,n+m\}} +$$
$$\det \begin{bmatrix} \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \tag{4.11}$$

with $i_j \in [1, n]$ and $j = 1, 2, \dots, m, \dots, n-1$. Herein $\sum_{i_1<i_2}$ is the sum over all i_1 under the condition that i_1 has lower value than i_2. Comparison of (4.11) with (4.8) and reordering yields

$$
\begin{aligned}
p_k &= \sum_{i_k} \dots \sum_{i_1<i_2} \begin{bmatrix} \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}_{\{i_1,\dots,i_k,n+1,\dots,n+m\}}^{\{i_1,\dots,i_k,n+1,\dots,n+m\}} \\
&= \sum_{i_k} \dots \sum_{i_1<i_2} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}_{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}}^{\{1,\dots m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}} \\
&= \sum_{i_k} \dots \sum_{i_1<i_2} \begin{bmatrix} \boldsymbol{0} & \text{-}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \text{-}\boldsymbol{A} & \text{-}\boldsymbol{B} \\ \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}_{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}}^{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}} \\
&= \sum_{i_k} \dots \sum_{i_1<i_2} (-1)^{k+2m} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{B} \\ \text{-}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}_{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}}^{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}} .
\end{aligned} \tag{4.12}
$$

Since

$$
\begin{bmatrix} 0 & \boldsymbol{C} & 0 \\ 0 & \boldsymbol{A} & \boldsymbol{B} \\ \text{-}\boldsymbol{I} & 0 & 0 \end{bmatrix} \tag{4.13}
$$

is the adjacency matrix of the feedback graph $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E}, \mathcal{W})$, by Lemma 4.1.2 it is now possible to calculate the minors

$$
\begin{bmatrix} \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{B} \\ \text{-}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}_{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}}^{\{1,\dots,m,i_1,\dots,i_k,n+m+1,\dots,n+2m\}} \tag{4.14}
$$

from the weights of the feedback cycle families in $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ of width k. Assuming the cycle families consist of d disjoint cycles, the sign factor is $(\text{-}1)^{k+2m-d}$. Hence the overall sign factor becomes

$$
(\text{-}1)^{k+2m}(\text{-}1)^{k+2m-d} = (\text{-}1)^d . \tag{4.15}
$$

□

The application of Theorem 4.3.1 is demonstrated by the following example.

Example 4.3.1. As an example system a loading bridge is considered, see Figure 4.2. The plant consist of three parts. A cart that runs on rails in the roof of a factory hall. A rod for attaching a load is connected to it by a pivot joint. At the bottom of the rod, the load is mounted. The rod is considered massless and rigid. All masses are considered as point masses. The cart is driven by a controllable force F and has mass m_2. Its horizontal position on the rail is given by s_2. The rod has the length l and its deviation is given by the angle θ. The load has mass m_1 and its horizontal position is measured by s_1. The system is subject to gravity g.

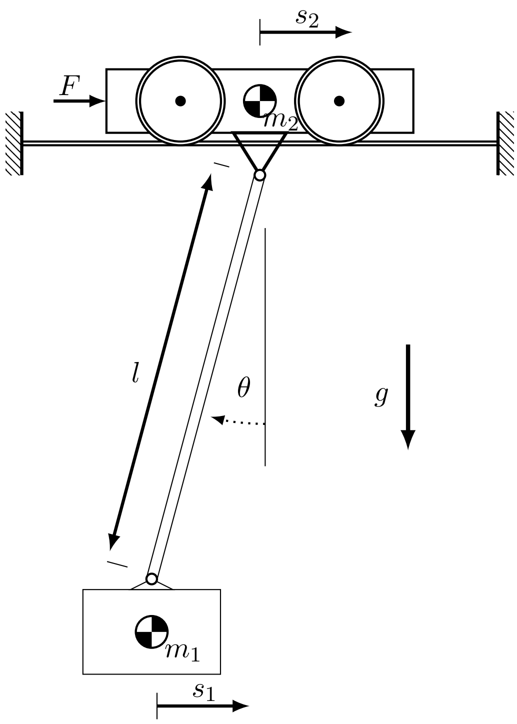

Figure 4.2: Sketch of the loading bridge.

The plant can be modelled by the equations

$$\begin{aligned}(m_1+m_2)\ddot{s}_2 + m_1 l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) &= F\\ \ddot{s}_2\cos\theta + l\ddot{\theta} - g\sin\theta &= 0\\ s_1 &= s_2 - \sin\theta l\ .\end{aligned} \tag{4.16}$$

This system has four state variables $\boldsymbol{x} = \begin{bmatrix} s_2 & \dot{s}_2 & \theta & \dot{\theta}\end{bmatrix}^T$, the force on the cart as input, i. e. $u = F$, and the horizontal position of the load as output, i. e. $y = s_1$. A linearization around the equilibrium $\bar{\boldsymbol{x}} = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$ yields the linear parameterized system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_1}{m_2}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m_1+m_2}{-m_2}\frac{g}{l} & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ \frac{-1}{m_2 l} \end{bmatrix} u \tag{4.17}$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} \, . \tag{4.18}$$

The feedback graph of (4.18) is depicted in Figure 4.3. Herein is $a_1 = \frac{m_1}{m_2}g$,

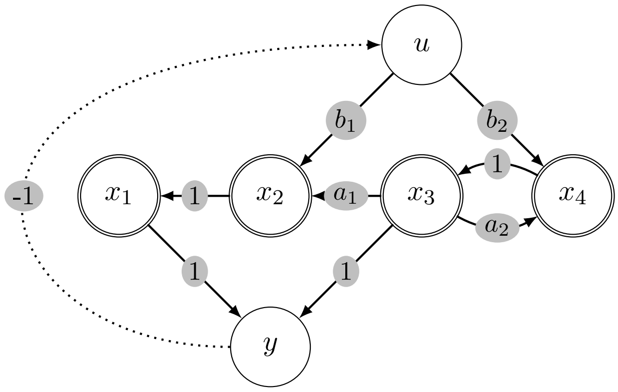

Figure 4.3: $\mathcal{G}_{fb}$ of the loading bridge.

$a_2 = \frac{m_1+m_2}{-m_2}\frac{g}{l}$, $b_1 = \frac{1}{m_2}$ and $b_2 = -\frac{1}{m_2 l}$. Applying Theorem 4.3.1 on the $\mathcal{G}_{fb}$ in in Figure 4.3 yields Table 4.2. From this table the invariant zeros polynomial

	cycle families $\mathcal{C}_i$	$\sum\prod \mathcal{W} \in \mathcal{C}_i$
p_1	–	–
p_2	$(u \to x_2 \to x_1 \to y), (u \to x_4 \to x_3 \to y)$	$b_1 + b_2$
p_3	–	–
p_4	$(u \to x_4 \to x_3 \to x_2 \to x_1 \to y)$, $(u \to x_2 \to x_1 \to y, x_3 \leftrightarrows x_4)$	$b_2 a_1 - b_1 a_2$

Table 4.2: Cycle families and their corresponding values for the coefficients of (4.8).

$$p_{IZ}(s) = (b_1 + b_2)s^2 + b_2 a_1 - b_1 a_2 \tag{4.19}$$

is obtained.

4.4 System Zeros Polynomials

If the considered system is non-square or degenerated, Theorem 4.3.1 is not directly applicable. However, it is still possible to determine the system zeros of such a system by graph-theoretic methods.

Suppose now that the system Σ_{LS} has rank $r = \text{norm-rank}\,\boldsymbol{G}(s)$ and consider its graph $\mathcal{G}_{sys}(\mathcal{V},\mathcal{E},\mathcal{W})$.

Theorem 4.4.1. For each distinct unordered combination ω_i of r input and r output vertices, which are contained in r vertex disjoint simple paths from input to output in $\mathcal{G}_{sys}(\mathcal{V},\mathcal{E},\mathcal{W})$, a polynomial

$$p_{\omega_i}(s) := p_r s^{n-r} + p_{r+1}s^{n-r-1} + \ldots + p_{n-1}s + p_n = \sum_{k=r}^{n} p_k s^{n-k} \tag{4.20}$$

can be obtained by constructing a feedback graph $\mathcal{G}_{fb}^{\omega_i}(\mathcal{V},\mathcal{E},\mathcal{W})$ and applying Theorem 4.3.1 on it. The system zeros and their multiplicity are determined by the roots of the monic greatest common divisor $p_{SZ}(s)$ of the polynomials (4.20) of all combinations ω_i.

Proof. Each distinct unordered combination ω_i of r input and r output vertices yields, by deleting all other input and output vertices and their connecting edges that are not part of the considered combination, a square non-degenerated system. Hence Theorem 4.3.1 can be used to obtain the polynomials $p_{\omega_i}(s)$. As shown by the proof of Theorem 4.2.1 each combination ω_i refers to a minor

$$p_{\omega_i}(s) = \boldsymbol{P}(s)^{\{1,2,\ldots,n,n+i_1,n+i_2,\ldots,n+i_r\}}_{\{1,2,\ldots,n,n+j_1,n+j_2,\ldots,n+j_r\}}\,. \tag{4.21}$$

Thus by Lemma 2.2.4, the system zeros of Σ_{LS} can be determined by the roots of the monic greatest common divisor $p_{SZ}(s)$ of the polynomials of all combinations ω_i. □

Remark 4.4.1. For Theorem 4.4.1 the knowledge of the rank of Σ_{LS} is a precondition. However, r can be estimated by the structural system rank of Σ_{LS}, which is an upper bound. In the case that $\text{s-rank}\,\Sigma_{LS} > \text{g-rank}\,\boldsymbol{G}(s)$, all polynomials $p_{\omega_i}(s)$ are identical zero. By reducing the estimated rank $\tilde{r}_{i+1} := \tilde{r}_i - 1$, $\tilde{r}_0 := \text{s-rank}\,\Sigma_{LS}$, until at least one of the polynomials $p_{\omega_i}(s)$ doesn't vanish, the rank of Σ_{LS} r and the system zeros can be determined.

The application of Theorem 4.4.1 is shown by the next example.

Example 4.4.1. Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & -1 & 5 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix} u, \quad \boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \boldsymbol{x} \tag{4.22}$$

and its feedback graphs depicted in Figure 4.4. In this graph the output y_1 is

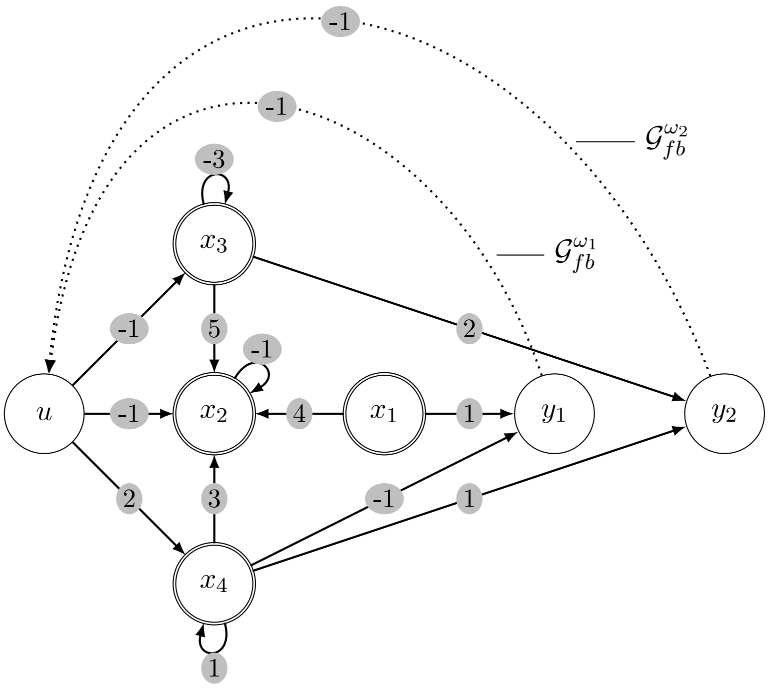

Figure 4.4: $\mathcal{G}_{fb}^{\omega_1}$ and $\mathcal{G}_{fb}^{\omega_2}$ of (4.22).

reachable by one simple path and the output y_2 is reachable by two simple paths from the input u. However, since there is only one input vertex, the maximal number of vertex disjoint simple paths from input to output is one, i. e. its structural system rank equals one, which is also the rank of the system $r = 1$. Since both outputs are contained in a simple path there are two possible combinations for the inputs and the outputs: (u, y_1) and (u, y_2). By Theorem 4.4.1, this means that there will be two polynomials, $p_{\omega_1}(s)$ and $p_{\omega_2}(s)$, for the determination of the system zeros. The results for the first combination ω_1 are given in Table 4.3. The corresponding polynomial is

$$p_{\omega_1}(s) = -2s^3 - 8s^2 - 6s \; . \tag{4.23}$$

	cycle families $\mathcal{C}_i$	$\sum\prod\mathcal{W}\in\mathcal{C}_i$
p_1	$(u \underleftarrow{\rightarrow x_4} \rightarrow y_1)$	-2
p_2	$(u \underleftarrow{\rightarrow x_4} \rightarrow y_1, x_2), (u \underleftarrow{\rightarrow x_4} \rightarrow y_1, x_3)$	-8
p_3	$(u \underleftarrow{\rightarrow x_4} \rightarrow y_1, x_2, x_3)$	-6
p_4	–	–

Table 4.3: Coefficients of $p_{\omega_1}(s)$ obtained from $\mathcal{G}_{fb}^{\omega_1}$.

For the second combination ω_1 the results are collected in Table 4.4. This yields

	cycle families $\mathcal{C}_i$	$\sum\prod\mathcal{W}\in\mathcal{C}_i$
p_1	$(u \underleftarrow{\rightarrow x_4} \rightarrow y_2), (u \underleftarrow{\rightarrow x_3} \rightarrow y_2)$	0
p_2	$(u \underleftarrow{\rightarrow x_3} \rightarrow y_2, x_2), (u \underleftarrow{\rightarrow x_3} \rightarrow y_2, x_4)$ $(u \underleftarrow{\rightarrow x_4} \rightarrow y_2, x_2), (u \underleftarrow{\rightarrow x_4} \rightarrow y_2, x_3)$	-8
p_3	$(u \underleftarrow{\rightarrow x_3} \rightarrow y_2, x_2, x_4), (u \underleftarrow{\rightarrow x_4} \rightarrow y_2, x_2, x_3)$	-8
p_4	–	–

Table 4.4: Coefficients of $p_{\omega_2}(s)$ obtained from $\mathcal{G}_{fb}^{\omega_2}$.

$$p_{\omega_2}(s) = -8s^2 - 8s \tag{4.24}$$

for the second polynomial. The monic greatest common divisor polynomial of $p_{\omega_1}(s)$ and $p_{\omega_2}(s)$ is

$$p_{SZ}(s) = s^2 + s \; , \tag{4.25}$$

which has the roots $s_1 = 0$ and $s_2 = -1$. These are the system zeros of the system (4.22).

5 Structural Approach to Non-Minimum Phase Systems

In this chapter, the methods for the determination of zeros and poles developed in the previous chapter will be used to analyze systems by their structure. The structure of a system is given by the mutual dependency between the state variables, the inputs and the outputs. This means the connections and effects in a system are considered only by their quality and not by their quantity. For instance, it is only considered if the parts in a mechanical system are connected or that acceleration, speed and distance of these parts are related. The structure that describes a dynamical system is sometimes called a "structural system" or "structured system".

Many system properties that can be investigated structurally are well-known from numerical analysis. Some information about a system is lost if only its structure is considered. Therefore, it is not always possible to give a precise answer whether the considered property holds numerically. However, the advantage of the structural analysis is that if a property holds structurally, it holds for almost all systems of the same structure.

There are several publications considering zeros of structural systems. First, the number of zeros a system possesses was investigated. In [Söt79] and [Rei82] it was shown that the number of transmission zeros and decoupling zeros is given by the structure of a system. In [Sva86] this was extended to the number of invariant zeros for non-degenerated square systems. Much later, in [Wou99] criteria were given to determine the number of invariant zeros for non-square systems. Furthermore, it was investigated if the position of zeros and poles is determined by the structure of a system. In [HCD99] it was shown that for structural systems the transmission zeros are either fixed at 0 or arbitrary located dependent on the numerical values. This was extended to the invariant zeros in [WCD03]. It is well-known, that the position of the zeros and poles determines the stability of the system and the zero dynamics. Indirectly, this relation was used to determine by its structure if a system can be stable provided that specific numeric values are chosen [Bel13].

