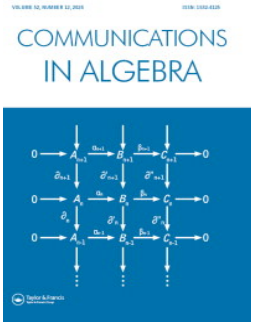

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Quaternionic *p*-adic continued fractions

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ABSTRACT

We develop a theory of p-adic continued fractions for a quaternion algebra B over $\mathbb{Q}$ ramified at a rational prime p. Many properties holding in the commutative case can be proven also in this setting. In particular, we focus our attention on the characterization of elements having a finite continued fraction expansion. By means of a suitable notion of quaternionic height, we prove a sufficient condition to estabilish the finiteness of the continued fraction. Furthermore, we draw some consequences about the solutions of a family of quadratic polynomial equations with coefficients in B.

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1. Introduction

The classical continued fraction algorithm over the field of real numbers provides an integer sequence that represents the given number by means of the following algorithm:

$$\begin{cases} \alpha_0 = \alpha, \\ a_n = \lfloor \alpha_n \rfloor, \\ \alpha_{n+1} = \frac{1}{\alpha_n - a_n} & \text{if } \alpha_n - a_n \neq 0, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the integral part of a real number. The a_n's and α_n's are called *partial* and *complete* quotients respectively. It is easy to see via the Euclidean algorithm that the procedure stops if and only if we start from a rational number, and it is a classical theorem of Lagrange that the continued fraction expansion is periodic if and only if α is a quadratic irrational. Moreover, in case of irrational numbers, the continued fraction expansion provides the best rational approximations of the number; this is one of the reasons why the study of continued fractions is very important in diophantine approximation and transcendence theory. For the same reason, one would like to find more general notions of continued fraction expansion, for example when we start from a division algebra B.

In this context, given a list $[a_0, \dots, a_n]$ of nonzero elements in B, the simple continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}, \qquad \text{with } a_0, \dots, a_n \in B,$$

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represents an element $\alpha \in B$. When B is also provided with a topology, under some hypotheses of completeness and convergence, one can also define *infinite continued fractions* as

$$[a_0, a_1, \ldots, a_n \ldots] = \lim_{n\to\infty} [a_0, \ldots, a_n].$$

In this very general setting, several questions arise, concerning first the representability of elements in the topological completion of B, the uniqueness of the representation, the existence of an algorithm to compute the continued fraction expansion of any representable element, the convergence and quality of the approximation. Moreover, one would be interested in characterizing those elements in B which are represented by finite continued fractions and those elements in B which have a periodic continued fraction.

As already mentioned before, the questions above have been extensively studied in the "classical" case, i.e. when B is the real field endowed with the Euclidean topology. The first contributions date back to the works of Wallis, Euler and Lagrange [3].

The theory of continued fractions has also been extended to p-adic fields; a complete and updated review on this topics is provided by [16]. In this context, however, there is no natural definition of a p-adic continued fraction, since there is no canonical definition for a p-adic floor function. A general formulation of floor function for a p-adic field, where partial quotients satisfy some integrality properties, is given in [5, Definition 3.1], inspired by the models proposed by [17] and [4]. In the same article, the authors give a criterion for a p-adic floor function to provide finite continued fraction expansions for every element of a given number field [5, Theorem 4.5]. This criterion applies, for instance, to norm-Euclidean number fields, for sufficiently large p [5, Theorem 5.6].

In this paper, we address the case where B is a quaternion algebra over $\mathbb{Q}$ ramified at a rational prime p, and the topology is the p-adic one.

The idea of studying continued fractions over algebras, and in particular quaternion algebras, goes back to Hamilton [8, 9], who investigated the main properties of continued fractions over the skew field of real quaternions and their relations with quadratic equations. Much work has been done on (real) matrix-valued continued fractions, especially concerning their convergence and approximation rate [1, 6, 12, 15, 18, 22]. In [13] the authors prove the convergence of a wide class of continued fractions, including generalized continued fractions over real quaternions and octonions.

However, a standard terminology for the p-adic case – and, more generally, an arithmetic approach to the study of quaternionic continued fractions – is still missing. In analogy with the case of number fields [5], we introduce the notion of *quaternionic type* associated to an order R in B. Namely, this is a quadruple $\tau = (B, R, p, s)$ where B is a quaternion algebra, R is an order in B, p a prime ≥ 3 and s a *p-adic floor function* taking values in $R[\frac{1}{p}]$. Each quaternionic type gives rise to an algorithm that computes the continued fraction expansion of every element in the p-adic completion of B. We show that the convergents of such continued fractions enjoy many of the properties holding also in the commutative case; in particular, their sequence converges with respect to the p-adic topology – which is the first property one would expect from a meaningful definition of continued fraction.

Given a quaternionic type τ, we approach the problem of characterizing all the elements of B whose continued fraction expansion is finite. Adopting a similar terminology as in [5], we say that a type τ satisfies the *Quaternionic Continued Fraction Finiteness* (QCFF) if every $\alpha \in B$ has a finite continued fraction expansion of type τ.

To study the QCFF property, we introduce a notion of *quaternionic height* for the pair (B, R), which is a particular instance of a height function associated to an adelic norm as defined in [19]. Using this notion of height and applying the Northcott property, we prove a criterion which provides a sufficient condition to establish the finiteness of the continued fraction algorithm.

Later on, we focus on the case of indefinite quaternion algebras of discriminant pq, with p and q two primes: in this case, we mimic the classic algorithm for continued fractions in $\mathbb{Q}_p$, given by Browkin [4], to construct a concrete example of a quaternionic type. While every element of $\mathbb{Q}$ has a finite continued fraction expansion via Browkin algorithm, we show that, surprisingly, the analogous property does not hold for this quaternionic type. Moreover, in some particular cases, we provide explicit examples of elements of B having purely periodic continued fraction expansion. In the final part of the paper, we show how the main result can be applied to study the existence of solutions of some families of quadratic polynomials with coefficients in B.

The paper is organized as follows: in Section 2, we recall some properties of quaternion algebras over $\mathbb{Q}$ that will be useful in the paper; in Section 4.4 we introduce the notion of p-adic floor function and then of quaternionic type; we study the main properties of these objects, proving that these provides a suitable notion of quaternionic continued fractions. In Section 4 we introduce a suitable notion of quaternionic height, which will be used in Section 5 to prove Theorem 5.2 which gives a criteria which gives a sufficient condition to estabilish the finiteness of the continued fraction algorithm. In Section 6 we give some examples in the case of quaternion algebras of discriminant the product of two primes. Finally, in Section 7 we show how to apply Theorem 5.2 to study the roots of some particular families of quadratic polynomials with coefficients in B.

2. Some generalities on quaternion algebras over $\mathbb{Q}$

We shall denote by $\mathcal{M}$ the set of rational places, and by $\mathcal{M}^0$ the subset of non-archimedean places. Therefore,every $v \in \mathcal{M}^0$ corresponds to a rational prime q and is associated to an absolute value $|\cdot|_q$ normalized in such a way that $|q|_q = \frac{1}{q}$. If $v = \infty$, then $|\cdot|_\infty$ is the usual absolute value.

We refer to [20, 21] for the basic definitions and properties of quaternion algebras and orders. In what follows:

- B is a quaternion algebra over $\mathbb{Q}$ of discriminant $\Delta > 1$;
- p is an odd prime dividing Δ;
- R is an order in B.

If $x \in B$, we shall denote by $\mathrm{nrd}\,(x)$ and $\mathrm{trd}(x)$ the reduced norm and trace of x, respectively.

For every rational place $v \in \mathcal{M}$, we put

$$B_v = B \otimes_{\mathbb{Q}} \mathbb{Q}_v$$

and, if v is a non-archimedean place,

$$R_v = R \otimes_{\mathbb{Z}} \mathbb{Z}_v.$$

Therefore, R_v is an order in B_v.

We say that B is *ramified at* v if B_v is a division algebra, and *split at* v if $B_v \cong M_2(\mathbb{Q}_v)$. Moreover, we say that B is *indefinite* if it is split at ∞, *definite* otherwise. As explained in [21, Section 4.1], the definiteness of B is equivalent to that of the quadratic form on $B \otimes \mathbb{R}$ given by the reduced norm.

Let us consider an odd prime q such that B is ramified at q. Following [21, Section 13.3], B_q contains a unique maximal order, that is

$$R_q^{max} = \{\alpha \in B_q \mid v_q(\mathrm{nrd}(\alpha)) \geq 0\}.$$

Notice that R_q^{max} is a local ring; moreover, the q-adic valuation $v_q\colon \mathbb{Q}_q \to \mathbb{R} \cup \{\infty\}$ can be extended to a valuation on B_q, defined as

$$w_q(\alpha) = \frac{v_q(\mathrm{nrd}\,(\alpha))}{2}.$$

The map w_q is a discrete valuation (see [21, Lemma 13.3.2]); this ensures that one can define q-adic non-archimedean absolute value over B_q:

$$|\alpha|_q = \left(\frac{1}{q}\right)^{w_q(\alpha)} \qquad \text{for each } \alpha \in B_q. \tag{1}$$

The following result shows that B_q is unique up to isomorphism.

Theorem 2.1. *Up to $\mathbb{Q}_q$-algebra isomorphism, we have*

$$B_q \cong \left(\frac{\mathbb{Q}_{q^2}, q}{\mathbb{Q}_q}\right) = \mathbb{Q}_{q^2} \oplus \mathbb{Q}_{q^2} j,$$

where $j^2 = q$ and $\mathbb{Q}_{q^2}$ denotes the unique quadratic unramified[1] *(separable) extension of $\mathbb{Q}_q$.*

Proof. See [21, Thm. 13.3.11]. □

Remark 2.2. Notice that the standard generators of B_q can be chosen in B. To see this, write $\mathbb{Q}_{q^2} \cong \mathbb{Q}_q[X]/f(X)$ for some irreducible quadratic polynomial $f(X) = X^2 + a_1X + a_0 \in \mathbb{Q}_q[X]$. As a consequence of Krasner's lemma [21, Cor. 13.2.9], there exists a constant $\delta > 0$ such that any polynomial $g(x) = X^2 + b_1X + b_0$ with $|b_i - a_i|_q < \delta$ satisfies

$$\mathbb{Q}_q[X]/f(X) \cong \mathbb{Q}_q[X]/g(X).$$

In particular, one can choose $g(X) \in \mathbb{Q}[X]$, and set the generator i as a root of $g(X)$. Up to "completing the square" (q is an odd prime), we can assume $i^2 \in \mathbb{Q}$.

3. Quaternionic continued fractions

In this section we introduce a suitable notion of p-adic quaternionic continued fractions. To do this, we first need to define an analogue of the usual floor function with good properties which guarantee the convergence of the algorithm. We start with the following definition.

Definition 3.1. A *p-adic floor function* for the pair (B, R) is a function $s\colon B_p \to B$ such that

- $|\alpha - s(\alpha)|_p < 1$ for every $\alpha \in B_p$;
- $s(\alpha) \in R_q$ for every prime $q \neq p$;
- $s(0) = 0$;
- $s(\alpha) = s(\beta)$ if $|\alpha - \beta|_p < 1$.

It is easy to see that, given a pair (B, R), the p-adic floor function is not unique, as the following result shows.

Theorem 3.2. *There exist infinitely many p-adic floor functions for the pair* (B, R).

Proof. Let e_1, e_2, e_3, e_4 be a basis of R over $\mathbb{Z}$. Let $\alpha \in B_p$ and write $\alpha = \sum_{i=1}^4 \alpha_i e_i$ with $\alpha_i \in \mathbb{Q}_p$. Fix $0 < \epsilon < 1$. By strong approximation, for $i = 1, \ldots, 4$, there exists $\beta_i \in \mathbb{Q}$ such that

- $|\beta_i - \alpha_i|_p < \epsilon$;
- $|\beta_i|_q \le 1$ if $p \neq q$.

Put $\beta = \sum_{i=1}^4 \beta_i e_i$. Then $\beta \in R_q$ for every prime $q \neq p$. Since $e_i \in R_q$ for $i = 1, \ldots, 4$ we also have

$$|\alpha - \beta|_p \le \max_i\{|\alpha_i - \beta_i|_p |e_i|_p\} \le \max_i\{|\alpha_i - \beta_i|_p\} < 1.$$

Then, we can define $s(\alpha) = \beta$. By letting ϵ tend to 0, we obtain infinitely many p-adic floor functions for (B, R). □

We now have the essentials tools for extending the classical continued fraction algorithm to our setting. We keep the notation of Section 2: let s be a p-adic floor function for the pair (B, R), and α_0 be an element in B_p.

[1] i.e. q is also a generator for the maximal ideal of the valuation ring of $\mathbb{Q}_{q^2}$.

The following recursive algorithm computes the continued fraction expansion of α_0:

$$\begin{cases} a_n = s(\alpha_n), \\ \alpha_{n+1} = (\alpha_n - a_n)^{-1} \qquad \text{if } \alpha_n - a_n \neq 0, \end{cases} \tag{2}$$

and the algorithm stops if $\alpha_n = a_n$. Thus, the *continued fraction expansion of* α_0 is the (possibly infinite) sequence $[a_0, a_1, \dots]$. The a_i-s are called *partial quotients*, while the α_i-s are called *complete quotients*.

Proposition 3.3. For every $n \geq 1$, we have $|a_n|_p > 1$.

Proof. By definition of a_n,

$$|a_n|_p = |s(\alpha_n)|_p = \left|s\big((\alpha_{n-1} - a_{n-1})^{-1}\big)\right|_p = \left|s\big((\alpha_{n-1} - s(\alpha_{n-1}))^{-1}\big)\right|_p.$$

Since s is a floor function, $|\alpha_n|_p = |(\alpha_{n-1} - s(\alpha_{n-1}))^{-1}|_p > 1$. Moreover, since $|\cdot|_p$ is non-archimedean, the ultrametric inequality holds:

$$\underbrace{|\alpha_n|_p}_{>1} \leq \max\big\{|\underbrace{s(\alpha_n) - \alpha_n|_p}_{<1}, |s(\alpha_n)|_p\big\}.$$

Therefore, $|a_n|_p$ must be > 1. □

We call a *quaternionic type* any quadruple $\tau = (B, R, p, s)$ such that

- B is a division quaternion algebra over $\mathbb{Q}$;
- R is an order of B;
- p is a prime number such that B is ramified at p;
- s is a p-adic floor function for the pair (B, R).