A similar approach will be followed in this chapter. The results of Chapter 4 are transferred to structural systems. This leads to sufficient conditions to determine if a system is not asymptotically stable or non-minimum phase by only considering its structure. Parts of the presented results were published in [DSS16a] and [DSS16c].

First, the terms “structural system” and “structural property” are defined. With these, it will be possible to state a theorem on structurally non-minimum phase systems. This will give a sufficient condition for the numerical realization of a system with this structure being non-minimum phase in almost all cases. Subsequently, it is investigated how this property can be modified so that it holds in all cases numerically. This will lead to properties, known as strong-structural, which will hold in all cases of a numerical realization of a structural system. In this context, the property “strong-structurally non-minimum phase” will be stated. Concluding, some extensions are discussed. First, a criterion for “strong-structurally not asymptotically stable” is given. In cases where the structural approach fails, an extension to signed systems is developed, where, besides the structure, also the signs of the dependencies between the state variables, inputs and outputs are considered. Finally, the applicability of the structural properties to nonlinear systems is discussed.

5.1 Structural Properties

The structure of a system is described by the mutual dependency of the state variables x_i, the inputs u_i and the outputs y_i. That means, it is a “yes or no” criterion if e. g. a state variable x_i depends on another state variable x_k or an input u_j. The two subsequent definitions are adopted from [Rei88, Definition 12.1 and 12.2].

> **Definition 5.1.1 (Structural System).** For a set of linear systems $\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_i\boldsymbol{u}_i$, $\boldsymbol{y}_i = \boldsymbol{C}_i\boldsymbol{x}_i$ with the same number of state variables, inputs and outputs, their common *structure* can be defined by the *structural system*, $\Sigma_{\circledast} : \dot{\boldsymbol{x}} = \boldsymbol{A}_{\circledast}\boldsymbol{x} + \boldsymbol{B}_{\circledast}\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{C}_{\circledast}\boldsymbol{x}$. An element in the *structure matrices*, $\boldsymbol{A}_{\circledast}$,$\boldsymbol{B}_{\circledast}$ and $\boldsymbol{C}_{\circledast}$ is zero, denoted by 0, if the element at the same position is identically zero respectively for all $\boldsymbol{A}_i$,$\boldsymbol{B}_i$ and $\boldsymbol{C}_i$. Otherwise, this element is nonzero, denoted by $\circledast$, if the element at the same position is nonzero and independent respectively for almost all $\boldsymbol{A}_i$,$\boldsymbol{B}_i$ and $\boldsymbol{C}_i$. Systems have an identical structure, if their states, inputs and outputs have identical dependencies of each other. Such systems are called *structurally equivalent*.

The terms "nonzero" and "independent" will be explained in detail by an example in Section 5.3. "Almost all" means all except of a set of measure zero, see "almost everywhere" in [Hal74].

In control engineering two types of Σ_{LS} appear regularly. If the model of the plant, which should be controlled, is generated by system identification methods using measurements, the result is one or a set of numerical matrices $(\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i)$. If, in contrast, the model of the plant is derived from physical equations, the result is a parametrized system $(\boldsymbol{A}[\boldsymbol{\mu}], \boldsymbol{B}[\boldsymbol{\mu}], \boldsymbol{C}[\boldsymbol{\mu}])$. Both types can be transferred to a structural system $\Sigma_{\circledast}$ as described by Definition 5.1.1.

This structural system can be investigated by structural methods as mentioned before. The properties that can be found using these methods are common to all numerical realizations of the structural system. Such properties are called structural properties and the following definition is given.

> **Definition 5.1.2 (Structural Property).** A *structural property* of a system is a property of a class of systems that are structurally equivalent. For this class the property under investigation holds numerically for almost all admissible numerical realizations.

According to [Rei88, Def. 12.1], "a numerically given [system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$] is called an admissible numerical realization with respect to $[(\boldsymbol{A}_{\circledast}, \boldsymbol{B}_{\circledast}, \boldsymbol{C}_{\circledast})]$, if it can be obtained by fixing all nonzero entries of $[(\boldsymbol{A}_{\circledast}, \boldsymbol{B}_{\circledast}, \boldsymbol{C}_{\circledast})]$ at some particular values."

In order to construct a *structural graph* $\mathcal{G}_{fb}^{\circledast}$ of a system or a class of systems, the first step is to create a structural system $\Sigma_{\circledast}$ according to Definition 5.1.1. Then by Definition 2.1.1 the structural graph $\mathcal{G}_{fb}^{\circledast}$ is generated. The connection to the graph-theoretic approach is immediate. The unweighted graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ of system Σ_{LS} reveals the structure of the system, since the edges in the graph coincide with the nonzero entries in the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. Thus, if the unweighted graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ complies with Definition 5.1.1,

$$\mathcal{G}_{sys}^{\circledast} = \mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}) \tag{5.1}$$

holds. From this point of view it is easy to conclude that systems that are structurally equivalent have the same unweighted graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$. Hence, structural properties can be analyzed using the unweighted graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ and a structural property is valid for all systems that have the same unweighted graph.

5.2 Structurally Non-Minimum Phase Systems

The determination of the invariant zeros for non-degenerated square systems is generally less complex compared to the case of non-square or degenerated systems. As explained in Chapter 4 for non-degenerated square systems the invariant zeros polynomial is given by a determinant whereas in the general case several minors have to be considered. This is also the case for the structural approach. Thus, it is an advantage to check structurally the squareness and degeneracy of a system. The squareness of a system is obvious since it depends only on the number of input vertices and output vertices in $\mathcal{G}_{sys}$. In order to determine the degeneracy of a system its rank has to be known. In Section 4.2 a rank criterion, the structural system rank, was already introduced. Since this criterion is independent of the actual value of the edges in $\mathcal{G}_{sys}$ or respectively entries in the matrices of Σ_{LS}, but relies on the structure of system, it is actually a structural property. This will be investigated in detail in Subsection 5.4.1. Nevertheless, the following corollary of Theorem 5.4.1 is anticipated to decide whether a system is structurally degenerated.

Corollary 5.2.1. A system Σ_{LS} is structurally non-degenerated if s-rank $\Sigma_{\circledast} = \min(m, p)$ holds. Otherwise, it is structurally degenerated.

5.2.1 Square Structurally Non-Degenerated Systems

The stability of a polynomial, i. e. the position of its roots, is usually checked by the Routh-Hurwitz criterion as explained in Subsection 2.2.3. Therefore, the plan is to find a criterion to check the stability of the invariant zeros polynomial (4.8) without knowing numeric values. The necessary condition for stability is given by Lemma 2.2.3. By the structural approach it is not possible to determine the numerical values of the coefficients but their existence can be checked. That means, if some coefficient of a polynomial misses, it is at least not asymptotically stable. Hence, in order to determine if a system is structurally non-minimum phase the following theorem can be applied considering its structural feedback graph $\mathcal{G}_{fb}^{\circledast}$.

Theorem 5.2.1. A square structurally non-degenerated system of order n with m inputs and outputs is structurally non-minimum phase if for its corresponding feedback graph $\mathcal{G}_{fb}^{\circledast}$ the following conditions hold:

1. There is a *smallest feedback cycle family* of width k_l with $m \leq k_l < n$, which is the feedback cycle family with the lowest number of state vertices.

2. There is at least one k with $k_l < k \leq n$ so that $\mathcal{G}_{fb}^{\circledast}$ contains no feedback cycle family of width k.

Proof. The property is structural since for the general absence or presence of a coefficient p_k in (4.8) only the absence or presence of cycle families of width k in $\mathcal{G}_{fb}^{\circledast}$ is relevant, regardless of the numerical values of the system.

If a smallest cycle family of width $k_l < n$ exists, it maps to the coefficient p_{k_l}. If there is a cycle family of width $k \in]k_l, n]$ missing in $\mathcal{G}_{fb}^{\circledast}$, the corresponding p_k in (4.8) is also missing. Thus, by Lemma 2.2.3 not all invariant zeros have strictly negative real parts. By Definition 3.2.1 a system with this property is non-minimum phase. □

In some cases it is possible to determine the position of some invariant zeros by structural investigation.

Corollary 5.2.2. If there exists a $k_u < n$ that is the width of the feedback cycle family in $\mathcal{G}_{fb}^{\circledast}$ with the highest number of state vertices, i. e. the *largest feedback cycle family*, the considered system has $n - k_u$ invariant zeros in 0.

A similar result for transmission zeros was found by [HCD99].

Remark 5.2.1. Theorem 5.2.1 is only a sufficient condition for the non-minimum phase property of a numerical realization of a system as reasoned next. Even if all cycle families of width $k \in [k_l, n]$ exist, it is not possible to conclude that the system of interest is minimum phase. Since the exact values of p_k have not been calculated they may be positive or negative. This opposes Lemma 2.2.3 and hence it is not possible to conclude a structural minimum phase property of a system. This can be easily shown by the following example.

Example 5.2.1. In Figure 5.1 an example of a system, which is structurally not non-minimum phase but its numerical realization is non-minimum phase, is depicted.

However, there is an exception, where it is possible to conclude a structural minimum phase property.

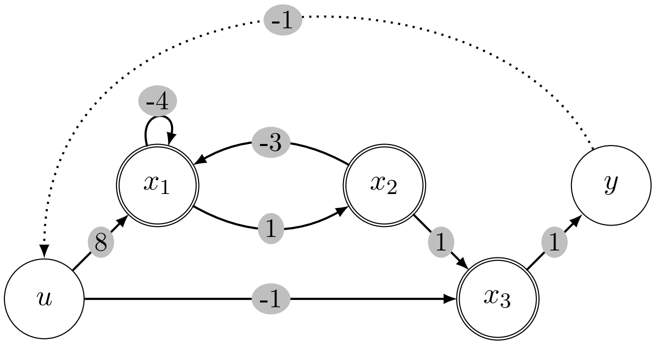

Figure 5.1: System with positive zeros that is not structurally non-minimum phase.

Corollary 5.2.3. If the feedback graph $\mathcal{G}_{fb}^{\circledast}$ of a system contains only feedback cycle families of width $k = n$, then the system is structurally minimum phase.

This is obvious, since in the mentioned case only the coefficient p_n in (4.8) exists and hence the polynomial has no roots. Almost all systems, which fulfill this condition, have a full relative degree or are flat.

5.2.2 Non-Square or Structurally Degenerated Systems

If the considered system is non-square or structurally degenerated, it is in many cases not possible to tell if it is structurally non-minimum phase. The reason is that the zeros polynomial is not given directly by a determinant (4.8) but by the greatest common divisor of several minors (Lemma 2.2.4), which by structural methods is not possible to find. However, it is still possible to detect structural zeros in 0 and thus conclude that a system is structurally non-minimum phase.

Theorem 5.2.2. Suppose a non-square or degenerated system Σ_{LS} of order n has structural rank r_s. Consider all feedback graphs $\mathcal{G}_{fb}^{\circledast\omega_i}$ of this system, where ω_i is a distinct unordered combination of r_s input and r_s output vertices that are contained in r_s vertex disjoint simple paths from input to output in $\mathcal{G}_{sys}^{\circledast}$. If there is no feedback cycle family of width n in any of the graphs $\mathcal{G}_{fb}^{\circledast\omega_i}$ the considered system is structurally non-minimum phase.

Proof. For every graph $\mathcal{G}_{fb}^{\circledast\omega_i}$ the feedback cycle family of width n refers to the coefficient p_n of $p_{\omega_i}(s)$. If these feedback cycle families are missing the coefficient p_n is missing in every $p_{\omega_i}(s)$. This implies that in the monic greatest common divisor $p_{SZ}(s)$ of all these polynomials the coefficient that precedes the zero order of s does not exist. This means that at least one system zero of the considered system is located at 0. By Definition 3.2.1 a system with this property is non-minimum phase. □

Note, Theorem 5.2.2 is weaker than Theorem 5.2.1 since only systems with zeros in 0 can be considered. Systems with positive zeros cannot be detected by this method.

5.2.3 Example: Vertical Take Off and Landing Aircraft

As an example a simple PVTOL (Planar Vertical Take Off and Landing) aircraft, see Figure 5.2, will be analyzed, which was presented in [HSM92] and whose non-minimum phase behavior was well examined in [MDP94]. The position of the plane is given by the coordinates x and z of its

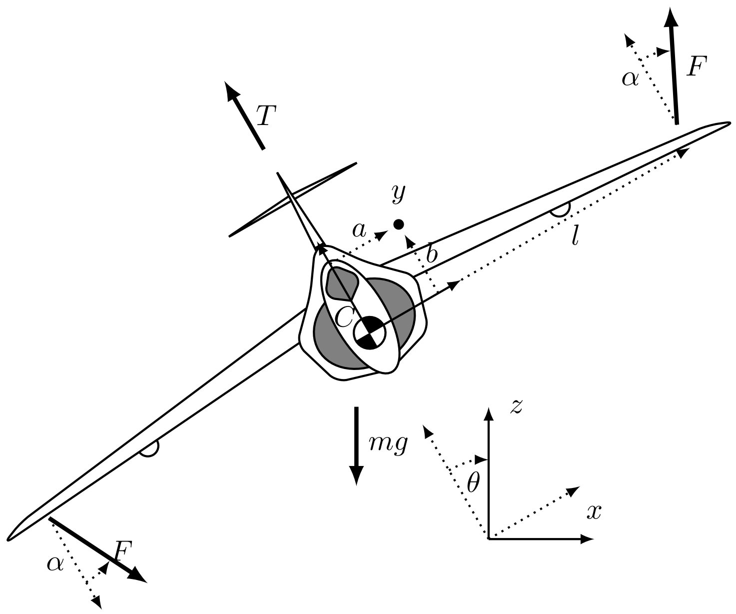

Figure 5.2: Planar Vertical Take Off and Landing aircraft.

center of mass C and its angle θ with respect to an inertial frame. The motion of the system is caused by the following forces. The buoyancy generated by the wings due to the relative forward speed is denoted by T. The variable F represents the force produced in equal amounts by the vertical thrusters, which are mounted rotated by fixed α with respect to the vertical axis of the plane. The gravity force is denoted by mg. Considering that the forward speed and the force of the vertical thrusters can be manipulated, the inputs can be defined as

$$u_1 := \frac{T}{m},\; u_2 := \frac{2F}{m}\cos\alpha\;. \tag{5.2}$$

With that the system dynamics can be described by

$$\begin{aligned}\ddot{\theta} &= \lambda u_2\\ \ddot{x} &= -u_1\sin\theta + \epsilon u_2\cos\theta\\ \ddot{z} &= u_1\cos\theta + \epsilon u_2\sin\theta - g\;,\end{aligned} \tag{5.3}$$

with $\epsilon = \tan\alpha$, $\lambda = \frac{ml}{J}$, where J is the inertia with respect to C and l the distance from C to the thrusters. An inertial position sensor is placed somewhere on the body of the aircraft. Its position $\boldsymbol{y}$ is given in the coordinates (a, b) with respect to the body fixed frame originated in the center of mass. Using the sensor as output yields

$$\begin{aligned}y_1 &= x + a\cos\theta - b\sin\theta\\ y_2 &= z + a\sin\theta + b\cos\theta\end{aligned} \tag{5.4}$$

for its position in inertial coordinates.

Hence, the system has $n = 6$ states $\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & x & \dot{x} & z & \dot{z}\end{bmatrix}^T$ and $m = 2$ inputs $\boldsymbol{u} = \begin{bmatrix}u_1 & u_2\end{bmatrix}^T$ and $p = 2$ outputs $\boldsymbol{y} = \begin{bmatrix}y_1 & y_2\end{bmatrix}^T$. Linearization around the equilibrium ($\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{u} = \begin{bmatrix}g & 0\end{bmatrix}^T$) yields the system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ -g & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}0 & 0\\ 0 & \lambda\\ 0 & 0\\ 0 & \epsilon\\ 0 & 0\\ 1 & 0\end{bmatrix}\boldsymbol{u}\\ \boldsymbol{y} &= \begin{bmatrix}-b & 0 & 1 & 0 & 0 & 0\\ a & 0 & 0 & 0 & 1 & 0\end{bmatrix}\boldsymbol{x}\;.\end{aligned} \tag{5.5}$$

The corresponding unweighted feedback graph $\mathcal{G}_{fb}^{\circledast}$ is drawn in Figure 5.3. Analyzing the graph and applying Theorem 5.2.1 on it yields the following values:

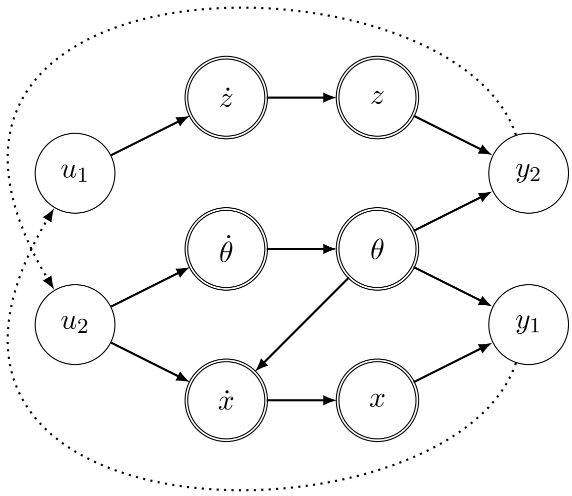

Figure 5.3: $\mathcal{G}_{fb}^{\circledast}$ of the linearized PVTOL.