Special types

Assume that R contains an element π such that $\mathrm{nrd}(\pi) = \pm p$. Then π is a uniformizer of R_p. Since R is dense in R_p, there is an isomorphism $R/\pi R \simeq R_p/\pi R_p \simeq \mathbb{F}_{p^2}$. Let $\mathcal{C} \subseteq R$ be a complete set of representatives for the quotient $R/\pi R$. Then, every $\alpha \in B_p$ can be expressed uniquely as a Laurent series $\alpha = \sum_{k=-n}^{\infty} c_k \pi^k$, where $c_k \in \mathcal{C}$ for every k. It is possible to define a p-adic floor function for the pair (B, R) by

$$s(\alpha) = \sum_{k=-n}^{0} c_k \pi^k \in B.$$

In analogy with [5, Section3.2], we shall denote the types $\tau = (B, R, p, s)$ obtained in this way by $\tau = (B, R, \pi, \mathcal{C})$, and we will usually call them *special types*.

For example, when B is indefinite and R is an Eichler order, then it is proven [21, Cor. 17.8.5] and [20, Cor. 5.9] that every ideal in a maximal order is principal, so that there exists at least an element $\pi \in R$ of reduced norm $\pm p$. In fact there are infinitely many such elements, since the group of elements of reduced norm 1 in R is infinite, as one can easily deduce from [20, Thm. 4.1.1].

If a quaternionic type τ is fixed, a *(quaternionic) continued fraction of type* τ is any sequence $[a_0, a_1 \dots]$ of elements of $\mathrm{Im}(s)$ such that $|a_i|_p > 1$ for each $i \neq 0$. A periodic sequence of the form $[a_0, a_1, \dots, a_k, a_0, a_1 \dots]$ is usually denoted by $[\overline{a_0, \dots, a_k}]$.

Similar definitions as for the classical case can be adopted in the (non-commutative) quaternionic setting: for any continued fraction $[a_0, a_1, \dots]$, we define the sequences

$$\begin{array}{llll} A_{-1} = 1, & A_0 = a_0, & A_n = A_{n-1}a_n + A_{n-2} & \text{for } n \geq 1, \\ B_{-1} = 0, & B_0 = 1, & B_n = B_{n-1}a_n + B_{n-2} & \text{for } n \geq 1, \end{array}$$

and the matrices

$$\mathcal{A}_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } n \geq 0,$$
$$\mathcal{B}_n = \begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} \qquad \text{for } n \geq 0.$$

Proposition 3.4. For each $n \geq 0$,

$$\mathcal{B}_n = \mathcal{A}_0 \cdots \mathcal{A}_n. \tag{3}$$

Proof. We first observe

$$\begin{aligned}
\mathcal{B}_n &= \begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} A_{n-1}a_n + A_{n-2} & A_{n-1} \\ B_{n-1}a_n + B_{n-2} & B_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{pmatrix} \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\
&= \mathcal{B}_{n-1}\mathcal{A}_n
\end{aligned}$$

for any $n > 0$. Therefore, (3) immediately follows by induction. □

As usual, the *n-th convergent* is

$$Q_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \qquad \text{for } n \geq 0,$$

where the notation $1/\alpha$ stands for α^{-1}.

As in the commutative case, Q_n can be expressed in terms of A_n and B_n.

Proposition 3.5. For $n \geq 0$

$$Q_n = A_n(B_n)^{-1}. \tag{4}$$

Proof. We prove the statement for *any* sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements in B, no matter if such sequence is a continued fraction or not. For $n = 0$, the equality is trivial. Assume that the statement holds for any sequence of length $n - 1$. Given $a_0, \ldots, a_n$, define a new sequence

$$\begin{cases} \tilde{a}_i = a_i & \text{for } 0 \leq i < n-1, \\ \tilde{a}_{n-1} = a_{n-1} + \frac{1}{a_n}, \end{cases}$$

and denote by $\tilde{Q}_i$ the corresponding partial convergents. Then

$$\begin{aligned}
Q_n &= \tilde{Q}_{n-1} \\
&= \tilde{A}_{n-1}(\tilde{B}_{n-1})^{-1} \\
&= (A_{n-2}\tilde{a}_{n-1} + A_{n-3})(B_{n-2}\tilde{a}_{n-1} + B_{n-3})^{-1},
\end{aligned}$$

where the second equality follows by inductive hypothesis. On the other hand,

$$\begin{aligned}
A_n(B_n)^{-1} &= (A_{n-1}a_n + A_{n-2})(B_{n-1}a_n + B_{n-2})^{-1} \\
&= \big((A_{n-2}a_{n-1} + A_{n-3})a_n + A_{n-2}\big)\big((B_{n-2}a_{n-1} + B_{n-3})a_n + B_{n-2}\big)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \big(A_{n-2}(a_{n-1}a_n+1)+A_{n-3}a_n\big)\big(B_{n-2}(a_{n-1}a_n+1)+B_{n-3}a_n\big)^{-1}\\
&= \big(A_{n-2}(a_{n-1}a_n+1)+A_{n-3}a_n\big)\cdot a_n^{-1}\cdot a_n\cdot\big(B_{n-2}(a_{n-1}a_n+1)+B_{n-3}a_n\big)^{-1}\\
&= (A_{n-2}\tilde{a}_{n-1}+A_{n-3})(B_{n-2}\tilde{a}_{n-1}+B_{n-3})^{-1}.
\end{aligned}
$$

□

Proposition 3.6. For any $n > 0$, the following equalities hold:

$$|B_n|_p = \prod_{j=1}^{n} |a_j|_p, \tag{5}$$

$$|Q_n - Q_{n-1}|_p = \frac{1}{|B_n|_p|B_{n-1}|_p}. \tag{6}$$

Proof. We first prove (5) by induction: the case $n = 1$ holds trivially since $B_1 = a_1$. For $n = 2$ we have $|B_2|_p = |a_1a_2+1|_p = |a_2a_1|_p$, where the second equality is granted by the fact that $|a_1|_p, |a_2|_p > 1$ and the ultrametric inequality. Similarly, let us assume inductively that (5) holds for $n-1$ and $n-2$; then, we have

$$|B_n|_p = \max\{|a_nB_{n-1}|_p, |B_{n-2}|_p\} = |a_n|_p\cdot\prod_{j=1}^{n-1}|a_j|_p = \prod_{j=1}^{n}|a_j|_p.$$

As for (6), if $n = 1$ we have

$$|Q_1 - Q_0|_p = \frac{1}{|a_1|_p} = \frac{1}{|B_1|_p|B_0|_p}.$$

If $n \geq 2$, then we have:

$$
\begin{aligned}
Q_n - Q_{n-1} &= A_nB_n^{-1} - A_{n-1}B_{n-1}^{-1}\\
&= (A_{n-1}a_n + A_{n-2})(B_{n-1}a_n+B_{n-2})^{-1} - A_{n-1}B_{n-1}^{-1}\\
&= \big(A_{n-1}a_n + A_{n-2} - A_{n-1}B_{n-1}^{-1}(B_{n-1}a_n+B_{n-2})\big)\cdot(B_{n-1}a_n+B_{n-2})^{-1}\\
&= \big(A_{n-1}a_n + A_{n-2} - A_{n-1}a_n - A_{n-1}B_{n-1}^{-1}B_{n-2}\big)\cdot(B_{n-1}a_n+B_{n-2})^{-1}\\
&= (A_{n-2}B_{n-2}^{-1} - A_{n-1}B_{n-1}^{-1})B_{n-2}\cdot(B_{n-1}a_n+B_{n-2})^{-1}\\
&= (Q_{n-2}-Q_{n-1})\cdot(B_{n-1}a_nB_{n-2}^{-1}+1)^{-1}.
\end{aligned}
\tag{7}
$$

Let us consider the second factor of the latter equality: by (5) and the ultrametric inequality we have

$$|B_{n-1}a_nB_{n-2}^{-1}+1|_p = |B_{n-1}a_nB_{n-2}^{-1}|_p = |a_{n-1}a_n|_p > 1.$$

Thus we may conclude

$$|Q_n - Q_{n-1}|_p = \frac{1}{|a_1|_p}\cdot\frac{1}{\prod_{i=2}^{n}|a_{i-1}a_i|_p} = \frac{1}{|B_n|_p|B_{n-1}|_p}.$$ □

This implies easily the following result.

Corollary 3.7. *The sequence $\{Q_n\}_{n\in\mathbb{N}}$ is convergent with respect to the p-adic topology.*

The previous corollary ensures that a sequence $\{Q_n\}_{n\in\mathbb{N}}$ is convergent. However, we still need to check that, given a sequence of convergents defined by applying Algorithm (2) to some element, the p-adic limit coincides with the element itself. To prove this, some preliminary results are needed.

Lemma 3.8. *For each $n \geq 0$,*

$$|\alpha_n|_p = |a_n|_p.$$

Proof. The case $n = 0$ is immediate. Fix any $n \geq 1$. Algorithm (2) yields

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}},$$

and we already showed that (see the proof of Proposition 3.3)

$$|\alpha_{n+1}|_p > 1.$$

Therefore, the thesis follows from the ultrametric inequality. □

Lemma 3.9. *For each $n \geq 1$,*

$$\alpha_0 = (A_n\alpha_{n+1} + A_{n-1})(B_n\alpha_{n+1} + B_{n-1})^{-1}. \tag{8}$$

Proof. For every $n \geq 0$, let us substitute a_n by α_n in the expression of Q_n and denote by $\tilde{Q}_n$ the resulting element, i.e.

$$\begin{cases} \tilde{Q}_0 = \alpha_0, \\ \tilde{Q}_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_n}}} & \text{for } n \geq 1. \end{cases}$$

Algorithm (2) yields

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}},$$

so that $\tilde{Q}_n = \tilde{Q}_{n-1}$ and therefore $\tilde{Q}_n = \alpha_0$ for each $n \geq 1$. On the other hand, by (4),

$$\tilde{Q}_{n+1} = (A_n\alpha_{n+1} + A_{n-1})(B_n\alpha_{n+1} + B_{n-1})^{-1}$$

for $n \geq 1$.

□

Proposition 3.10. If the continued fraction expansion of $\alpha_0 \in B_p$ is infinite, it converges p-adically to α_0.

Proof. By (1) we have

$$\begin{aligned} \alpha_0 - A_n(B_n)^{-1} &= (A_n\alpha_{n+1} + A_{n-1})(B_n\alpha_{n+1} + B_{n-1})^{-1} - A_n(B_n)^{-1} \\ &= \big(A_n\alpha_{n+1} + A_{n-1} - A_n(\alpha_{n+1} + B_n^{-1}B_{n-1})\big)(B_n\alpha_{n+1} + B_{n-1})^{-1} \\ &= (A_{n-1} - A_nB_n^{-1}B_{n-1})(B_n\alpha_{n+1} + B_{n-1})^{-1} \\ &= (A_{n-1}B_{n-1}^{-1} - A_nB_n^{-1})B_{n-1}(B_n\alpha_{n+1} + B_{n-1})^{-1} \\ &= (Q_{n-1} - Q_n)(B_n\alpha_{n+1}B_{n-1}^{-1} + 1)^{-1}. \end{aligned}$$

The first factor p-adically converges to 0 thanks to Corollary 3.7, while the second factor converges to 0 by (5) and Lemma 3.8. □

Proposition 3.11. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of elements in B such that, for each $n \in \mathbb{N}$,

- $|a_n|_p > 1$,
- there exists a maximal order $R \subset B$ such that $a_n \in R_q$ for every prime $q \neq p$,
- if $|a_i - a_j|_p < 1$, then $a_i = a_j$.

Then there exists a floor function s such that $[a_0, a_1, \dots]$ is a continued fraction of type $\tau = (B, R, p, s)$.

Moreover, let $\alpha_0 \in B_p$ be the p-adic limit of $[a_0, a_1, \dots]$. Then, $a_0, a_1, \dots$ are exactly the partial quotients of the continued fraction expansion of α_0.

Proof. For each $\alpha \in B_p$, let us define $s(\alpha) = a_i$ if $|\alpha - a_i|_p < 1$ for some $i \in \mathbb{N}$; otherwise, let us construct s as in Theorem 3.2. It is immediate to check that the resulting function is a floor function.

For each $n \in \mathbb{N}$, let Q_n be n-th convergent of $[a_0, a_1, \dots]$. Since we are assuming that $\{Q_n\}_{n\in\mathbb{N}}$ converges p-adically to α_0, there exists $m \geq 1$ such that $|\alpha_0 - Q_m|_p < 1$. Then, ultrametric inequality and (6) yield

$$|\alpha_0 - Q_{m-1}|_p \leq \max\{|\alpha - Q_m|_p, |Q_m - Q_{m-1}|_p\} < 1.$$

Thus, after iterating the above argument m times, we conclude $|\alpha_0 - a_0|_p < 1$ since $Q_0 = a_0$. In particular, $s(\alpha_0) = a_0$ by definition of s. Setting $\alpha_n = (\alpha_{n-1} - a_{n-1})^{-1}$ for each $n \geq 1$, one can inductively check that α_n is the p-adic limit of $[a_n, a_{n+1}, \dots]$. Therefore, $|\alpha_n - a_n|_p < 1$ by the same argument. This proves that $s(\alpha_n) = a_n$ and $\{a_n\}_{n\in\mathbb{N}}$ (resp. $\{\alpha_n\}_{n\in\mathbb{N}}$) are the partial (resp. complete) quotients of the continued fraction expansion of α_0, as wanted. □

For any $n \geq -1$, let us define

$$V_n = A_n - \alpha B_n.$$

Then, one can prove the following useful equalities.

Proposition 3.12. For each $n \geq 1$, the following relations hold:

i) $V_n = V_{n-1}a_n + V_{n-2}$.
ii) $V_{n-1}\alpha_n + V_{n-2} = 0$.
iii) $V_{n-1} = (-1)^n \alpha_1^{-1} \cdots \alpha_n^{-1}$.
iv) $|V_{n-1}|_p = \prod_{j=1}^{n} \frac{1}{|a_j|_p}$.