- The smallest width of a feedback cycle family is $k_l = 4$,
  e. g. $(u_1 \rightarrow \dot{z} \rightarrow z \rightarrow y_2 \rightarrow u_2 \rightarrow \dot{\theta} \rightarrow \theta \rightarrow y_1)$.

- The largest width of a feedback cycle family is $k_u = 6 = n$,
  e. g. $(u_2 \rightarrow \dot{\theta} \rightarrow \theta \rightarrow \dot{x} \rightarrow x \rightarrow y_1 \rightarrow u_1 \rightarrow \dot{z} \rightarrow z \rightarrow y_2)$.

- There is no cycle family of width $k = 5$.

Thus, it is possible to conclude that the PVTOL is structurally non-minimum phase.

Indeed, the invariant zeros of (5.5) are given by

$$s_{1,2} = \pm\sqrt{\frac{\lambda g}{\epsilon - \lambda b}}\,, \tag{5.6}$$

which are under no circumstance simultaneously negative and consequently the zero dynamics are not asymptotically stable.

5.3 Strong-Structural Properties

The properties considered until now are valid for almost all numerical realizations. The natural question that arise is whether there exist structural properties that hold for all numerical realizations of a system. This kind of properties were introduced by the strong-structural controllability [MY79].

Definition 5.3.1 (**Strong-Structural Property**). A *strong-structural property* of a system is a property of a class of systems that are structurally equivalent. For this class the property under investigation holds numerically for all admissible numerical realizations.

This definition implies that the considered property holds generically for all structurally equivalent systems.

Some former results about strong-structural controllability were reinvestigated by [JSA11] and shown to be wrong. This lead to a new graph-theoretic characterization of strong-structural controllability [SJA11]. Recently [HRS13; RHS14] have extended the results about strong-structural controllability of [MY79] to the time-variant case. In addition, the works [TD13; TD14; TD15] have to be considered.

Definition 5.1.2 states that a structural property holds for almost all numerical realizations. This raises the question, under which conditions a system possesses a specific (non-strong) structural property but does not have it in the numerical sense. Consider again Example 4.3.1 and its parametrized feedback graph, depicted in Figure 5.4. From that its parametrized characteristic polynomial

$$p(s) = \underbrace{(\mu_1\mu_3\mu_7 + \mu_2\mu_5\mu_8)}_{p_2} s^2 + \underbrace{\mu_2\mu_5\mu_4\mu_3\mu_7 - \mu_1\mu_3\mu_7\mu_5\mu_6}_{p_4} \tag{5.7}$$

can be derived. As a result of Lemma 2.2.3 the system is (structurally) non-minimum phase because the coefficient p_3 in (5.7) is missing. However, for certain values for the parameters μ_i the system might be minimum phase. This problem is the background of the statement in Definition 5.1.1, that the entries $\circledast$ should be "nonzero and independent". The origin of these two requirements will be investigated now.

First, consider the term "nonzero" in Definition 5.1.1. According to [Rei88; DCW03; RHS14] for numerical realizations entries in $\mathbb{R}$ are allowed for the nonzero entries $\circledast$ in the structural

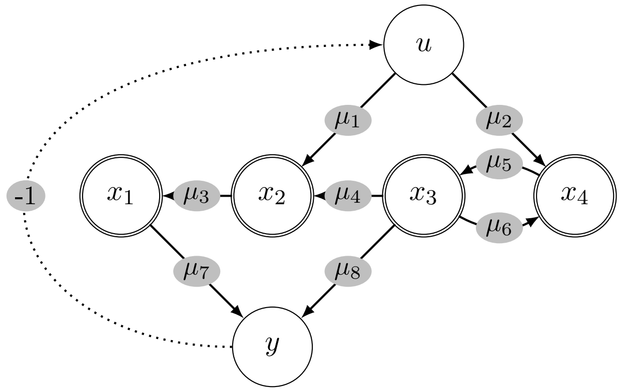

Figure 5.4: $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ of the loading bridge with $\mu_i \in \mathcal{W}$.

system. In that case, the conditions under which a numeric zero 0 value is allowed for a nonzero entry $\circledast$ are the following:

1. If the structural system is given by a set of structural equivalent numerical realizations $(\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i)$, the considered entry has to be different from 0 for almost all of them.

2. If the structural system is given by a parametrized system $(\boldsymbol{A}[\boldsymbol{\mu}], \boldsymbol{B}[\boldsymbol{\mu}], \boldsymbol{C}[\boldsymbol{\mu}])$, the corresponding parameter μ_i has to be different from 0 almost everywhere.

Suppose now that only the parameter μ_3 in Figure 5.4 is zero in a specific numerical realization. This leads to the coefficient p_4 of (5.7) being also zero. Nevertheless, this particular numerical realization will also be non-minimum phase, since the conditions of Lemma 2.2.3 are not met. Further, consider only the parameters μ_1 and μ_8 to be zero. From this follows that the coefficient p_2 is zero. Now the numerical realization is minimum phase since it has no zeros. However, the structural property is still valid, since only a particular numerical realization is considered and in almost all cases $\mu_1 \neq 0$ and $\mu_8 \neq 0$ is required. Nevertheless, for strong-structural properties all numerical realizations have to be non-minimum phase. So the discussed case must be avoided by forbidding the value 0 in the numerical realizations for a nonzero entry.

Further, according to Definition 5.1.1 the nonzero entries $\circledast$ should be “independent” in the corresponding numerical realization of $\Sigma_{\circledast}$. In the literature [Rei88] “assumes [the entries] to be independent of one another” and in [Mur09; Wou91b] the parameter values have to lie outside of some proper algebraic variety in the parameter space. To explain the problem, consider again the polynomial (5.7). The coefficient p_2 consists of a sum. That means that under certain conditions a (numerical) cancellation $\mu_1\mu_3\mu_7 + \mu_2\mu_5\mu_8 = 0$ may occur. As described above this will render the system minimum phase. In order to avoid these numerical cancellations the following three approaches can be taken.

The first one is to demand that all nonzero entries $\circledast$ in $\Sigma_\circledast$ have to be algebraically independent, see [Mur09, p.32]. That means no entry $\circledast$ can be expressed by an algebraic combination of the other nonzero entries. Considering the example this implies that the equation $\mu_1\mu_3\mu_7+\mu_2\mu_5\mu_8 = 0$ will have no solution, and thus no cancellation occurs. In general algebraic independence of the nonzero entries leads to avoidance of cancellations in the minors or determinants of numerical realizations. With that many structural properties, including the ones proposed in this work, will hold also in the strong sense. However, the demand of algebraic independence of the nonzero entries is in many cases not desirable. For instance, it would be not possible to investigate Example 4.3.1 by structural methods since the elements in (4.18) may be algebraic dependent [1].

The second approach, followed by e. g. [Wou91b; HCD99; DCW03], is to exclude some proper algebraic varieties [2] in the parameter space. Basically this means, that in contrast to the algebraic independence approach, only some algebraic combination of the parameters are not allowed. Namely, exactly these which lead to numerical cancellations in determinants or minors.

For almost all choices of numerical values $\in \mathbb{R}$ for the nonzero entries $\circledast$, these will be algebraic independent and will not form a proper algebraic variety. Also, for almost all parameterized systems, the parameters will lie outside of a proper algebraic variety that leads to numerical cancellations in the considered determinants or minors. Nevertheless, in some cases the parameters of a system are dependent such that a structural property may not hold for any numerical realization, compare e. g. [JSA11]. Therefore, independence of the nonzero entries $\circledast$ is required in Definition 5.1.1.

However, both mentioned approaches require more information about the considered system for avoiding cancellations than just the zero/nonzero pattern. This means, for the investigation of strong-structural properties the relation between the nonzero entries has to be considered a priori, which has no benefit compared to applying numerical methods directly. Therefore, the approach taken in this work is to avoid sums in the critical terms in the determinants or minors. This also makes the here discussed strong-structural properties independent of dependencies between the nonzero entries $\circledast$ of $\Sigma_\circledast$.

[1] Let be $g = 1$ then $a_2 = b_2(\frac{a_1}{b_1} + \frac{1}{b_1})$.
[2] See [Won79].

Hence, for the investigation of strong-structural properties a less strict version of Definition 5.1.1 is used.

Definition 5.3.2 (**Strong-Structural System**). For a set of linear systems $\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_i\boldsymbol{u}_i$, $\boldsymbol{y}_i = \boldsymbol{C}_i\boldsymbol{x}_i$ with the same number of state variables, inputs and outputs, their common *structure* can be defined by the *(strong-)structural system*, $\Sigma_* : \dot{\boldsymbol{x}} = \boldsymbol{A}_*\boldsymbol{x} + \boldsymbol{B}_*\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{C}_*\boldsymbol{x}$. An element in the *(strong-)structure matrices*, $\boldsymbol{A}^*$,$\boldsymbol{B}^*$ and $\boldsymbol{C}^*$ is zero, denoted by 0, if the element at the same position is identically zero respectively for all $\boldsymbol{A}_i$,$\boldsymbol{B}_i$ and $\boldsymbol{C}_i$. Otherwise, this element is nonzero, denoted by $*$, if the element at the same position is nonzero respectively for all $\boldsymbol{A}_i$,$\boldsymbol{B}_i$ and $\boldsymbol{C}_i$. Systems have an identical structure, if their states, inputs and outputs have identical dependencies of each other. Such systems are called *strong-structurally equivalent.*

Similar to the description in Section 5.1, a graph can be generated for strong-structurally equivalent systems i. e. a *strong-structural* graph $\mathcal{G}^*_{sys}$. Thus, if an unweighted system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ complies with Definition 5.3.2,

$$\mathcal{G}^*_{sys} = \mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}) \tag{5.8}$$

holds.

5.4 Strong-Structurally Non-Minimum Phase Systems

In order to investigate the strong version of the already discussed structural properties a new type of cycle family will be introduced. The reason is that sums in the coefficients of the considered polynomials have to be avoided as discussed in Section 5.3. Each summand of a coefficient in these polynomials is given by distinct cycle families of identical width. The method to avoid sums in the coefficient will make use of unique cycle families, which are defined as follows.

Definition 5.4.1 (**Unique Cycle Family**). A cycle family in a graph is *unique* if there exists no other cycle family of same width in this graph.

With unique cycle families it is possible to determine the strong-structural properties as will be explained subsequently.

5.4.1 Generic Rank of a Structural System

As mentioned in Section 4.2 and Section 5.2 the structural rank of a system Definition 4.2.2 is a structural property. Applied to the structural system $\Sigma_{\circledast}$, it is coincident with the structural norm rank Definition 4.2.1 by [Rei88, Theorem 32.7]. Furthermore, it will be a strong-structural property if s-rank Σ_* equals the system rank for any numerical realization of Σ_*. Speaking in terms of matrices this is the case when the term rank of $\boldsymbol{P}[\boldsymbol{\mu}](s)$ equals the generic rank of $\boldsymbol{P}[\boldsymbol{\mu}](s)$ considering the nonzero entries as parameters $\boldsymbol{\mu}$ and thus equals the norm-rank $\boldsymbol{G}(s)+n$ for any numerical realization of Σ_*. This is achieved by the following theorem.

Theorem 5.4.1. The structural system rank s-rank Σ_* equals the system rank of all numerical realizations Σ_{LS}, if at least one of the corresponding feedback graphs $\mathcal{G}_{fb}^{*\omega_i}$, defined in Theorem 4.4.1, contains at least one unique cycle feedback family. In this case, it is a strong-structural property.

Proof. Consider a system of rank r and its system matrix $\boldsymbol{P}(s)$. Then it exists at least one minor (2.46) of $\boldsymbol{P}(s)$ that does not vanish. These minors are polynomials in s and can be obtained by applying Theorem 4.4.1. The coefficients of (4.20) are created from the feedback cycle families in $\mathcal{G}_{fb}^{*\omega_i}$. If there is at least one unique feedback cycle family of some width k, the corresponding coefficient p_k is not given by a sum. Thus, no numerical cancellation may occur and the corresponding minor does not vanish generically. In this case the structural rank equals the normal rank of the transfer function, i. e.

$$\text{s-rank}\,\Sigma_{LS} = \text{norm-rank}\,\boldsymbol{G}(s)\ , \tag{5.9}$$

and thus the rank of the system. This holds for all strong-structurally equivalent numerical realizations and hence this property is generic. □

With this property, it is now possible to safely determine structurally if a system is degenerated.

Remark 5.4.1. The generic rank of a structural system is also described in [Wou91b; Wou91a]. The structural normal rank Definition 4.2.1 is identical to this generic rank, if, for the independence of the entries in $\Sigma_{\circledast}$, the condition is met that the entries lie outside of some proper algebraic variety.

5.4.2 Square Non-Degenerated Systems

In order to determine if a system is strong-structurally non-minimum phase Theorem 5.2.1 can be adapted as follows.

Theorem 5.4.2. A square system of order n with m inputs and outputs is strong-structurally non-minimum phase if for its corresponding feedback graph $\mathcal{G}^*_{fb}$ the following properties hold:

1. There is a *smallest unique feedback cycle family* of width k_l with $m \leq k_l < n$, which is the unique feedback cycle family with the lowest number of state vertices.

2. There is at least one k with $k_l < k \leq n$ so that $\mathcal{G}^*_{fb}$ contains no feedback cycle family of width k.

Proof. In addition to Theorem 5.2.1 it has to be shown that this is a strong-structural property. Since the cycle family of width k_l has to be unique, the numerical value of the coefficient p_{k_l} of (4.8) is given only by the product of the weights of this cycle family. By Definition 5.3.2 there is no weight of value 0, hence the coefficient p_{k_l} cannot be zero. Thus, all numerical realizations are non-minimum phase. □

In the same manner Corollary 5.2.3 can be extended.

Corollary 5.4.1. If the feedback graph $\mathcal{G}^*_{fb}$ of a system contains only one feedback cycle family and this feedback cycle family is of width $k = n$ then the system is also strong-structurally minimum phase.

5.4.3 Non-Square or Degenerated Systems

As it was the case in Subsection 5.2.2 for non-square or degenerated systems only zeros in 0 can be found structurally. With a small addition to Theorem 5.2.2, the following corollary can be stated.

Corollary 5.4.2. If a system Σ_{LS} has rank r and if Theorem 5.2.2 applies with $r_s := r$ on $\mathcal{G}^*_{fb}$, then the system is strong-structurally non-minimum phase.

This means, if it is possible to determine the rank of a system with Theorem 5.4.1, Theorem 5.2.2 can be applied directly to get the strong-structural property.

Remark 5.4.2. Note, the considered strong-structural properties are not coordinate free. Consider a system with two numerical representations with different coordinate base. If these two representations are transferred to a structural system one may be strong-structurally non-minimum phase and the other not. However, if a system is strong-structurally non-minimum phase all its numerical realizations are non-minimum phase regardless in which coordinate base they are transformed. The reason is that the (numerical) non-minimum phase property is coordinate free since it is determined by the invariant zeros polynomial which is invariant under coordinate transformations. Nevertheless, the sum of products the coefficients of the polynomial are generally formed of, refer to (2.19) and Lemma 4.1.1, is not coordinate free. This means, it depends on the selected coordinate basis if a certain coefficient is formed by only one product, i. e. a unique cycle family, or by a sum of products, i. e. more than one cycle family of identical width. Hence, the structural non-minimum phase property is coordinate free but not the strong-structurally non-minimum phase property.