Proof.

i) Let us prove it by induction on n. The case $n = 1$ is an easy verification. Assume that the claim is true for every $m < n$; then by definition we have

$$\begin{aligned} V_n &= A_n - \alpha B_n \\ &= A_{n-1}a_n + A_{n-2} - \alpha B_{n-1}a_n - \alpha B_{n-2} \\ &= \underbrace{(A_{n-1} - \alpha B_{n-1})}_{V_{n-1}}a_n + \underbrace{A_{n-2} - \alpha B_{n-2}}_{V_{n-2}}, \end{aligned}$$

as wanted.

ii) We can mimic the proof of Lemma 3.9: for every $n \geq 0$, let us substitute a_n by α_n in the expression of V_n (resp. Q_n) and denote by $\tilde{V}_n$ (resp. $\tilde{Q}_n$) the resulting element. We have already observed $\tilde{Q}_n = \alpha$ for each $n \geq 1$. Thus,

$$\tilde{V}_n = (\alpha - \tilde{Q}_n)B_n = 0.$$

On the other hand, (i) ensures

$$\tilde{V}_n = V_{n-1}\alpha_n + V_{n-2},$$

proving the claim.

iii) From (ii) we get

$$\alpha_n = -V_{n-1}^{-1} V_{n-2},$$

so that

$$\alpha_n \cdots \alpha_1 = (-1)^n V_{n-1}^{-1}$$

since $V_{-1} = 1$.

iv) Follows immediately from (iii) and Lemma 3.8.

□

4. Quaternionic heights

In this section we define a suitable notion of quaternionic height, using the work of Talamanca [19] in which the author defines a height function associated to an adelic norm.

We begin by revising some basic definitions and properties related to such heights. For this, we follow [19].

4.1. Local norms over a vector space

Let q be a rational prime. Let V be a finite-dimensional vector space over $\mathbb{Q}_q$. A subset $\Omega \subseteq V$ is a $\mathbb{Z}_q$*-lattice* if it is a compact open $\mathbb{Z}_q$-module.

Every $\mathbb{Z}_q$-lattice $\Omega \subseteq V$ defines a *norm* N_Ω on V by

$$N_\Omega(\mathbf{v}) = \inf_{\lambda \in \mathbb{Q}_q, \lambda\mathbf{v} \in \Omega} |\lambda|_q^{-1}.$$

N_Ω is an ultrametric norm on V, that is

- $N_\Omega(\lambda\mathbf{v}) = |\lambda|_q N_\Omega(\mathbf{v})$ for all $\mathbf{v} \in V$;
- $N_\Omega(\mathbf{v}_1 + \mathbf{v}_2) \leq \max\{N_\Omega(\mathbf{v}_1), N_\Omega(\mathbf{v}_2)\}$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$.

4.2. Adelic norms on vectors spaces over global fields

Let now V be a $\mathbb{Q}$-vector space. Let $M \subseteq V$ be a lattice, that is a finitely generated subgroup containing a basis of V over $\mathbb{Q}$.

Put $V_q = V \otimes_{\mathbb{Q}} \mathbb{Q}_q$, for every $q \in \mathcal{M}$, and $M_q = M \otimes_{\mathbb{Z}} \mathbb{Z}_q$, for every $v \in \mathcal{M}_K^0$.

A family of norms $\mathcal{F} = \{N_v : V_v \to \mathbb{R}, v \in \mathcal{M}_K\}$ is said to be an *adelic norm* on V if

- every N_v is a norm on K_v, ultrametric if $v \in \mathcal{M}^0(K)$;
- there exists an $\mathcal{O}_K$-lattice $M \subseteq V$ such that $N_v = N_{M_v}$ for all but finitely many $v \in \mathcal{M}^0(K)$.

Remark 4.1. The last condition implies that if $\mathbf{x} \in V$ then $N_v(\mathbf{x}) = 1$ for all but finitely many v.

4.3. Height function associated to an adelic norm

Given an adelic norm $\mathcal{F}$, the *height function* on V associated to $\mathcal{F}$ is

$$\mathcal{H}_{\mathcal{F}}(\mathbf{x}) = \prod_{v \in \mathcal{M}(K)} N_v(\mathbf{x})^{d_v}.$$

Notice that this is well-defined since the product is finite by the remark above.

We will denote by

$$\mathcal{H}(V) = \{\mathcal{H}_{\mathcal{F}} \mid \mathcal{F} \text{ is an adelic norm on } V\}$$

the set of height functions associated to adelic norms.

The following properties are proven in [19, Prop. 1.1].

- By the product formula,

$$\mathcal{H}_{\mathcal{F}}(\lambda \mathbf{x}) = \mathcal{H}_{\mathcal{F}}(\mathbf{x}) \text{ if } \lambda \in \mathbb{Q}^{\times},$$

  so that $\mathcal{H}_{\mathcal{F}}$ descends on a function on $\mathbb{P}(V)$, i.e., the projective space associated to V.
- If $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}(V)$, there exists a constant $C = C(\mathcal{H}_1, \mathcal{H}_2) > 1$ such that, for all $\mathbf{x} \in V$,

$$\frac{1}{C}\mathcal{H}_1(\mathbf{x}) \leq \mathcal{H}_2(\mathbf{x}) \leq C\mathcal{H}_1(\mathbf{x}).$$

- **Northcott property:** For all $C > 0$ the set

$$\{[\mathbf{x}] \in \mathbb{P}(V) \mid \mathcal{H}(\mathbf{x}) < C\}$$

  is finite.

4.4. The quaternionic case

Now let B be quaternion algebra over $\mathbb{Q}$, and R be an order in B. Then, R is by definition a lattice in B.

We consider the adelic norm $\mathcal{F} = \{N_v, v \in \mathcal{M}\}$ on B, where

- $N_v = N_{R_v}$ if v is non-archimedean and B is unramified at v;
- $N_v = |\cdot|_v$ (induced by the discrete valuation) if v is non-archimedean and B is ramified at v;
- N_v is the operator norm on $M_2(\mathbb{R})$ or $M_2(\mathbb{C})$ if v is archimedean.

For these norms, it is straightforward to verify that the following multiplicative properties hold.

Proposition 4.2. For every v the norm N_v is submultiplicative, that is

$$N_v(\mathbf{x} \cdot \mathbf{y}) \leq N_v(\mathbf{x})N_v(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in B.$$

Moreover, if v is non-archimedean and ramified, then N_v is multiplicative

$$N_v(\mathbf{x} \cdot \mathbf{y}) = N_v(\mathbf{x})N_v(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in B.$$

5. A criterion for finiteness

Let $\tau = (B, R, p, s)$ be a quaternionic type. In this section, we prove a sufficient criterion to decide whether a type satisfies the QCFF. To do this, we first need to prove an elementary result regarding real linear recurrence sequences.

For any $x \in \mathbb{C}$, let us define

$$\theta(x) = \frac{1}{2}(|x|_\infty + \sqrt{|x|_\infty^2 + 4});$$

then, we have the following inequality:

$$|x|_\infty \leq \theta(x) \leq |x|_\infty + 1,$$

and the map θ is a bijection from $[0, +\infty)$ to $[1, +\infty)$ whose inverse is given by $y \mapsto y - \frac{1}{y}$.

Lemma 5.1. *Let $(c_n)_{n\geq 1}$ be any sequence of real numbers ≥ 0 and let $(t_n)_{n\geq -1}$ be a sequence of real numbers ≥ 0 satisfying, for every $n \geq 1$ the inequality:*

$$t_n \leq c_n t_{n-1} + t_{n-2}.$$

Then, there exists $c > 0$ such that, for every $n \geq 0$,

$$\max\{t_n, t_{n-1}\} \leq c \cdot \prod_{j=1}^{n} \theta(c_j).$$

Proof. For any complex matrix M, let us consider the operator norm

$$||M|| = \sup_{\mathbf{v}\neq \mathbf{0}} \frac{||M\mathbf{v}||}{||\mathbf{v}||},$$

where $||\mathbf{v}||$ denotes the Euclidean norm of a complex vector. The following facts are well known (see for example [10, Section 5]):

- $||M_1 \cdot M_2|| \leq ||M_1|| \cdot ||M_2||$;
- $||M|| = \sqrt{|\gamma|_\infty}$, where γ is the dominant eigenvalue of $M \cdot M^*$ (here M^* denotes the transpose conjugate of M).