5.4.4 Example: ECP-Pendulum

As an example the ECP Inverted Pendulum, see [Edu95], is considered. It is a mechanical model of the lateral movement that a tightrope walker has to perform to balance on the rope. The pendulum consists of two rods, one pole and one crossbar, see Figure 5.5. The pole is mounted by a pivot bearing on the bottom plate, which allows rotation around one axis measured by $\theta(t)$. On top of the pole the crossbar, which has two weights at its ends, is mounted. The crossbar can be moved orthogonal to the pole in its rotational plane by a linear motor. The position between the tip of the pole and the center of gravity of the crossbar is measured by $s(t)$. The poles overall length is denoted by l_0, its mass is m_1 and its center of gravity is located l_c from the bottom joint. The crossbar has the mass m_2 and its center of gravity is in its geometric center. The system is actuated by the force F applied by the linear motor on the rods. The whole system is affected by gravity g. Because the masses are considered to be concentrated at the center of gravity of the rods, the inertias can be neglected. Hence, the system can be

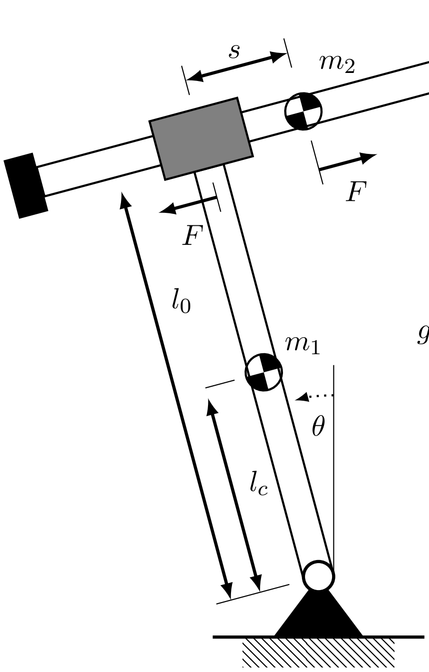

Figure 5.5: The ECP Inverted Pendulum.

modelled by the differential equations

$$
\begin{aligned}
m_1\ddot{s} + m_1 l_0 \ddot{\theta} - m_1 s \dot{\theta}^2 - m_1 g \sin\theta &= F \\
m_1 l_0 \ddot{s} + \left(m_1\left(l_0^2 + s^2\right) + m_2 l_c^2\right)\ddot{\theta} + 2 m_1 s \dot{s} \dot{\theta} & \\
- \left(m_1 l_0 + m_2 l_c\right) g \sin\theta - m_1 g s \cos\theta &= 0\,.
\end{aligned}
\tag{5.10}
$$

The system has $n = 4$ state variables $\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta} & s & \dot{s}\end{bmatrix}^T$, the input $u = F$ and the output $y = \theta$. Linearization around the equilibrium ($s = 0$, $\theta = 0$, $F = 0$) yields the system

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{l_c} & 0 & \frac{g m_1}{l_c^2 m_2} & 0 \\ 0 & 0 & 0 & 1 \\ g - \frac{g l_0}{l_c} & 0 & -\frac{g l_0 m_1}{l_c^2 m_2} & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \frac{-l_0}{l_c^2 m_2} \\ 0 \\ \frac{1}{m_1} \frac{l_0^2}{l_c^2 m_2} \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}\,.
\end{aligned}
\tag{5.11}
$$

The corresponding feedback graph $\mathcal{G}_{fb}^*$ is drawn in Figure 5.6.

Applying Theorem 5.4.2 on $\mathcal{G}_{fb}^*$ yields the following:

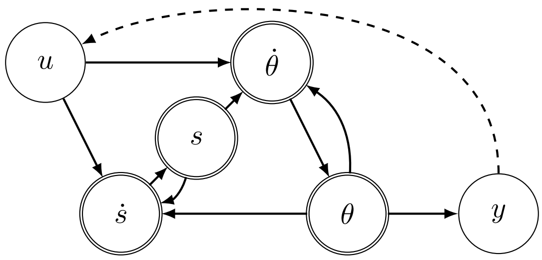

Figure 5.6: Strong structural feedback graph $\mathcal{G}^*_{fb}$ of the inverted pendulum.

- The smallest width of a feedback cycle family is $k_l = 2$,
  $u \rightleftarrows \dot{\theta} \rightarrow \theta \rightarrow y$, which is the only one with width 2.

- The largest width of a feedback cycle family is $k_u = 4 = n$,
  e. g. $u \rightleftarrows \dot{s} \rightarrow s \rightarrow \dot{\theta} \rightarrow \theta \rightarrow y$.

- There is no cycle family of width $k = 3$.

This leads to the conclusion that the inverted pendulum is strong-structurally non-minimum phase.

Indeed, if the nonlinear system (5.10) is analyzed, it is possible to isolate the zero dynamics

$$\ddot{s} = \frac{g}{l_0} s \,, \tag{5.12}$$

which are linear. This dynamical system is not asymptotically stable, regardless which numerical nonzero values are chosen for the parameters g and l_0. Thus, the example system is non-minimum phase for all numerical realizations.

5.5 Extensions

In this section three extensions of the considered structural properties will be discussed. First, the strong-structural approach is used to decide whether a system is strong-structurally not asymptotically stable. Further, the case is considered, where only by structural methods, it is not possible to determine the stability or non-minimum phase property of a system but by considering the signs of the relations between the state variables, inputs and outputs.

Finally, the applicability of the introduced structural properties to nonlinear systems (2.5) is investigated.

5.5.1 Strong-Structurally Not Asymptotically Stable Systems

The idea is fundamentally identical to the one presented in Subsection 5.2.1. The structural approach is used to determine whether not all coefficients of the characteristic polynomial (2.33) are present. By the Routh-Hurwitz criterion, this is a precondition for stability of the system. Hence, in order to determine if a system is strong-structurally not asymptotically stable the following theorem can be applied considering its graph $\mathcal{G}^*_{sys}$.

> **Theorem 5.5.1.** A system Σ_{LS} of order n is strong-structurally not asymptotically stable if there is at least one k with $1 \leq k \leq n$ so that its system graph $\mathcal{G}^*_{sys}$ contains no cycle family of width k.

Proof. If there is a cycle family of the width $k \in [1, n]$ missing in $\mathcal{G}^*_{sys}$ the corresponding α_k in (2.33) is also missing for all numerical realizations. Thus, by Lemma 2.2.3 not all poles have strictly negative real parts. Hence, the system with this property is not asymptotically stable. □

This is basically an inversion of [Bel13, Theorem 2].

Again, it is also possible to determine the number of eigenvalues in 0.

> **Corollary 5.5.1.** Let k_u with $0 \leq k_u \leq n$ be the width of the cycle family in $\mathcal{G}^*_{sys}$ with the highest number of state vertices, i. e. the largest cycle family. Then the system has $n - k_u$ eigenvalues in 0.

The following example shows the applicability of Theorem 5.5.1.

> **Example 5.5.1.** Consider the system graph depicted in Figure 5.7. Counting the cycle families of the widths $k \in [1, n]$ yields Table 5.1. Since there is no cycle family of width $k = 4$ the considered system is strong-structurally not asymptotically stable. This means there does not exist any numerical realization of the system that is asymptotically stable.

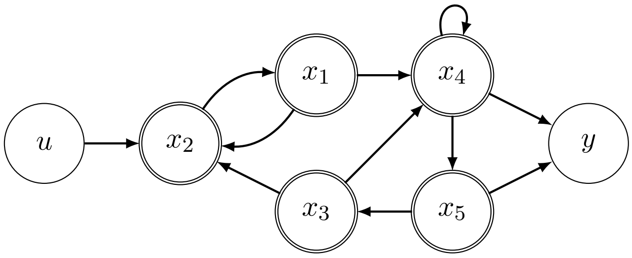

Figure 5.7: Structural graph $\mathcal{G}^*_{sys}$ of a strong-structurally not asymptotically stable system.

	cycle families $\mathcal{C}_i$	$\lvert\mathcal{C}_i\rvert$
α_1	(x_4)	1
α_2	$(x_1 \rightleftarrows x_2)$	1
α_3	$(x_1 \rightleftarrows x_2, x_4), (\underleftarrow{x_3 \to x_4 \to x_5})$	2
α_4	–	0
α_5	$(\underleftarrow{x_3 \to x_4 \to x_5}, x_1 \rightleftarrows x_2),$ $(\underleftarrow{x_1 \to x_4 \to x_5 \to x_3 \to x_2})$	2

Table 5.1: Cycle families in $\mathcal{G}^*_{sys}$(Figure 5.7) and their counts.

5.5.2 Sign Non-Minimum Phase Systems

Often not only the dependencies in a system are known, but also the sign of directions and parameters and therefore the sign of the dependencies. For example in a parametrized system like the loading bridge, refer to Example 4.3.1, many parameters are masses or dimensions, which are naturally positive. Further, it is known in which direction a part moves applying a force in a specific direction on it. The question is, may this additional information be used to investigate further properties that cannot be determined structurally.

This is actually strongly related to the term “sign stability” introduced in [QR65]. There the issue of concluding the stability of linear systems only knowing the signs of the parameters is investigated. This was originally considered by [Sam47]. A graph-theoretic method for determination of the sign stability was given by [Jef74]. A survey about this topic is presented in [Qui80]. Recently, this was extended to “unsafe signed systems” in [Har16] and “sign stabilizability” in [HS14].

Since the structural non-minimum phase property is only sufficient, there are cases where it does not apply. Consider for instance the next example.

Example 5.5.2. Depicted in Figure 5.8 is a lift truck as seen from above. The

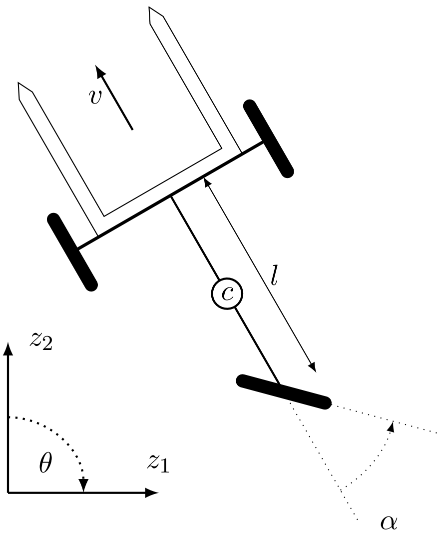

Figure 5.8: Sketch of a lift truck from above.

task is to place the center c of the truck at a specified position in the plain. The position in the plane is measured by the coordinates z_1 and z_2 and the orientation is measured by the angle θ. The truck can be steered by its rear wheel changing the angle α and moved in forward direction by the velocity v. The distance between the axle of the front wheels and the hub of the rear wheel is l, where c is right in the middle. The dynamics of the system are given by

$$\dot{z}_1 = v\sin\theta - \frac{l}{2}\cos\theta\dot{\theta} \tag{5.13}$$

$$\dot{z}_2 = v\cos\theta + \frac{l}{2}\sin\theta\dot{\theta} \tag{5.14}$$

$$\dot{\theta} = \frac{v}{l}\sin\alpha\,. \tag{5.15}$$

This system has 3 state variables $\boldsymbol{x} = [z_1\, z_2\, \theta]^T$ two inputs $\boldsymbol{u} = [v\, \alpha]^T$ and two outputs $\boldsymbol{y} = [z_1\, z_2]^T$. Linearization around $\boldsymbol{x}_0 = [0\,0\,0]^T$ and $\boldsymbol{u}_0 = [1\,0]^T$ leads to the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \\ 0 & \frac{1}{l} \end{bmatrix} \boldsymbol{u}\,, \boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}\,. \tag{5.16}$$

This system has one transmission zero $s_0 = \frac{2}{l}$ hence it is non-minimum phase for every positive wheelbase l. For structural analysis the graph $\mathcal{G}^*_{fb}$ is shown in Figure 5.9. Since the systems contains $n = 3$ state variables and $m = p = 2$ inputs

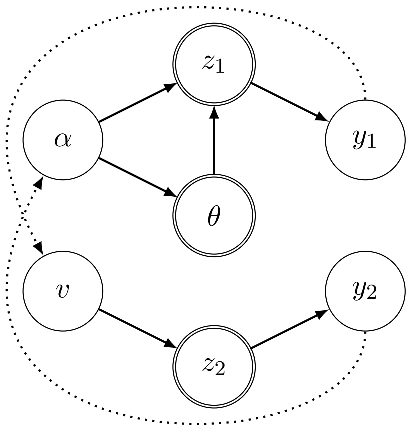

Figure 5.9: $\mathcal{G}^*_{fb}$ of the system (5.16).

and outputs, the maximal order of the invariant zero polynomial is one. Hence, the graph in Figure 5.9 may only contain feedback cycle families of width $k = 2$ and $k = 3$. Investigation of the graph yields one feedback cycle family of width $k = 2$, i. e. $(\alpha \rightarrow z_1 \rightarrow y_1 \rightarrow v \rightarrow z_2 \rightarrow y_2)$, and one feedback cycle family of width $k = 3$, i. e. $(\alpha \rightarrow \theta \rightarrow z_1 \rightarrow y_1 \rightarrow v \rightarrow z_2 \rightarrow y_2)$. Since both cycle families exist, neither Theorem 5.4.2 nor Theorem 5.2.1 is applicable. Thus, the system is not structurally non-minimum phase.

However, if the signs of the nonzero elements of the system (5.16) are considered it is possible to create a *signed graph* as given by Figure 5.10. The cycle families in this graph are identical with these of the graph in Figure 5.9. Now for each feedback cycle family a sign can be determined by multiplying the signs of its edges weights as shown in Table 5.2. As can be seen by this table, in (4.8) p_2 will be

	cycle families $\mathcal{C}_i$	$\prod \operatorname{sign} \mathcal{W} \in \mathcal{C}_i$	$\operatorname{sign} \mathcal{C}_i$
p_2	$(\alpha \rightarrow z_1 \rightarrow y_1 \rightarrow v \rightarrow z_2 \rightarrow y_2)$	$(-1)(+1)(-1)(+1)(+1)(-1)$	$-$
p_3	$(\alpha \rightarrow \theta \rightarrow z_1 \rightarrow y_1 \rightarrow v \rightarrow z_2 \rightarrow y_2)$	$(+1)(+1)(+1)(-1)(+1)(+1)(-1)$	$+$

Table 5.2: Feedback cycle families in $\mathcal{G}^{\pm}_{fb}$ (Figure 5.7) and their counts.

negative and p_3 will be positive. Thus, by Lemma 2.2.3 this system will not have a

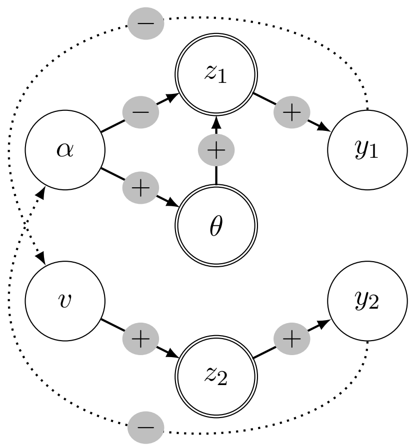

Figure 5.10: $\mathcal{G}_{fb}^{\pm}$ of the system (5.16).

negative zero and hence it is non-minimum phase by its signs.

In order to formalize the methods sketched in this example some terms have to be defined first.

Definition 5.5.1 (**Signed System**)**.** A set of linear systems $\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_i\boldsymbol{u}_i$, $\boldsymbol{y}_i = \boldsymbol{C}_i\boldsymbol{x}_i$ with the same number of state variables, inputs and outputs, has *common sign structure*, given by the *signed system* $\Sigma_\pm : \dot{\boldsymbol{x}} = \boldsymbol{A}_\pm\boldsymbol{x} + \boldsymbol{B}_\pm\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{C}_\pm\boldsymbol{x}$ if the following holds. In all matrices $\boldsymbol{A}_i$,$\boldsymbol{B}_i$ and $\boldsymbol{C}_i$ of the set, the elements at the same position are either 0, positive or negative. Then an element in the *signed matrices*, $\boldsymbol{A}^\pm$,$\boldsymbol{B}^\pm$ and $\boldsymbol{C}^\pm$ at this position is either zero, denoted by 0, positive, denoted by + or negative, denoted by −.

Again, it is possible to transfer a signed system into a graph as introduced for structural systems.

Definition 5.5.2 (**Signed System Graph**)**.** The *signed system graph* $\mathcal{G}_{sys}^{\pm}$ of a system Σ_{LS} is determined by its weighted system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$. The weight of an edge in $\mathcal{G}_{sys}^{\pm}$ is +1, denoted by +, if the corresponding weight in $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ is positive. Otherwise, the weight in $\mathcal{G}_{sys}^{\pm}$ is −1, denoted by −. In $\mathcal{G}_{sys}^{\pm}$ the *sign of a cycle family* is given by the sign of the product of the weights of the edges the cycle family contains multiplied by $(-1)^d$, where d is the number of cycles of which the cycle family consists.