In particular, we have that, for every $a \in \mathbb{C}$,

$$\left|\left|\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\right|\right| = \theta(a).$$

Let $\mathcal{M}_n = \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}$; then, for every $n \geq 1$ we have

$$\left|\left|\begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix}\right|\right| \leq \left|\left|\mathcal{M}_n \begin{pmatrix} t_{n-1} \\ t_{n-2} \end{pmatrix}\right|\right| \leq ||\mathcal{M}_n|| \left|\left|\begin{pmatrix} t_{n-1} \\ t_{n-2} \end{pmatrix}\right|\right| = \theta(c_n) \left|\left|\begin{pmatrix} t_{n-1} \\ t_{n-2} \end{pmatrix}\right|\right|,$$

so that

$$\max\{|t_n|_\infty, |t_{n-1}|_\infty\} \leq \left|\left|\begin{pmatrix} t_n \\ t_{n-1} \end{pmatrix}\right|\right| \leq c \cdot \prod_{j=1}^{n} \theta(c_j),$$

with $c = \left|\left|\begin{pmatrix} t_0 \\ t_{-1} \end{pmatrix}\right|\right|$, as wanted. □

Consider the adelic norm $\mathcal{F}$ defined in Section 4.4 and let $\mathcal{H}$ be the height function associated to $\mathcal{F}$ as in Section 4.3. Let $\alpha \in B$ and put $V_n = A_n - \alpha B_n$; then, by Proposition 3.12 *iv*),

- $N_p(V_n) = \left|\prod_{j=1}^n \frac{1}{a_j}\right|_p = \prod_{j=1}^n \frac{1}{|a_j|_p} = (\frac{1}{p})^{\sum_{j=1}^n w_p(a_j)}$, where w_p is the discrete valuation on B_p;
- for a non-archimedean $q \neq p$,
$$N_q(V_n) \leq \max\{N_q(A_n), N_q(\alpha)N_q(B_n)\} \leq \max\{N_q(\alpha), 1\};$$
- for the archimedean $v = \infty$,

$$\begin{aligned} N_\infty(V_n) &= N_\infty(V_{n-1}a_n + V_{n-2}) \\ &\leq N_\infty(V_{n-1})N_\infty(a_n) + N_\infty(V_{n-2}) && \text{(by triangular inequality and submultiplicativity)} \\ &\leq C_\infty(\alpha) \prod_{j=1}^{n} \theta(N_\infty(a_j)) && \text{(by Lemma 5.1).} \end{aligned}$$

These estimates for the norms allow us to prove a criterion to characterize the elements of the quaternion algebra having finite continued fraction expansion.

Following the terminology introduced in [14] in the real case and in [5], we shall say that a quaternionic type τ satisfies the *Quaternionic Continued Fraction Finiteness (*QCFF*)* property if every $\alpha \in B$ has a finite expansion of type τ. We say that a pair (B, R) has the *p-adic* QCFF *property* if there exists a quaternionic type (B, R, p, s) enjoying the QCFF property. We have the following result.

Theorem 5.2. *Let $\tau = (B, R, p, s)$ be a quaternionic type and let $\alpha \in B$ having an infinite continued fraction expansion $[a_0, a_1, a_2 \ldots]$ of type τ. Assume that there exists an eventual upper bound μ_α for the sequence*

$$\left\{\frac{\theta(N_\infty(a_n))}{|a_n|_p}\right\}_{n\in\mathbb{N}};$$

then, $\mu_\alpha \geq 1$.

Proof. We denote by

$$C(\alpha) = C_\infty(\alpha) \cdot \prod_{q \neq p} \max\{N_q(\alpha), 1\};$$

then

$$\begin{aligned} \mathcal{H}(V_n) &= \prod_{v \in \mathcal{M}} N_v(V_n) \\ &\leq C_\infty(\alpha) \prod_{j=1}^{n} \theta(N_\infty(a_j)) \cdot \prod_{q \neq p} \max\{N_q(\alpha), 1\} \cdot \prod_{j=1}^{n} \frac{1}{|a_j|_p} \\ &\leq C(\alpha) \cdot \mu_\alpha^n. \end{aligned}$$

If $\mu_\alpha < 1$, then $\mathcal{H}(V_n) \to 0$. By the Northcott property, we have that $V_n = 0$ for $n \gg 0$ and the continued fraction is finite, giving a contradiction. □

This implies the following criterion for the QCFF property.

Corollary 5.3. *Let $\tau = (B, R, p, s)$ be a quaternionic type. Define*

$$\mu = \sup\left\{ \frac{\theta(N_\infty(a))}{|a|_p} \mid a \in s(B), |a|_p > 1 \right\}.$$

If $\mu < 1$, then τ satisfies the QCFF property.

5.1. Bounded types

A type $\tau = (B, R, p, s)$ is said to be *bounded* if there exists a real number $C > 0$ such that $N_\infty(s(B)) = \sup\{N_\infty(a) \mid a \in s(B)\} < C$.

Proposition 5.4. For every triple (B, R, p) there exists a floor function s such that the type (B, R, p, s) is bounded.

Proof. Let

$$P = \{x \in R \mid |x|_p < 1\}.$$

Then, P is a lattice in B_∞, so that there is a bounded fundamental domain $\mathcal{D} \subseteq B_\infty$ for the quotient B_∞/P. We construct a p-adic floor function s for B as follows. Let π be a uniformizer in R_p, and let us consider a non trivial coset $\alpha + \pi R_p \subseteq B_p$; by strong approximation, it contains an element $\alpha' \in B$ such that $\alpha' \in R_q$ for every rational prime $q \neq p$. Possibly translating α' by a suitable element of P, we find a $\beta \in R[\frac{1}{p}]$ such that $\beta \in \mathcal{D}$ and $\alpha' \equiv \beta \pmod{P}$. Then, for every $\gamma \in \alpha + \pi R_p$ we put $s(\gamma) = \beta$ and $\tau = (B, R, p, s)$. □

Theorem 5.5. *Assume that τ is a bounded type; then, there exists a positive integer K such that every infinite continued fraction $[a_0, a_1, \dots]$ of type τ either represents an element $\alpha \in B_p \setminus B$ or the set $\{i \mid |a_i|_p \leq \sqrt{p}^K\}$ is infinite.*

Proof. Since τ is bounded, there exists a real number $C > 0$ such that $N_\infty(a_n) < C$ for every $n \geq 0$. Choose $K > 0$ such that $C < \sqrt{p}^K$. Assume that $\alpha \in B_p$ has an infinite expansion of type τ in which only finitely many partial quotients have absolute value $\leq \sqrt{p}^K$; we want to show that $\alpha \notin B$. By hypothesis

$N_\infty(a_n) \leq C$, so that, for $n \gg 0$,

$$\frac{\theta(N_\infty(a_n))}{|a_n|_p} \leq \frac{C+1}{\sqrt{p}^{K+1}} \leq \frac{\sqrt{p}^{K}+1}{\sqrt{p}^{K+1}} < 1.$$

Therefore $\mu_\alpha < 1$ and we can apply Theorem 5.2 to get the conclusion. □

6. Construction of a p-adic type when $\Delta = pq$

Fix a prime $p \geq 3$. We construct an indefinite division quaternion algebra B ramified at p, with the further requirement that B is ramified only at p and at another place $q \neq \infty$ (we recall that the number of ramified places of a quaternion algebra must be even [21, Thm. 14.6.1]). To ensure this, it is enough [2, Lem. 1.21] to choose a prime $q \equiv 1 \mod 4$ such that $\left(\frac{p}{q}\right) = -1$ and set $B = \left(\frac{q,p}{\mathbb{Q}}\right)$. If $p \equiv 3 \mod 8$, also $q = 2$ can be chosen.

Consider the *standard order* $R' = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij$. One can prove [21, Ex. 15.2.10] that the reduced discriminant of this order is $2pq$, while the discriminant of B (i.e. the product of all ramified places) is pq by construction. Therefore R' is not maximal [21, Thm. 15.5.5]. However, it is contained in a unique maximal order R by [2, Prop. 1.32.iii]. By [2, Prop. 1.60], R has an explicit expression of the form:

$$R = \begin{cases} \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+ij}{2} & \text{if } q = 2, \\ \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}\frac{1+j}{2} + \mathbb{Z}\frac{i+ij}{2} & \text{otherwise.} \end{cases}$$

We mimic the classic construction of continued fractions in $\mathbb{Q}_p$ given by Browkin [4]. A natural choice is to use a special type as described in Section 3.

First, we choose a set $\mathcal{C}$ of representatives for R/jR:

$$\mathcal{C} = \left\{ a + bi \mid a, b \in \left\{ 0, \pm 1, \dots, \pm \frac{1}{2}(p-1) \right\} \right\}.$$

Thus, given $\alpha \in B$, we can define a floor function as follows:

- write α as the series

$$\alpha = \sum_{\ell=r}^{\infty} \alpha_\ell j^\ell$$

  where $\alpha_\ell \in \mathcal{C}$ for each ℓ.
- set

$$s(\alpha) = \begin{cases} 0 & \text{if } r > 0, \\ \sum_{\ell=r}^{0} \alpha_\ell j^\ell & \text{if } r \leq 0. \end{cases}$$

It is clear that the elements of $\mathbb{Q}$, seen as a subset of B, enjoy a finite continued fraction expansion with respect to the special type $\tau = (B, R, j, \mathcal{C})$ constructed above. In fact, their expansion coincides with the classic Browkin continued fraction expansion considered in [4].

However, we claim that there exist elements in B whose continued fraction expansion is infinite. Some natural candidates to prove this claim, in analogy with the classic construction by Browkin, would be the square roots in $\mathbb{Q}_p \setminus \mathbb{Q}$. However, none of them can be in fact seen as an element of B: a square-free integer d is a square in $\mathbb{Q}_p$ if and only if it is a quadratic residue modulo p, while $\mathbb{Q}(\sqrt{d})$ can be embedded in B if and only if d is *not* a quadratic residue modulo every ramified prime – including p [21, Prop. 14.6.7]. Therefore, we need a subtler construction to find explicit examples of elements with infinite continued fraction expansion.

Theorem 6.1. *The type $\tau = (B, R, j, \mathcal{C})$ constructed above does not satisfy the QCFF property.*

Proof. Suppose that the n-th complete quotient of some continued fraction has the following form:

$$\alpha_n = \frac{k_1}{k_2 p^r}\left(i + \frac{1}{p} ij\right),$$

where k_1, k_2 are coprime integers not divided by p, $k_2 \notin \{-1, 1\}$ and $r \geq 0$. We write the Bézout's identity for p^{r+1} and k_2, i.e.

$$vp^{r+1} + wk_2 = k_1,$$

choosing the integers v, w in such a way that $w \in \{-(p^{r+1}-1)/2, \ldots, (p^{r+1}-1)/2\}$. Therefore, one can check that

$$a_n = s(\alpha_n) = \frac{w}{p^r}\left(i + \frac{1}{p} ij\right).$$

Moreover, since $k_2 \notin \{-1, 1\}$, we have that $v \neq 0$, so the continued fraction does not terminate and the next complete quotient is

$$\alpha_{n+1} = (\alpha_n - a_n)^{-1} = \frac{k_2 p^r}{vp^{r+1}} \cdot \left(i + \frac{1}{p} ij\right)^{-1} = \frac{k_2}{q(p-1)v}\left(i + \frac{1}{p} ij\right). \tag{9}$$

We claim that the continued fraction expansion of

$$\alpha_0 = \frac{1}{q}\left(i + \frac{1}{p} ij\right)$$

is infinite. Indeed, by (9),

$$\alpha_1 = \frac{1}{(p-1)v_1}\left(i + \frac{1}{p} ij\right),$$

where v_1 satisfies Bézout's identity

$$v_1 p + w_1 q = 1 \tag{10}$$

for some integer w_1. After writing $v_1 = v_1' p^{r_1}$ with $p \nmid v_1'$ and $r_1 \geq 0$, the next complete quotient can be computed using (9):

$$\alpha_2 = \frac{v_1'}{qv_2}\left(i + \frac{1}{p} ij\right),$$

where v_2 satisfies the equality

$$v_2 p^{r_1+1} + w_2(p-1)v_1' = 1 \tag{11}$$

for some integer w_2. Notice that q cannot divide v_1': otherwise, the contradiction $1 \equiv 0 \mod q$ would follow from (10). Moreover, v_1' and v_2 have no common factors because of (11). Similarly, from (9) we get

$$\alpha_3 = \frac{v_2'}{(p-1)v_3}\left(i + \frac{1}{p} ij\right)$$

where $v_2 = v_2' p^{r_2}$ with $p \nmid v_2'$ and $r_2 \geq 0$,

$$v_3 p^{r_2+1} + w_3 q v_2' = v_1'$$

for some integer w_3, and $p-1$ does not divide v_2' because $v_2 p + w_2(p-1)v_1' = 1 \not\equiv 0 \mod (p-1)$. Moreover, v_3 and v_2' have no common factors (otherwise, such factor would be also a common factor of v_1' and v_2). The form of each α_n can now be derived by induction. Namely, first define the sequence $\{v_n\}_{n\in\mathbb{N}}$ as follows: $v_{-1} = v_0 = 1$, and, for each $n \geq 1$, v_n is the unique integer such that

$$v_n p^{r_{n-2}+r_{n-1}+1} + w_n C v_{n-1} p^{r_{n-2}} = p^{r_{n-1}} v_{n-2} \qquad \text{with} \quad C = \begin{cases} p-1 & \text{if } n \text{ is even,} \\ q & \text{if } n \text{ is odd,} \end{cases}$$

for some $w_n \in \{-(p^{r_{n-2}+r_{n-1}+1}-1)/2, \dots, (p^{r_{n-2}+r_{n-1}+1}-1)/2\}$, where r_n denotes the p-adic valuation of v_n. Then we have, for every $n \geq 0$,

$$\alpha_n = \frac{v'_{n-1}}{C' v_n}\left(i + \frac{1}{p} ij\right) \qquad \text{with} \quad C' = \frac{(p-1)q}{C} \quad \text{and} \quad v'_{n-1} = \frac{v_{n-1}}{p^{r_{n-1}}}.$$

In particular, the denominator in α_n is always a multiple of either q or $p-1$, so that it cannot be either 1 or -1. As a consequence, the continued fraction expansion of α_0 never stops, which proves the claim. □

Corollary 6.2. *If q divides $p-1$, then the element*

$$\alpha_0 = \frac{1}{q}\left(i + \frac{1}{p} ij\right)$$

has purely periodic continued fraction expansion. The period has length 1 *if $p = 3$ and $q = 2$, and* 2 *otherwise.*

Proof. This is just a specialization of the proof of Theorem 6.1. Namely, we can write Bézout's identity explicitly as

$$p + \frac{1-p}{q} \cdot q = 1,$$

so that, by (9),

$$\alpha_1 = \frac{1}{p-1}\left(i + \frac{1}{p} ij\right).$$

Bézout's identity is now

$$p + (-1)(p-1) = 1,$$

which gives $\alpha_2 = \alpha_0$. □

7. Roots of some quadratic quaternionic polynomials

In this section we show how Theorem 5.2 can be exploited to study the roots of a family of quadratic polynomials[2] with coefficients in B.