With these definitions, it is now possible to state a theorem for a sign non-minimum phase property.

Theorem 5.5.2. A square non-degenerated system of order n with m inputs and outputs is *sign non-minimum phase*, meaning all systems with same sign structure are non-minimum phase in the numerical sense, if for its corresponding signed feedback graph $\mathcal{G}_{fb}^{\pm}$ the following properties hold:

1. Let k_l with $m \leq k_l < n$ be the width of the feedback cycle family with the lowest number of state vertices, i. e. the *smallest feedback cycle family.*
2. For each $k \in]k_l, n]$ there exists a feedback cycle family.
3. All feedback cycle families of same width $k \in [k_l, n]$ have identical sign.
4. There is at least one $k \in [k_l, n]$ for that the sign of a feedback cycle family of width k differs from the sign of a feedback cycle family of width $\bar{k} \in [k_l, n] \setminus \{k\}$.

Proof. If a smallest feedback cycle family of width $k_l < n$ exists, it maps to the coefficient p_{k_l} in (4.8). If all feedback cycle families of width $k \in]k_l, n]$ exist, all corresponding coefficients p_k in (4.8) are present. The sign of every p_k is uniquely determined, because in the case that there are more than one feedback cycle family of identical width, these must have identical signs. If, furthermore, not all feedback cycle families in $\mathcal{G}_{fb}^{\pm}$ have the same sign, at least one coefficient will differ in its sign from another coefficient in (4.8). Thus, by Lemma 2.2.3 not all invariant zeros have strictly negative real parts. By Definition 3.2.1, a system with this property is non-minimum phase. □

Furthermore, this method can also be used to identify sign not asymptotically stable systems.

Corollary 5.5.2. A linear system of order n is *sign not asymptotically stable*, meaning all systems with same sign structure are not asymptotically stable in the numerical sense, if for its corresponding signed system graph $\mathcal{G}_{sys}^{\pm}$ the following properties hold:

1. For each $k \in [1, n]$ exists a cycle family.

2. All cycle families of same width $k \in [1, n]$ have identical sign.

3. There is at least one $k \in [1, n]$ for that the sign of a cycle family of width k differs from the sign of a cycle family of width $\bar{k} \in [1, n] \setminus \{k\}$.

This corollary is the inversion of the theorem in [Jef74] or [Har16, Satz 3.18].

5.5.3 Applicability to Nonlinear Systems

Many plants considered in control engineering are modelled by physical equations, compare to Example 4.3.1, Subsection 5.2.3 and Subsection 5.4.4. Usually the obtained differential equations can be brought in the form of the nonlinear system (2.5). The question is, may the introduced graph-theoretic methods be directly applied to nonlinear systems, i. e. without determining the equilibria and linearization? For several other system-theoretic properties, graph-theoretic criteria were established. For instance in [Wey02], observability, rank, invertibility, zeros at infinity and disturbance decoupling were considered. In this section, it will be discussed if the criterions for structurally non-minimum phase systems are directly applicable for nonlinear systems of form (2.5).

In order to investigate nonlinear systems structurally a system graph is required. The following method for its generation is proposed.

Definition 5.5.3 (Graph of a Nonlinear System). Similar to Definition 2.1.1 the (weighted) system graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ ($\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$) of the nonlinear system (2.5) can be generated by following rules considering its Jacobians $\frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}}$, $\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ and $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h}$:

1. There exists a directed edge $e_{k,l} \in \mathcal{E}$ from input vertex u_j to state vertex x_i if in $\frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}}$ the element $b_{i,j}$ in the i-th row and j-th column is nonzero. (Then the weight $w_{k,l} \in \mathcal{W}$ of this edge is given by $b_{i,j}$.)

2. There exists a directed edge $e_{k,l} \in \mathcal{E}$ from state vertex x_j to state vertex x_i if in $\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ the element $a_{i,j}$ in the i-th row and j-th column is nonzero. (Then the weight $w_{k,l} \in \mathcal{W}$ of this edge is given by $a_{i,j}$.)

3. There exists a directed edge $e_{k,l} \in \mathcal{E}$ from state vertex x_j to output vertex y_i

if in $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h}$ the element $c_{i,j}$ in the i-th row and j-th column is nonzero. (Then the weight $w_{k,l} \in \mathcal{W}$ of this edge is given by $c_{i,j}$.)

As introduced for linear systems, the (strong-)structural graph is incident with the unweighted system graph if the entries of the Jacobians comply with Definition 5.1.1 or Definition 5.3.2. Similarly, for the system graph of a nonlinear system (2.5) $\mathcal{G}^{\circledast}_{sys} = \mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ or respectively $\mathcal{G}^{*}_{sys} = \mathcal{G}_{sys}(\mathcal{V}, \mathcal{E})$ holds.

Remark 5.5.1. Since the Jacobians $\frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}}$, $\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ and $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h}$ have zero entries whenever there is no dependency of a state variable on another or an input, or an output on a state variable, the structural graph, $\mathcal{G}^{\circledast}_{sys}$ or $\mathcal{G}^{*}_{sys}$, might be constructed without determining the Jacobians. It is sufficient to consider the dependencies between the state variables, the inputs and the outputs to draw the edges in the graph.

Now reconsider the example of Subsection 5.2.3. The weighted feedback graph $\mathcal{G}_{sys}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ of the nonlinear system (5.3,5.4) can be created as described above. The result of this is given in Figure 5.11. In this graph the feedback cycle families of every admissible width are counted

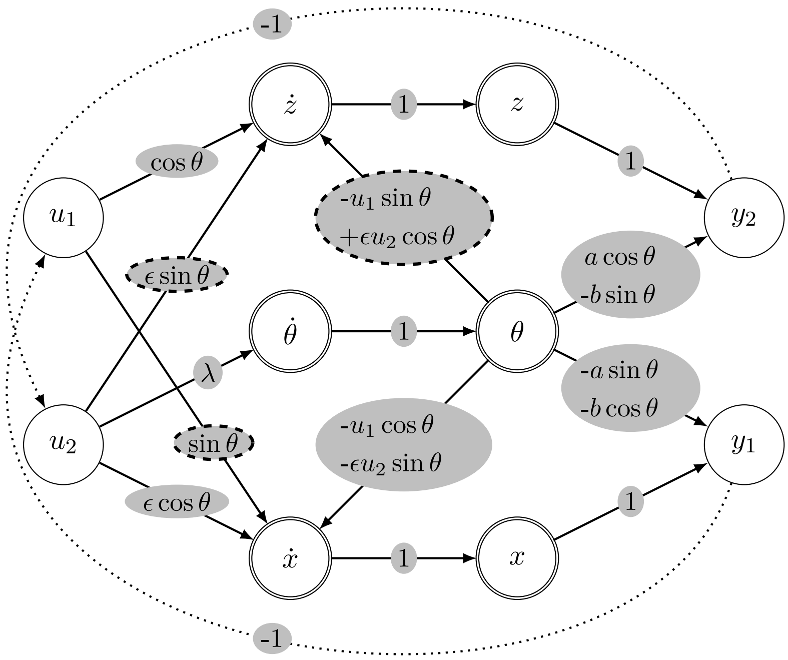

Figure 5.11: $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ of the PVTOL.

in Table 5.3. Applying Theorem 5.2.1 on this table yields that systems with a graph $\mathcal{G}_{sys}^{\circledast}$ as

	cycle families $\mathcal{C}_i$	$\lvert\mathcal{C}_i\rvert$
p_2	–	0
p_3	–	0
p_4	$(u_1 \to \dot{x} \to x \to y_1, u_2 \to \dot{z} \to z \to y_2)$, $(u_1 \to \dot{x} \to x \to y_1, u_2 \to \dot{\theta} \to \theta \to y_2)$, $(u_1 \to \dot{z} \to z \to y_2 \to u_2 \to \dot{x} \to x \to y_1)$, $(u_1 \to \dot{z} \to z \to y_2 \to u_2 \to \dot{\theta} \to \theta \to y_1)$	4
p_5	–	0
p_6	$(u_2 \to \dot{\theta} \to \theta \to \dot{x} \to x \to y_1 \to u_1 \to \dot{z} \to z \to y_2)$, $(u_1 \to \dot{x} \to x \to y_1, u_2 \to \dot{\theta} \to \theta \to \dot{z} \to z \to y_2)$	2

Table 5.3: Cycle families in $\mathcal{G}_{sys}^{\circledast}$(Figure 5.11) and their counts.

depicted in Figure 5.11, i. e. without weights and feedback edges, are structurally non-minimum phase.

Indeed, if the nonlinear system (5.3,5.4) is analyzed by the methods described in Section 2.3, it is possible to isolate the zero dynamics

$$\ddot{\theta} = \frac{\lambda}{\epsilon - \lambda b}\left(g \sin\theta + a\dot{\theta}^2\right) \,. \tag{5.17}$$

It can be shown that this nonlinear differential equation is unstable i. e. not asymptotically stable ([MDP94]). Thus, the example system is non-minimum phase in the nonlinear sense.

The result of this structural analysis is consistent with the analysis of the linearized system (5.5). However, the (structural) graph of the nonlinear system Figure 5.11 is distinct from the graph of the linearized system Figure 5.3. In the linearized graph, the edges that are marked by a dashed border around the weights in Figure 5.11 are missing. Hence, also the cycle families containing these edges are missing. This originates from the fact that some entries in the Jacobians, $\frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}}$, $\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ and $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h}$, become zero in the considered equilibrium of the system (5.3,5.4). In the graph this leads to the fact that the weights of the marked edges become zero. This can be seen easily since these weights contain sines of the variable θ, which is 0 in the considered equilibrium.

Comparing the graph of a nonlinear system with the graphs of the (at the equilibria) linearized system, it can be observed that some edges vanish in the equilibria but new edges will never appear. The reason is that the Jacobians $\frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}}$, $\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ and $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h}$ have zeros only there, where no dependency between the variables of $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{y}$ exist. In the other case, the entries may contain functions of the state variables or the inputs. Thus, in certain points of the state space or input space they will vanish. However, a zero entry will never turn into a nonzero entry.

Despite the result above, the theorems Theorem 5.2.1 or Theorem 5.4.2 can not be applied to the graph of the nonlinear system from Definition 5.5.3. The following example demonstrates the case when the application on the graph of the nonlinear system and on the graph of the linearized system in an equilibrium is contradicting.

Example 5.5.3. Consider the system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix} x_2 \\ \frac{1}{2}\sin(2x_3) \\ -6x_1 - 11x_2 - 3\sin(2x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= x_1 + \sin x_3 \ , \end{aligned} \tag{5.18}$$

and its Jacobians

$$\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \cos(2x_3) \\ -6 & -11 & -6\cos(2x_3) \end{bmatrix}, \quad \frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h} = \begin{bmatrix} 1 & 0 & \cos x_3 \end{bmatrix} . \tag{5.19}$$

This system has two equilibria for $u = 0$, $\bar{\boldsymbol{x}}_1 = [0 \ \ 0 \ \ k\pi]^T$ and $\bar{\boldsymbol{x}}_2 = [0 \ \ 0 \ \ \frac{\pi}{2} + k\pi]^T$ with $k \in \mathbb{Z}$.

The feedback graph resulting form the Jacobians is depicted in Figure 5.12. The feedback graph of the system linearized at $\bar{\boldsymbol{x}}_2$ is shown in Figure 5.13. Application of Theorem 5.4.2 to both graphs yields that a system with the first graph is strong-structurally non-minimum phase but a system with the second graph not. In the first graph the feedback cycle family of width 2 is missing whereas the feedback cycle families of width 1 and 3 are present. The second graph has only one feedback cycle family of width 3, since the edge that is connecting the vertex x_3 to the output is missing. In fact, it is strong-structurally minimum phase by Corollary 5.4.1.

In conclusion that means, that in general, the theorems Theorem 5.4.2 and Theorem 5.4.2 are not applicable to graphs of nonlinear systems generated by their Jacobians according to

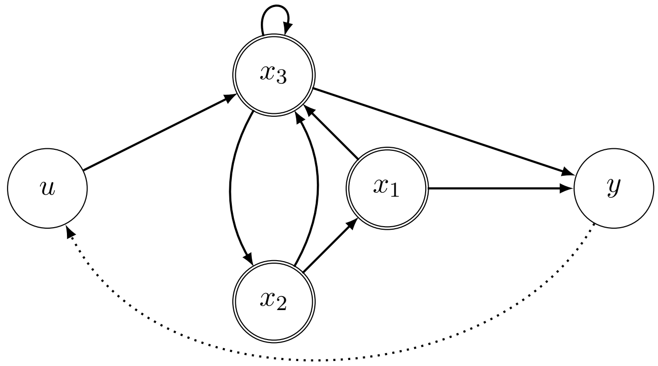

Figure 5.12: $\mathcal{G}_{fb}(\mathcal{V}, \mathcal{E})$ of (5.18) identical with $\mathcal{G}^*_{fb}$ at $\bar{\boldsymbol{x}}_1$.

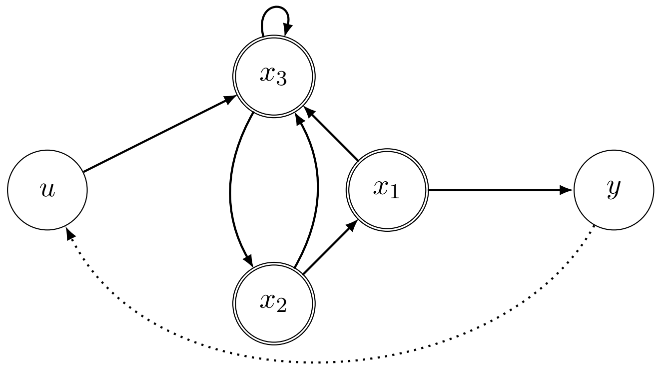

Figure 5.13: $\mathcal{G}^*_{fb}$ at $\bar{\boldsymbol{x}}_2$.

Definition 5.5.3. As shown, the reason is that the graph generated by the Jacobians differs from the graph of the linearized system. Furthermore, nonlinear systems can have an arbitrary number of equilibria and for nonlinear systems the investigated properties are only valid within a region around these equilibria. However, the graph created by Definition 5.5.3 is constant across the whole state space. Therefore, if a property is determined by one of the theorems on this graph, it should be also valid in any at the equilibrium linearized system which is very unlikely.

Interestingly, there is a possibility to determine a property by the graph given by Definition 5.5.3. As discussed before, the difference between this graph and the graph of the system linearized at the equilibrium is that the latter might have fewer edges. That leads to the fact that it has less cycle families. For the structurally non-minimum phase property Theorem 5.2.1 and Theorem 5.4.2 this is a problem, since these cycle families might be the smallest ones and hence the results of the analysis of the two graphs may be contradicting. The source of the problem is that these smallest cycle families are related to coefficients of the highest order in the zero polynomial, compare to the proof of Theorem 5.2.1. However, if the characteristic polynomial (4.8) is considered, the coefficient that is in front of the highest order is always one. In that case, the discussed differences in the graphs are not a problem to determine the structural stability properties, introduced by Theorem 5.5.1, of the nonlinear system. The following theorem can be given for structurally not asymptotically stable systems.

Theorem 5.5.3. A nonlinear system (2.5) of order n has only strong-structurally not asymptotically stable equilibria, if there is at least one k with $1 \leq k \leq n$ so that its system graph $\mathcal{G}_{sys}^*$ contains no cycle family of width k.

Proof. Since the structural system graph $\mathcal{G}_{sys}^*$ of a nonlinear system is generated by its Jacobians $\frac{\partial}{\partial \boldsymbol{u}}\dot{\boldsymbol{x}}$, $\frac{\partial}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ and $\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{h}$ it has the maximum count $\tilde{e}$ of edges across all points of its state and input space. Only at some distinct points $\boldsymbol{u}_i$ and $\boldsymbol{x}_i$ entries of the Jacobians will vanish. At these points a system graph $\mathcal{G}_{sys}^*|_{\boldsymbol{u}_i,\boldsymbol{x}_i}$ of the linearized system will have fewer edges than $\tilde{e}$. Since the cycle families in a graph are made of edges at the points $\boldsymbol{u}_i$ and $\boldsymbol{x}_i$, the structural graph $\mathcal{G}_{sys}^*|_{\boldsymbol{u}_i,\boldsymbol{x}_i}$ may have less cycle families than $\mathcal{G}_{sys}^*$. So if $\mathcal{G}_{sys}^*$ contains no cycle family of width k, also any $\mathcal{G}_{sys}^*|_{\boldsymbol{u}_i,\boldsymbol{x}_i}$ will not contain a cycle family of width k. Thus, if the numerical realization of $\mathcal{G}_{sys}^*$ is strong-structurally not asymptotically stable, also the numerical realization of $\mathcal{G}_{sys}^*|_{\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}}$ at any equilibrium $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{x}})$ will be strong-structurally not asymptotically stable according to Theorem 5.5.1. □

5.6 Complexity of the Determination of Structural Non-Minimum Phase Systems

In this section an algorithm for the determination whether a system is structural non-minimum phase is presented and its complexity is analyzed. In order to determine the introduced properties of a system, the (non-)existence of cycle families in the corresponding graph has to be checked. For the analysis of the complexity, the case of a square non-degenerated system is considered. The following algorithm checks the conditions of Theorem 5.2.1 for a given structural feedback graph.

In the first line, a flag is initialized that indicates whether a smallest cycle family is already

Input: structural feedback graph of a system $\mathcal{G}_{fb}^{\circledast}$, order of the system n and number of inputs and outputs m.

Output: *true* if the considered system is structural non-minimum phase else *false*.

```
smallestFBFamily = false;
c[] = FindAllCycles(G_fb^⊛);
for k ∈ [m, n] do
    C_k[] = ∅;
    CombineFBCycleFamilyOfWidth(∅,c[],k, C_k[]);
    if smallestFBFamily = false & |C_k[]| > 0 then
        smallestFBFamily = true;
    else if smallestFBFamily & |C_k[]| = 0 then
        return true;
    end
end
return false;
define CombineFBCycleFamilyOfWidth(C_i, c_r[], k, C_k[])
    c_p[] = ∅;
    for z ∈ c_r[] do
        C_{i+1} = C_i ⋃ z;
        c_p[] = c_p[] ⋃ z;
        if |C_{i+1}| < k then
            c_d[] = GetVertexDisjointCycles(C_{i+1}, c[]-c_p[]);
            CombineFBCycleFamilyOfWidth(C_{i+1}, c_d[], k, C_k[]);
        else if |C_{i+1}| = k then
            if NumberOfFeedbackEdges(C_{i+1}) = m then
                C_k[] = C_k[] ⋃ C_{i+1};
            end
        end
    end
end
```

Algorithm 1: Checking of the conditions of Theorem 5.2.1.

found. In the second line, a list is initialized by the function `FindAllCycles`. This function receives a structural feedback graph as argument and returns all cycles in this graph. An implementation of this function is given by [Joh75]. The for-loop from line 3 to line 11 iterates over an interval from m to n corresponding to the coefficients p_k in the invariant zeros polynomial (4.8). In every loop, first an empty list of cycle families is initialized. The function `CombineFBCycleFamilyOfWidth` then finds all feedback cycle families of the width k and saves them in the list $\mathcal{C}_k[]$. This function is defined in line 13 and will be described in the next paragraph. In order to find the smallest feedback cycle family, as required by the first condition of Theorem 5.2.1, in line 6 and 7, the flag smallestFBFamily is set to true when the list $\mathcal{C}_k[]$ is non-empty for the first time. In line 8, both conditions of Theorem 5.2.1 are checked and if they are met, the result true is returned in line 9. If the conditions are not met during the evaluation of the for-loop, the considered system is not structurally non-minimum phase and false is returned in line 12.

The function `CombineFBCycleFamilyOfWidth` defined in line 13ff. works recursively. As arguments, it takes a feedback cycle family $\mathcal{C}_i$ to which it appends cycles from a list $c_r[]$ until the width of the cycle family equals k. When this condition is met, it appends the found feedback cycle family to the list $\mathcal{C}_k[]$, which is given as last argument. In line 14, an empty list is initialized. In this list, the already considered cycles from the input cycle list $c_r[]$ will be put during execution. Then over the list $c_r[]$ is iterated. In every loop, one cycle z is taken from this list and appended to the cycle family $\mathcal{C}_i$ and to the list of used cycles $c_p[]$. In line 18, it is checked whether the width of this extended cycle family is still smaller than k. If this is true, a new list of cycles $c_d[]$ that is vertex disjoint to the extended cycle family $\mathcal{C}_{i+1}$ is generated from the list of cycles $c[]$ reduced by the list of already used $c_p[]$. This cycle family, the new list of cycles $c_d[]$ and two arguments, k and $\mathcal{C}_k[]$, are then passed to a recursive call of `CombineFBCycleFamilyOfWidth`. In line 21, the case that the extended cycle family has the desired width k is checked. Since the function should only find valid feedback cycle families, the number of feedback edges has to be counted. If this number equals m, the considered cycle family is valid and it is added to the list of feedback cycle families $\mathcal{C}_k[]$.

In order to determine the complexity of the whole algorithm the complexity of the functions `FindAllCycles` and `CombineFBCycleFamilyOfWidth` will be analyzed. The problem of finding all possible cycles in a graph can be solved by a depth-first search algorithm. The algorithm provided in [Joh75] solves this problem in $\mathcal{O}\left((|\mathcal{V}| + |\mathcal{E}|)(|c| + 1)\right)$ time, i. e. it scales with the sum of the vertices and edges multiplied by the number of cycles $|c|$ in the graph. The number of cycles of width k, $k \in [1, n]$, in a fully connected graph is given by

$$|c_k| = \binom{n}{k}(k-1)! \tag{5.20}$$

For the worst case of a fully connected graph of a system of order n with m inputs and outputs this yields $|\mathcal{V}| = n + 2m$ plus $|\mathcal{E}| = n^2 + 2mn + m^2$ and hence $\mathcal{O}\left((n+m)^2 \sum_k \binom{n}{k}(k-1)!\right)$.

The function `CombineFBCycleFamilyOfWidth` combines these cycles to cycle families. It scales proportionally with the number of feedback cycle families in the graph. Considering the fully connected graph the number of cycle families of width $k = n$ is $|\mathcal{C}_n| = n!$. There are n possibilities to get a fully connected subgraph with $n-1$ vertices. That means that the number of cycle families of width $k < n$ can be determined by counting the number of cycle families in the subgraphs. This yields the recursive formula

$$|\mathcal{C}(n)| = n! + n|\mathcal{C}(n-1)| \tag{5.21}$$

for the number of cycle families of all possible widths. In a fully connected system graph every edge has a parallel feedback path $x_i \to y_j \to u_j \to x_l$. This means that for every cycle family of width k there are $\binom{k}{m}$ feedback cycle families. Thus, (5.21) becomes to

$$|\mathcal{C}(n)| = \binom{n}{m} n! + n|\mathcal{C}(n-1)| \; . \tag{5.22}$$

Consequently, the number of feedback cycle families grows at least with the lower boundary $\mathcal{O}\left((n+1)!\right)$.

Hence, solving this problem with the above algorithm is clearly NP-hard supposing $N \neq NP$. However, in general there is at least some sparsity in the considered graphs reducing the evaluation time of the algorithm significantly.

6 Conclusion and Further Research

In this work sufficient conditions for the determination whether a structural system is non-minimum phase were developed. With these conditions, it is possible to check whether an arbitrary linear system, from which only the dependencies between its inputs, state variables and outputs are known but not the numerical values, is non-minimum phase by its structure. This means, if the considered system is structurally non-minimum phase, there is almost no combination of numerical values for which it is minimum phase in the numerical sense. An application for the developed method is early system analysis in the control design process, when only the structure of the system to control is known. If the considered system is structurally non-minimum phase, the inputs and outputs can be reselected to get a minimum phase system, or an appropriate method of control can be chosen.

In order to obtain these conditions, the following approach was taken. Commonly the term non-minimum phase is used for (stable) linear systems that possess zeros with non-negative real parts or for nonlinear systems with not asymptotically stable zero dynamics. In order to unify this term, the relation between zeros and zero dynamics of linear systems was investigated in Chapter 3. For the determination of the zero dynamics by the Byrnes-Isidori normal form the relative degree of a system has to be obtained. Since the relative degree has only been defined for square systems, the definition was extended to non-square right invertible, left invertible and degenerated systems. This was achieved using results from linear decoupling control. With the generalized definition of the relative degree, it was possible to state the Byrnes-Isidori normal form for all linear systems with relative degree. For systems without relative degree, the dynamic extension algorithm for nonlinear systems was simplified for the linear case. It was then possible to show that all zeros of a linear system are contained in its zero dynamics. The conclusion is that, at least for systems that have relative degree or that can be made to have one by the dynamic extension algorithm, the non-minimum phase property, as defined by Isidori, applies, provided that these systems have non-negative real parts in their zeros. In Chapter 4 the graph-theoretic approach for the determination of the characteristic polynomial and of the zeros polynomials by cycle families in the graph was introduced. The known results for SISO systems were generalized to arbitrary linear systems. With the developed method, it is possible to determine the invariant zeros polynomial of square MIMO systems or the systems zero polynomial of non-square or degenerated systems. Further, the introduced graph-theoretic tools were used to investigate structural systems in Chapter 5. Sufficient conditions for structural systems to be non-minimum phase for almost all numerical realizations, i. e. a structurally non-minimum phase system, were given. By investigating under which cases this property does not hold numerically, a method using unique cycle families was developed to avoid numerical cancellations in the coefficients of the considered polynomials. With the unique cycle families, it was possible to state the strong-structurally non-minimum phase property, i. e. all numerical realizations of a structural system with this property are non-minimum phase in the numerical sense. Furthermore, the introduced approach was extended in three ways. First, it was applied to stability leading to a sufficient property for numerical systems to be not asymptotically stable, i. e. strong-structurally not asymptotically stable. Second, in cases where the pure structural approach fails, i. e. it cannot be determined by structural analysis if a certain property holds, the signs of the mutual dependencies in the system were considered. This lead to the sign non-minimum phase property and to the sign not asymptotically stable property. Finally, the extension to nonlinear systems was discussed. It was not possible to find a structural non-minimum phase property for nonlinear systems but conditions for a nonlinear system to be not asymptotically stable could be given.

The different sets of non-minimum phase systems are related as depicted in Figure 6.1. Since

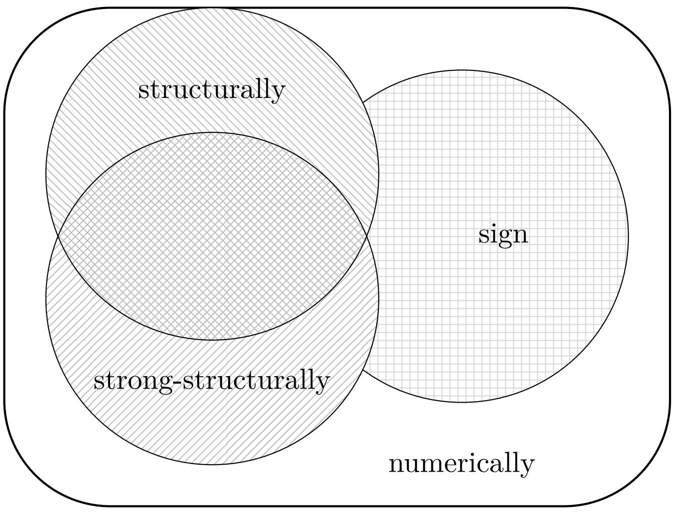

Figure 6.1: Relation between the different types of the non-minimum phase property.

the structurally non-minimum phase, the strong-structurally non-minimum phase and the sign non-minimum phase property are sufficient conditions for the numerical non-minimum phase property, their sets lie all in the set of numerical non-minimum phase systems. The set of structurally non-minimum phase systems touches the border of the numerical non-minimum phase systems because it holds only for almost all numerical realizations. It is remarkable that the set of strong-structurally non-minimum phase systems is not completely inside the set of structurally non-minimum phase systems. This results from the different definitions of the structural systems, Definition 5.1.1 and Definition 5.3.2, on which the considered property is based on. This means that there may exist some systems whose nonzero entries are not independent and thus they cannot be structurally non-minimum phase. Nevertheless, since independence of the nonzero entries is not a precondition for the strong-structural properties these systems may be strong-structurally non-minimum phase.

Investigations of several systems showed that for many of them the non-minimum phase property is determined by their structure. Examples are, steering a bicycle [ÅKL05], the vertical take-off and landing aircraft Subsection 5.2.3 [MDP94], the ECP Pendulum Subsection 5.4.4 [Edu95], the loading bridge Example 4.3.1 and the inverse pendulum with cart [SL91, p. II.2]. Two further examples, the lift truck Example 5.5.2 and all-pass filters [Bod45, Ch. 11], were sign non-minimum phase. There were also two systems, the turbo charged engine [DSS16b] and the vertical dynamics of an airplane [SL91, Ch. 6], where the non-minimum phase property could only be determined numerically, i. e. the property was related to certain value ranges of the parameters.

Further Research

The following issues are starting points for further investigations. The relative degree has relationships to the structure at infinity [PR79], the number of invariant zeros and system zeros [MK76] and the Morse indices [Mor73]. To the author's knowledge, there is no publication available that shows the links between these named properties in the general case. It would be convenient to have these terms linked in the geometric control approach [Won79] to the dimensions of appropriate subspaces, as for instance described in [AS84], to obtain a coordinate free description. Moreover, in Chapter 3 the zero dynamics for non-square linear systems were defined. In reference to [Isi11], an extension to nonlinear systems (2.5) may be considered.

Since this work is a discourse of the theoretical basis of structurally non-minimum phase systems and structurally not asymptotically stable systems, application to non-academic examples would be interesting. Examples are the stability of electrical power grids [AAN04], the polarization in social networks [Sla07] or several forms of stability in supply chains [PTC13].

Often it is possible to define the structure of a system not only by nonzero entries and zeros but also by ones. This is, for instance, the case in the relation between state variables that represent acceleration, speed and distance. This approach is introduced by [Mur09] as “mixed matrices”. The properties presented in this work could be extended to this kind of structural systems. Finally, the use of the considered properties in an algorithm, to identify automatically inputs and outputs that render a system structurally non-minimum phase, may be considered.

Bibliography

[AAN04] Réka Albert, István Albert, and Gary L. Nakarado. „Structural Vulnerability of the North American Power Grid“. In: *Phys. Rev. E* 69 (2 Jan. 7, 2004). DOI: 10.1103/PhysRevE.69.025103. arXiv: cond-mat/0401084v1 [cond-mat.dis-nn] (cit. on p. 107).

[ÅKL05] Karl J. Åström, Richard E. Klein, and Anders Lennartsson. „Bicycle Dynamics and Control“. In: *IEEE Control Systems Magazine* (Aug. 1, 2005). DOI: 10.1109/MCS.2005.1499389 (cit. on pp. 1, 106).

[AM07] Panos J. Antsaklis and Anthony N. Michel. *A Linear Systems Primer*. 2007. ISBN: 9780817644604 (cit. on p. 9).

[AS84] H. Aling and J. M. Schumacher. „A nine-fold canonical decomposition for linear systems“. In: *International Journal of Control* 39.4 (Apr. 1984). DOI: 10.1080/00207178408933206 (cit. on p. 106).

[Bel13] M.-A. Belabbas. „Sparse stable systems“. In: *Systems & Control Letters* 62.10 (Oct. 2013). DOI: 10.1016/j.sysconle.2013.07.004 (cit. on pp. 3, 71, 89).

[BI10] Thomas Berger and Achim Ilchmann. *Zero dynamics of time-varying linear systems.* Tech. rep. Institute for Mathematics, Ilmenau University of Technology, 2010 (cit. on p. 32).

[BI84] C. I. Byrnes and A. Isidori. „A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control“. In: *Proceedings of the 23rd IEEE Conference on Decision and Control.* Dec. 1984. DOI: 10.1109/CDC.1984.272345 (cit. on pp. 22, 27).

[Bod40] H. W. Bode. „Relations Between Attenuation and Phase in Feedback Amplifier Design“. In: *Bell System Technical Journal* 19.3 (July 1940). DOI: 10.1002/j.1538-7305.1940.tb00839.x (cit. on p. 1).

[Bod45] Hendrik W. Bode. *Network analysis and feedback amplifier design.* D. van Nostrand New York, 1945 (cit. on pp. 26, 106).

[CDP90] C. Commault, J.-M. Dion, and A. Perez. „Disturbance rejection for structured systems“. In: *Proceedings of the 29th IEEE Conference on Decision and Control*. IEEE. 1990. DOI: 10.1109/cdc.1990.203944 (cit. on p. 62).

[CDP91] C. Commault, J.-M. Dion, and A. Perez. „Disturbance rejection for structured systems“. In: *IEEE Transactions on Automatic Control* 36.7 (1991). DOI: 10.1109/9.85072 (cit. on p. 62).

[Che67] Wai-Kai Chen. „On directed graph solutions of linear algebraic equations“. In: *SIAM Review* 9.4 (1967). DOI: 10.1137/1009110 (cit. on p. 58).

[Che76] Wai-Kai Chen. *Applied Graph Theory: Graphs and Electrical Networks*. Ed. by H. A. Lauwerier and W. T. Koiter. 2nd ed. North-Holland Series in Applied Mathematics and Mechanics. North-Holland Publishing Company, 1976. ISBN: 9781483164151 (cit. on pp. 7, 58, 59).

[CLS12] B. M. Chen, Z. Lin, and Y. Shamash. *Linear Systems Theory: A Structural Decomposition Approach*. Control Engineering. Birkhäuser Boston, 2012. ISBN: 9781461220466 (cit. on pp. 11, 13).

[Daa16] Andreas Daasch. *Zero Dynamics of Linear Square Systems*. Tech. rep. University of the Federal Armed Forces Munich, Department of Aerospace Engineering, Chair of Control Engineering, 2016. DOI: 10.13140/RG.2.1.3665.5769/1 (cit. on pp. 3, 31, 57).

[DCW03] Jean-Michel Dion, Christian Commault, and Jacob van der Woude. „Generic properties and control of linear structured systems: a survey“. In: *Automatica* 39.7 (2003). DOI: 10.1016/S0005-1098(03)00104-3 (cit. on pp. 3, 7, 80, 82).

[Die10] Reinhard Diestel. *Graph Theory*. 4th ed. Springer-Verlag Berlin Heidelberg, 2010. DOI: 10.1007/978-3-642-14279-6 (cit. on p. ix).

[DSS16a] A. Daasch, M. Schultalbers, and F. Svaricek. „Structural non-minimum phase systems“. In: *2016 American Control Conference (ACC)*. July 2016. DOI: 10.1109/ACC.2016.7525498 (cit. on pp. 3, 72).

[DSS16b] Andreas Daasch, Erik Schulz, and Matthias Schultalbers. „Intrinsic Performance Limitations of Torque Generation in a Turbocharged Gasoline Engine“. In: *IFAC-PapersOnLine* 49.11 (2016). 8th IFAC Symposium on Advances in Automotive Control AAC 2016 Norrköping, Sweden, 20—23 June 2016. DOI: 10.1016/j.ifacol.2016.08.104 (cit. on pp. 1, 106).

[DSS16c] Andreas Daasch, Ferdinand Svaricek, and Matthias Schultalbers. „Strong Structural Non-Minimum Phase Systems“. In: *IFAC-PapersOnLine* 49.9 (2016). 6th IFAC Symposium on System Structure and Control SSSC 2016 Istanbul, Turkey, 22—24 June 2016. DOI: 10.1016/j.ifacol.2016.07.489 (cit. on pp. 3, 72).

[Dwo11] Paweł Dworak. „Dynamic decoupling of left-invertible MIMO LTI plants“. In: *Archives of Control Sciences* 21.4 (Jan. 2011). DOI: 10.2478/v10170-011-0006-2 (cit. on p. 32).

[Edu95] Educational Control Products. *Model 505 ECP Inverted Pendulum*. Fact Sheet. 1995 (cit. on pp. 86, 106).

[Est+07] Moisés Bonilla Estrada et al. „Left invertibility and duality for linear systems“. In: *Linear Algebra and its Applications* 425.2-3 (Sept. 2007). DOI: 10.1016/j.laa.2007.02.006 (cit. on pp. 10, 12).

[Fre75] E. Freund. „The structure of decoupled non-linear systems“. In: *International Journal of Control* 21.3 (Mar. 1975). DOI: 10.1080/00207177508922001 (cit. on p. 32).

[FW67] P. Falb and W. Wolovich. „Decoupling in the design and synthesis of multivariable control systems“. In: *IEEE Transactions on Automatic Control* 12.6 (Dec. 1967). DOI: 10.1109/TAC.1967.1098737 (cit. on p. 32).

[Gan59] F. R. Gantmacher. *The Theory of Matrices*. Vol. 1. Chealsea Publishing Company, 1959. ISBN: 0828401314 (cit. on p. 18).

[Gil69] Elmer G. Gilbert. „The Decoupling of Multivariable Systems by State Feedback“. In: *SIAM Journal on Control* 7.1 (Feb. 1969). DOI: 10.1137/0307004 (cit. on pp. 32, 39).

[Gir29] A. Girard. *Invention nouvelle En L'Algebre*. Blauew, 1629 (cit. on p. 18).

[GS10] Håvard Fjær Grip and Ali Saberi. „Structural decomposition of linear multivariable systems using symbolic computations“. In: *International Journal of Control* 83.7 (July 2010). DOI: 10.1080/00207171003758760 (cit. on p. 31).

[GS76] K. Glover and L. Silverman. „Characterization of structural controllability“. In: *IEEE Transactions on Automatic Control* 21.4 (Aug. 1976). DOI: 10.1109/TAC.1976.1101257 (cit. on p. 2).

[Hal74] Paul Richard Halmos. *Measure theory*. Graduate texts in mathematics 18. New York, NY [u.a.]: Springer, 1974. ISBN: 3540900888 (cit. on p. 73).

[Har16] Christoph Hartung. *Zur algebraischen Untersuchung der Steuerbarkeit und der Stabilisierbarkeit linearer zeitinvarianter Systeme*. Fortschritt-Berichte VDI. Düsseldorf, 2016. ISBN: 9783185247088 (cit. on pp. 90, 95).

[HCD99] V. Hovelaque, C. Commault, and J. M. Dion. „Zeros of structured linear systems“. In: *Automatica* 35.10 (1999). DOI: 10.1016/S0005-1098(99)00085-0 (cit. on pp. 71, 75, 82).

[HH83] M. Hautus and M. Heymann. „Linear feedback decoupling–Transfer function analysis“. In: *IEEE Transactions on Automatic Control* 28.8 (Aug. 1983). DOI: 10.1109/TAC.1983.1103320 (cit. on p. 32).

[HL12] Xiaoming Hu and Anders Lindquist. „Geometric Control Theory“. Lecture notes. Optimization and Systems Theory Royal institute of technology SE-100 44 Stockholm, Sweden. 2012 (cit. on pp. 31, 51).

[HRS13] C. Hartung, G. Reißig, and F. Svaricek. „Necessary conditions for structural and strong structural controllability of linear time-varying systems“. In: *2013 European Control Conference (ECC)*. July 2013 (cit. on p. 80).

[HS14] Christoph Hartung and Ferdinand Svaricek. „Sign Stabilizability“. In: *22nd Mediterranean Conference on Control and Automation*. Institute of Electrical and Electronics Engineers (IEEE), June 2014. DOI: 10.1109/MED.2014.6961362 (cit. on p. 90).

[HSM92] John Hauser, Shankar Sastry, and George Meyer. „Nonlinear Control Design for Slightly Non-minimum Phase Systems: Application to V/STOL Aircraft“. In: *Automatica* 28.4 (July 1992). DOI: 10.1016/0005-1098(92)90029-F (cit. on p. 77).

[Hur95] A. Hurwitz. „Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt“. In: *Mathematische Annalen* 46.2 (1895). DOI: 10.1007/BF01446812 (cit. on p. 18).

[IK15] Mitsuaki Ishitobi and Sadaaki Kunimatsu. „Normal forms of linear multivariable square systems and their application“. In: *2015 Proceedings of the Conference on Control and its Applications*. 2015. DOI: 10.1137/1.9781611974072.3 (cit. on p. 32).

[Isi11] Alberto Isidori. *The zero dynamics of a nonlinear system: from the origin to the latest progresses of a long successful story*. Presentation. 30th Chinese Control Conference, July 23, 2011 (cit. on pp. 31, 106).

[Isi13] Alberto Isidori. „The zero dynamics of a nonlinear system: From the origin to the latest progresses of a long successful story“. In: *European Journal of Control* 19.5 (Sept. 2013). DOI: 10.1016/j.ejcon.2013.05.014 (cit. on p. 22).

[Isi85] Alberto Isidori. *Nonlinear Control Systems: An Introduction*. Ed. by Alberto Isidori. Springer-Verlag, 1985. DOI: 10.1007/bfb0006368 (cit. on p. 25).

[Isi95] Alberto Isidori. *Nonlinear Control Systems*. 3rd ed. Springer London, 1995. ISBN: 978-1-4471-3909-6. DOI: 10.1007/978-1-84628-615-5 (cit. on pp. 1, 23, 25, 27, 29, 31, 39, 40, 47).

[ITY71] Masao Iri, Junkichi Tsunekawa, and Keiji Yajima. „The Graphical Techniques Used for a Chemical Process Simulator "JUSE GIFS"“. In: *Information Processing 71 (Proc. IFIP Congr. 71)* 2 (1971) (cit. on p. 2).

[IW13] Achim Ilchmann and Fabian Wirth. „On Minimum Phase“. In: *at – Automatisierungstechnik* 61.12 (Jan. 2013). DOI: 10.1515/auto.2013.1002 (cit. on p. 31).

[Jag95] B. de Jager. „The use of symbolic computation in nonlinear control: is it viable?“ In: *IEEE Transactions on Automatic Control* 40.1 (1995). DOI: 10.1109/9.362897 (cit. on p. 26).

[Jef74] Clark Jeffries. „Qualitative Stability and Digraphs in Model Ecosystems“. In: *Ecology* 55.6 (1974). DOI: 10.2307/1935470 (cit. on pp. 90, 95).

[Joh75] Donald B. Johnson. „Finding All The Elementary Circuits Of A Directed Graph“. In: *SIAM Journal on Computing* 4.1 (Mar. 1, 1975). DOI: 10.1137/0204007 (cit. on p. 102).

[JSA11] J. C. Jarczyk, F. Svaricek, and B. Alt. „Strong structural controllability of linear systems revisited“. In: *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*. Dec. 2011. DOI: 10.1109/CDC.2011.6160392 (cit. on pp. 80, 82).

[Kai80] Thomas Kailath. *Linear systems*. Vol. 1. Prentice-Hall Englewood Cliffs, NJ, 1980. ISBN: 978-0135369616 (cit. on pp. 13, 19).

[Kal62] Rudolf E. Kalman. „Canonical structure of linear dynamical systems“. In: *Proceedings of the National academy of Sciences of the United States of America* 48.4 (1962). DOI: 10.1073/pnas.48.4.596 (cit. on p. 54).

[Kha02] Hassan K. Khalil. *Nonlinear Systems*. 3rd ed. Prentice Hall, 2002. ISBN: 0-13-067389-7 (cit. on pp. 17, 18, 25).

[Kho16] Saman Khodaverdian. „Vollständige und teilweise Ein-/Ausgangsentkopplung: Ein Transformationsansatz“. In: *at - Automatisierungstechnik* 64.12 (Jan. 2016). DOI: 10.1515/auto-2016-0032 (cit. on p. 32).

[KKK95] Miroslav Krstic, Petar V. Kokotovic, and Ioannis Kanellakopoulos. *Nonlinear and Adaptive Control Design*. 1st ed. New York, NY, USA: John Wiley & Sons, Inc., 1995. ISBN: 0471127329 (cit. on p. 1).

[KM76] B. Kouvaritakis and A. G. J. Macfarlane. „Geometric approach to analysis and synthesis of system zeros Part 2. Non-square systems“. In: *International Journal of Control* 23.2 (1976). DOI: 10.1080/00207177608922150 (cit. on p. 57).

[Kön16] D. König. „Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre“. ger. In: *Mathematische Annalen* 77 (1916) (cit. on p. 58).

[Lin74] Ching-Tai Lin. „Structural controllability“. In: *IEEE Transactions on Automatic Control* 19.3 (June 1974). DOI: 10.1109/tac.1974.1100557 (cit. on p. 2).

[LSB11] Yang-Yu Liu, Jean-Jacques Slotine, and Albert-Laszlo Barabasi. „Controllability of complex networks“. In: *Nature* 473.7346 (May 2011). DOI: 10.1038/nature10011 (cit. on pp. 3, 7).

[LSB13] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabasi. „Observability of complex systems“. In: *Proceedings of the National Academy of Sciences* 110.7 (Jan. 2013). DOI: 10.1073/pnas.1215508110 (cit. on p. 3).

[LT85] Peter Lancaster and Miron Tismenetsky. *The theory of matrices*. 2nd ed. Orlando: Academic Press, 1985. ISBN: 0124355609 (cit. on p. 18).

[Lun13] Jan Lunze. *Regelungstechnik 2. Mehrgrößensysteme, Digitale Regelung*. 7th ed. Springer Berlin Heidelberg, 2013. ISBN: 978-3-642-29561-4. DOI: 10.1007/978-3-642-29562-1 (cit. on p. 13).

[Lya92] Aleksandr Mikhailovich Lyapunov. „The general problem of the stability of motion“. In: *International Journal of Control* 55.3 (1992). Translation. DOI: 10.1080/00207179208934253 (cit. on p. 17).

[MDP94] P. Martin, S. Devasia, and B. Paden. „A different look at output tracking: control of a VTOL aircraft“. In: *Proceedings of the 33rd IEEE Conference on Decision and Control*. Vol. 3. Dec. 1994. DOI: 10.1109/CDC.1994.411460 (cit. on pp. 1, 77, 97, 106).

[Mir55] L. Mirsky. *An Introduction To Linear Algebra*. London: Oxford University Press, 1955. ISBN: 978-0486664347 (cit. on pp. 12, 59).

[Mit77] T. Mita. „On maximal unobservable subspace, zeros and their applications“. In: *International Journal of Control* 25.6 (June 1977). DOI: 10.1080/00207177708922276 (cit. on p. 33).

[MK76] A. G. J. MacFarlane and N. Karcanias. „Poles and zeros of linear multivariable systems : a survey of the algebraic, geometric and complex-variable theory“. In: *International Journal of Control* 24.1 (1976). DOI: 10.1080/00207177608932805 (cit. on pp. 19, 51, 53, 63, 106).

[Mor73] A. S. Morse. „Structural Invariants of Linear Multivariable Systems“. In: *SIAM Journal on Control* 11.3 (Aug. 1973). DOI: 10.1137/0311037 (cit. on p. 106).

[Moy77] P. Moylan. „Stable inversion of linear systems“. In: *IEEE Transactions on Automatic Control* 22.1 (Feb. 1977). DOI: 10.1109/TAC.1977.1101430 (cit. on p. 10).

[Mül09] Markus Müller. „Normal form for linear systems with respect to its vector relative degree“. In: *Linear Algebra and its Applications* 430.4 (2009). DOI: 10.1016/j.laa.2008.10.014 (cit. on p. 32).

[Mur09] K. Murota. *Matrices and Matroids for Systems Analysis*. Algorithms and Combinatorics. Springer Berlin Heidelberg, 2009. ISBN: 9783642039942 (cit. on pp. 3, 12, 14, 61, 81, 82, 107).

[MY79] Hirokazu Mayeda and Takashi Yamada. „Strong Structural Controllability“. In: *SIAM Journal on Control and Optimization* 17.1 (1979). DOI: 10.1137/0317010 (cit. on p. 80).

[PBB12] F. Pasqualetti, A. Bicchi, and F. Bullo. „Consensus Computation in Unreliable Networks: A System Theoretic Approach“. In: *IEEE Transactions on Automatic Control* 57.1 (Jan. 2012). DOI: 10.1109/tac.2011.2158130 (cit. on p. 3).

[Pol03] Jan Polzer. „Nutzung unterschiedlicher mathematischer Theorien zur Analyse und Regelung nichtlinearer Systeme“. VDI Fortschritt-Berichte. Reihe 12 Nr. 551, Düsseldorf: VDI-Verlag 2003. PhD thesis. Gerhard-Mercator-Universität, GH Duisburg Meß- Steuer- und Regelungstechnik, 2003 (cit. on p. 31).

[Pon66] J. Ponstein. „Self-avoiding paths and the adjacency matrix of a graph“. In: *SIAM Journal on Applied Mathematics* 14.3 (1966). DOI: 10.1137/0114051 (cit. on p. 58).

[Por70] W. A. Porter. „Diagonalization and inverses for non-linear systems“. In: *International Journal of Control* 11.1 (Jan. 1970). DOI: 10.1080/00207177008905882 (cit. on p. 32).

[PR79] A. C. Pugh and P. A. Ratcliffe. „On the zeros and poles of a rational matrix“. In: *International Journal of Control* 30.2 (Aug. 1979). DOI: 10.1080/00207177908922769 (cit. on p. 106).

[PTC13] Omkar D. Palsule-Desai, Devanath Tirupati, and Pankaj Chandra. „Stability issues in supply chain networks: Implications for coordination mechanisms“. In: *International Journal of Production Economics* 142.1 (Mar. 2013). DOI: 10.1016/j.ijpe.2012.11.003 (cit. on p. 107).

[QR65] James Quirk and Richard Ruppert. „Qualitative Economics and the Stability of Equilibrium“. In: *The Review of Economic Studies* 32.4 (1965) (cit. on p. 90).

[Qui80] James Quirk. „Qualitative stability of matrices and economic theory: A survey article“. In: *Staff Papers in Economics - Montana State University, College of Agriculture, College of Letters and Science, Agricultural Economics and Economics Department (USA)* (1980) (cit. on p. 90).

[Rei82] K. Reinschke. „Strukturelle Pol-Nullstellen-Analyse von Systemen in Zustandsraumdarstellung“. In: *Messen-Steuern-Regeln* 25 (1982) (cit. on p. 71).

[Rei88] Kurt J. Reinschke. *Multivariable Control. A Graph-Theoretic Approach.* Vol. 108. Springer Berlin Heidelberg, 1988. ISBN: 978-3-540-18899-5. DOI: `10.1007/BFb0051585` (cit. on pp. ix, 3, 7, 12, 13, 59, 61, 63, 72, 73, 80, 81, 84).

[RHS14] G. Reißig, C. Hartung, and F. Svaricek. „Strong Structural Controllability and Observability of Linear Time-Varying Systems“. In: *IEEE Transactions on Automatic Control* 59.11 (2014). DOI: `10.1109/tac.2014.2320297` (cit. on p. 80).

[Röb17] Klaus Röbenack. „On the Direct Computation of the Byrnes-Isidori Normal Form“. In: *PAMM* 17.1 (Dec. 2017). DOI: `10.1002/pamm.201710373` (cit. on p. 26).

[Ros70] H. H. Rosenbrock. *State-Space and Multivariable Theory.* Nelson-Wiley, 1970 (cit. on p. 36).

[Ros74] H. H. Rosenbrock. „Correspondence : Correction to 'The zeros of a system'“. In: *International Journal of Control* 20.3 (1974). DOI: `10.1080/00207177408932759` (cit. on p. 21).

[Rou77] E. J. Routh. *A Treatise on the Stability of a Given State of Motion, Particularly Steady Motion.* Macmillan, 1877 (cit. on p. 18).

[Sam47] P. A. Samuelson. *Foundations of Economic Analysis.* Foundations of economic analysis. Harvard University Press, 1947 (cit. on p. 90).

[Sas99] Shankar Sastry. *Nonlinear Systems. Analysis, Stability and Control.* Springer New York, 1999. ISBN: 978-1-4419-3132-0. DOI: `10.1007/978-1-4757-3108-8` (cit. on p. 47).

[Sch91] Helmut Schwarz. *Nichtlineare Regelungssysteme. Systemtheoretische Grundlagen.* München: R. Oldenbourg Verlag, 1991. ISBN: 3-486-21833-6 (cit. on p. 47).

[SJA11] Ferdinand Svaricek, Jan Cristian Jarczyk, and Benedikt Alt. „Strukturelle und streng strukuturelle Steuerbarkeit: Übersicht und neue Ergebnisse“. In: *at - Automatisierungstechnik* 59.6 (Jan. 2011). DOI: `10.1524/auto.2011.0928` (cit. on p. 80).

[SL91] J. J. E. Slotine and W. Li. *Applied Nonlinear Control.* Prentice Hall, 1991. ISBN: 0130408905 (cit. on pp. 1, 106).

[Sla07] Michael D. Slater. „Reinforcing Spirals: The Mutual Influence of Media Selectivity and Media Effects and Their Impact on Individual Behavior and Social Identity". In: *Communication Theory* 17.3 (Aug. 2007). DOI: `10.1111/j.1468-2885.2007.00296.x` (cit. on p. 107).

[Sma06] Ye. M. Smagina. *System zeros*. May 3, 2006. arXiv: `math/0605092v1 [math.DS]` (cit. on p. 22).

[Söt79] Werner Söte. „Eine strukturorientierte Untersuchung zur Approximation von linearen zeitinvarianten Systemen". PhD thesis. Universität Hannover, 1979 (cit. on p. 71).

[SP05] Sigurd Skogestad and Ian Postlethwaite. *Multivariable Feedback Control: Analysis and Design*. 2nd ed. Wiley, 2005. ISBN: 978-0-470-01168-3 (cit. on pp. 6, 26, 33).

[SP71] L. M. Silverman and H. J. Payne. „Input–Output Structure of Linear Systems with Application to the Decoupling Problem". In: *SIAM Journal on Control* 9.2 (May 1971). DOI: `10.1137/0309017` (cit. on p. 32).

[SS87] P. Sannuti and A. Saberi. „Special coordinate basis for multivariable linear systems - finite and infinite zero structure, squaring down and decoupling". In: *International Journal of Control* 45.5 (1987). DOI: `10.1080/00207178708933840` (cit. on pp. 16, 32, 34, 36, 57).

[Sva13] Ferdinand Svaricek. *On the Computation of State-Space Zeros of Linear Systems with Matlab*. Tech. rep. Universität der Bundeswehr München, Institut für Steuer- und Regelungstechnik, Oct. 1, 2013. DOI: `10.13140/2.1.4721.9842` (cit. on p. 19).

[Sva86] F. Svaricek. „Graphentheoretische Ermittlung der Anzahl von strukturellen und streng strukturellen Invarianten Nullstellen / Graph-theoretical determination of the number of structural and strictly structural invariant zeros". In: *at - Automatisierungstechnik* 34.1-12 (Jan. 1986). DOI: `10.1524/auto.1986.34.112.488` (cit. on p. 71).

[TD13] M. Trefois and J.-C. Delvenne. „Strong controllability of networked systems and minimum rank of a graph: what can we compute ?" In: *Proceedings of the European Conference on Complex Systems (ECCS)*. Sept. 1, 2013 (cit. on p. 80).

[TD14] M. Trefois and J.-C. Delvenne. „Strong structural controllability of trees". In: *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS)*. Feb. 7, 2014 (cit. on p. 80).

[TD15] Maguy Trefois and Jean-Charles Delvenne. „Zero forcing number, constrained matchings and strong structural controllability". In: *Linear Algebra and its Applications* 484 (Nov. 2015). DOI: `10.1016/j.laa.2015.06.025` (cit. on p. 80).

[TS11] K. Thulasiraman and M. N. S. Swamy. *Graphs: Theory and Algorithms.* Wiley, 2011. DOI: 10.1002/9781118033104 (cit. on p. x).

[TSH01] Harry Trentelman, Anton A. Stoorvogel, and Malo Hautus. *Control Theory for Linear Systems.* 1st ed. Springer-Verlag London, 2001. DOI: 10.1007/978-1-4471-0339-4 (cit. on pp. 10, 13, 16).

[Var80] Antonis I. G. Vardulakis. „Zero placement and the 'squaring down' problem: a polynomial matrix approach". In: *International Journal of Control* 31.5 (May 1980). DOI: 10.1080/00207178008961085 (cit. on p. 57).

[WA86] T. W. C. Williams and P. J. Antsaklis. „A unifying approach to the decoupling of linear multivariate systems". In: *International Journal of Control* 44.1 (July 1986). DOI: 10.1080/00207178608933589 (cit. on pp. 32, 40, 42).

[WCD03] W. J. Woude, C. Commault, and M. J. Dion. „Zero Orders and Dimensions of Some Invariant Subspaces in Linear Structured Systems". In: *Mathematics of Control, Signals and Systems* 16.2 (2003). DOI: 10.1007/s00498-003-0134-8 (cit. on p. 71).

[Wey02] Torsten Wey. *Nichtlineare Regelungssysteme. Ein differentialalgebraischer Ansatz.* Vieweg+Teubner Verlag, 2002. ISBN: 978-3-519-00395-3. DOI: 10.1007/978-3-663-12258-6 (cit. on pp. 3, 7, 62, 95).

[Won79] W. Murray Wonham. *Linear Multivariable Control: a Geometric Approach.* 2nd ed. 1979. ISBN: 978-1-4684-0070-0. DOI: 10.1007/978-1-4684-0068-7 (cit. on pp. 82, 106).

[Wou91a] J. W. van der Woude. „A graph-theoretic characterization for the rank of the transfer matrix of a structured system". In: *Mathematics of Control, Signals and Systems* 4.1 (Mar. 1991). DOI: 10.1007/bf02551379 (cit. on p. 84).

[Wou91b] J. W. van der Woude. „On the structure at infinity of a structured system". In: *Linear Algebra and its Applications* 148 (1991). DOI: 10.1016/0024-3795(91)90091-A (cit. on pp. 12, 81, 82, 84).

[Wou99] Jacob van der Woude. „The generic number of invariant zeros of a structured linear system". In: *SIAM Journal on Control and Optimization* 38.1 (1999). DOI: 10.1137/s0363012996310119 (cit. on p. 71).

[ZDG96] Kemin Zhou, John C. Doyle, and Keith Glover. *Robust and Optimal Control.* Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996. ISBN: 0-13-456567-3 (cit. on p. 9).

[Zei14] Michael Zeitz. „Minimalphasigkeit - keine relevante Eigenschaft für die Regelungstechnik!" In: *at - Automatisierungstechnik* 62.1 (2014). DOI: 10.1515/auto-2014-1067 (cit. on p. 27).

List of Acronyms

Acronym	Meaning	Definition
BIN	Byrnes-Isidori normal form	Section 2.3, p. 22
IDZ	input decoupling zeros	Subsection 2.2.4, p. 19
IODZ	input output decoupling zeros	Subsection 2.2.4, p. 19
IZ	invariant zeros	Subsection 2.2.4, p. 19
ODZ	output decoupling zeros	Subsection 2.2.4, p. 19
SZ	system zeros	Subsection 2.2.4, p. 19
TZ	transmission zeros	Subsection 2.2.4, p. 19

List of Symbols

Greek Letters

Symbol	Description	Definition
δ	relative degree	Definition 2.3.2, p. 25 Definition 3.1.1, p. 37
$\boldsymbol{\delta}$	vector relative degree	Definition 2.3.2, p. 25 Definition 3.1.1, p. 37
$\boldsymbol{\eta}$	internal state vector	Equation 2.49, p. 23 Equation 3.40, p. 45
$\boldsymbol{\mu}$	parameter vector	Notation, p. viii
Σ_{LS}	linear system without feed-through	Equation 2.2, p. 6
$\Sigma_{\circledast}$	structural system	Definition 5.1.1, p. 72
Σ_{*}	structural system	Definition 5.3.2, p. 83
$\Sigma_{\pm}$	signed system	Definition 5.5.1, p. 93
$\boldsymbol{\Phi}$	transformation to BIN	Equation 2.50, p. 24 Equation 3.45, p. 46
$\chi(\lambda)$	characteristic polynomial	Equation 2.33, p. 18

Latin Letters

Symbol	Description	Definition
$\boldsymbol{A}$	state matrix	Equation 2.1, p. 5
$\bar{\boldsymbol{A}}$	state matrix of the dual system	Equation 2.14, p. 10
$\boldsymbol{A}_{\circledast}$	structural state matrix	Definition 5.1.1, p. 72

Symbol	**Description**	**Definition**
$\boldsymbol{A}_{*}$	structural state matrix	Definition 5.3.2, p. 83
$\boldsymbol{A}_{\pm}$	signed state matrix	Definition 5.5.1, p. 93
$\mathcal{A}$	adjacency matrix	Equation 0.4, p. x
$\boldsymbol{B}$	input matrix	Equation 2.1, p. 5
$\bar{\boldsymbol{B}}$	input matrix of the dual system	Equation 2.14, p. 10
$\boldsymbol{B}_{\circledast}$	structural input matrix	Definition 5.1.1, p. 72
$\boldsymbol{B}_{*}$	structural input matrix	Definition 5.3.2, p. 83
$\boldsymbol{B}_{\pm}$	signed input matrix	Definition 5.5.1, p. 93
$\boldsymbol{C}$	output matrix	Equation 2.1, p. 5
$\bar{\boldsymbol{C}}$	output matrix of the dual system	Equation 2.14, p. 10
$\boldsymbol{C}_{\circledast}$	structural output matrix	Definition 5.1.1, p. 72
$\boldsymbol{C}_{*}$	structural output matrix	Definition 5.3.2, p. 83
$\boldsymbol{C}_{\pm}$	signed output matrix	Definition 5.5.1, p. 93
$\mathcal{C}$	cycle family	Notation, p. ix
$\mathbb{C}$	field of complex numbers	Notation, p. viii
diag	diagonal matrix	Notation, p. viii
det	determinant	Equation 2.19, p. 12
$\boldsymbol{D}$	decoupling matrix	Equation 2.54, p. 24 Equation 3.14, p. 37
$\tilde{\boldsymbol{D}}$	reduced decoupling matrix	Definition 3.1.1, p. 37
$\mathcal{E}$	set of edges	Notation, p. ix
$\mathbb{F}$	arbitrary field	Notation, p. viii
g-rank	generic rank	Definition 2.2.9, p. 13
$\boldsymbol{G}(s)$	transfer function	Equation 2.3, p. 6
$\bar{\boldsymbol{G}}(s)$	dual transfer function	Subsection 2.2.2, p. 10
$\mathcal{G}_{fb}$	feedback graph	Definition 4.3.1, p. 63
$\mathcal{G}_{sys}$	system graph	Definition 2.1.1, p. 7

Symbol	**Description**	**Definition**
$\mathcal{G}_{sys}^{\circledast}$	structural system graph	Equation 5.1, p. 73
$\mathcal{G}_{sys}^{*}$	structural system graph	Equation 5.8, p. 83
$\mathcal{G}_{sys}^{\pm}$	signed system graph	Definition 5.5.2, p. 93
$\mathcal{G}(\mathcal{V}, \mathcal{E})$	unweighted graph	Notation, p. ix
$\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$	weighted graph	Notation, p. ix
$\boldsymbol{I}$	identity matrix	Notation, p. viii
m	number of inputs	Equation 2.1, p. 5
n	number of state variables	Equation 2.1, p. 5
norm-rank	normal rank	Definition 2.2.10, p. 13
$\mathbb{N}$	set of natural numbers	Notation, p. viii
p	number of outputs	Equation 2.1, p. 5
$p_{IZ}(s)$	invariant zeros polynomial	Equation 4.8, p. 63
$p_{\omega_i}(s)$		Equation 4.20, p. 68
$p_{SZ}(s)$	system zeros polynomial	Theorem 4.4.1, p. 68
$\boldsymbol{P}(s)$	system matrix	Equation 2.3, p. 6
$\bar{\boldsymbol{P}}(s)$	dual system matrix	Subsection 2.2.2, p. 10
rank	(numerical) rank	Definition 2.2.8, p. 12
$\mathbb{R}$	field of real numbers	Notation, p. viii
s	complex frequency	Equation 2.3, p. 6
s-rank	structural system rank	Definition 4.2.2, p. 61
t	time	Notation, p. viii
term-rank	term rank	Definition 2.2.11, p. 14
$\boldsymbol{u}, \boldsymbol{u}(t)$	input vector	Equation 2.1, p. 5
$\boldsymbol{u}_z$	output zeroing feedback	Equation 3.50, p. 47
$\boldsymbol{U}(s)$	input vector (frequency domain)	Equation 2.3, p. 6
$\mathcal{U}$	set of input vertices	Definition 2.1.1, p. 7
U	input space	Equation 2.1, p. 5

Symbol	**Description**	**Definition**
$\mathcal{V}$	set of vertices	Notation, p. ix
$\mathcal{W}$	set of weights	Notation, p. ix
$\boldsymbol{x}, \boldsymbol{x}(t)$	state vector	Equation 2.1, p. 5
$\boldsymbol{X}(s)$	state vector (frequency domain)	Equation 2.3, p. 6
$\mathcal{X}$	set of state vertices	Definition 2.1.1, p. 7
X	state space	Equation 2.1, p. 5
$\boldsymbol{y}, \boldsymbol{y}(t)$	output vector	Equation 2.1, p. 5
$\boldsymbol{Y}(s)$	output vector (frequency domain)	Equation 2.3, p. 6
$\mathcal{Y}$	set of output vertices	Definition 2.1.1, p. 7
Y	output space	Equation 2.1, p. 5
$\boldsymbol{z}$	external state vector	Equation 2.48, p. 23 Equation 3.39, p. 45
$\mathbb{Z}$	set of integers	Notation, p. viii