A link between quadratic equations over quaternion algebras and continued fractions has already been considered by Hamilton [8] in the special case $B = \left(\frac{-1,-1}{\mathbb{R}}\right)$. The general case of quadratic equations over $\left(\frac{-1,-1}{\mathbb{R}}\right)$ has been dealt with in [11]. Namely, an equation

$$\sum_{\ell=1}^{n} \alpha_{0,\ell} X \alpha_{1,\ell} X \alpha_{2,\ell} + \sum_{\ell'=1}^{n'} \beta_{0,\ell'} X \beta_{1,\ell'} + \gamma_0 = 0, \tag{12}$$

with $\alpha_{0,\ell}, \alpha_{1,\ell}, \alpha_{2,\ell}, \beta_{0,\ell'}, \beta_{1,\ell'}, \gamma_0 \in B$, can be rewritten in terms of the components of X, say $x_0, \dots, x_3$, with respect to the standard generators of B, obtaining

$$f_0(x_0, \dots, x_3) + f_1(x_0, \dots, x_3) i + f_2(x_0, \dots, x_3) j + f_3(x_0, \dots, x_3) ij = 0$$

where $f_0, \dots, f_3$ are quadratic polynomials with real coefficients. Thus, $\alpha = t + xi + yj + zij \in B$ is a solution of (12) if and only if $(t, x, y, z) \in \mathbb{R}^4$ is a solution of the polynomial system

$$\begin{cases} f_0(x_0, \dots, x_3) = 0 \\ f_1(x_0, \dots, x_3) = 0 \\ f_2(x_0, \dots, x_3) = 0 \\ f_3(x_0, \dots, x_3) = 0. \end{cases}$$

Therefore, (12) has either no solution, up to 16 solutions or infinitely many.

[2]Here we generalize in an obvious way the notion of polynomial to this non-commutative setting.

Littlewood's arguments can be straightforwardly generalized to quaternion algebras over arbitrary fields. However, to the best of our knowledge, not much more is known about quadratic equations with coefficients in a quaternion algebra B over $\mathbb{Q}$.

7.1. Roots of $X^2 - aX - 1$

An element $a \in B$ is said to be *integral* if its minimum polynomial over $\mathbb{Q}$ has integral coefficients; similarly, $a \in B$ is said to be *p-integral* if its minimum polynomial over $\mathbb{Q}$ has coefficients in $\mathbb{Z}[\frac{1}{p}]$.

Lemma 7.1.
a) *Let $a \in B$ be integral. Then, there is an order R in B such that $a \in R$.*
b) *Let $a \in B$ be p-integral. Then there exists an order $R \in B$ such that $a \in R[\frac{1}{p}]$.*
c) *Let $a \in B$ be a p-integral element such that $|a|_p > 1$. Then, there exists a quaternionic type $\tau = (B, R, p, s)$ such that $a \in s(B)$.*

Proof. If $a \in \mathbb{Z}$, then a lies in every order, so we can assume $a \notin \mathbb{Z}$. In this case $\mathbb{Z}[a]$ is an order in a quadratic field $K = \mathbb{Q}(a) \subseteq B$. Then, we can write $a = n + \sqrt{d}$ with $n, d \in \mathbb{Z}$, so that it suffices to show that there is an order R in B containing $x = \sqrt{d}$. By the Skolem-Noether theorem [21, Thm. 1.2.1], there exists an element $y \in R$ such that the $yxy^{-1} = -x$. Then, it is immediate to see that $\mathbb{Z}[x, y]$ is an order containing x, proving the first part.

Point b) is an immediate consequence of a) applied to $p^k a$ for a suitable $k \in \mathbb{N}$. Finally, we deduce part c) by considering an order R such that $a \in R[\frac{1}{p}]$ and a floor function s such that $s(a + pR_p) = a$. □

Thanks to the previous lemma, we are able to prove a result about the existence of roots of certain quadratic polynomials.

Proposition 7.2. Let $a \in B$ and p an odd ramified prime such that
- $|a|_p > 1$,
- a is p-integral,
- $\frac{\theta(N_\infty(a))}{|a|_p} < 1$.

Then, the polynomial $f(X) = X^2 - aX - 1$ has no root in B.

Proof. Lemma 7.1 and Proposition 3.11 ensure that there exist an order R and a floor function s such that the periodic continued fraction $[\overline{a}]$ is a continued fraction of type $\tau = (B, R, p, s)$. The p-adic limit of $[\overline{a}]$, say α', annihilates $f(X)$ and does not lie in B by Theorem 5.2. It is immediate to check that the same holds for $\alpha'' = -1/\alpha'$. In order to conclude, we only need to prove that $f(X)$ has at most two roots. Let $\alpha \in B_p$ be another root; up to replacing α by α^{-1}, we can assume that $|\alpha|_p = |a|_p > 1$ and $|-\alpha^{-1}|_p < 1$. It follows that $s(\alpha) = a$, hence α and α' have the same continued fraction expansion with respect to the type τ. Therefore $\alpha = \alpha'$. □

Remark 7.3. If $a \notin \mathbb{Q}$, proving that $f(X)$ has only two roots is even easier. In fact, if α is a root of $f(X)$, then $\alpha \neq 0$ and $a = \alpha - 1/\alpha \in \mathbb{Q}(\alpha)$. Equivalently, each root of $f(X)$ belongs to $\mathbb{Q}(a)$. Since $\mathbb{Q}(a)$ is a field, it contains no more than two roots of $X^2 - aX - 1$, proving the claim directly.

7.2. More general quadratic polynomials

The previous argument can be exploited to prove a more general version of Proposition 7.2 for a larger class of quadratic polynomials.

Theorem 7.4. *Let B be a division quaternion algebra over $\mathbb{Q}$, p an odd prime at which B ramifies, and $a_0, \ldots, a_n$ a sequence of elements in B for some $n \geq 0$ such that, for each $i \in \{0, \ldots, n\}$,*

- $|a_i|_p > 1$,
- *there exists a maximal order $R \subset B$ such that $a_i \in R_q$ for every prime $q \neq p$,*
- *if $|a_i - a_j|_p < 1$, then $a_i = a_j$.*
- $\frac{\theta(N_\infty(a_i))}{|a_i|_p} < 1$.

Define the sequences $A_0, \ldots, A_n$ and $B_0, \ldots, B_n$ as in Section 3. Then, the polynomial

$$XB_nX + XB_{n-1} - A_nX - A_{n-1} \tag{13}$$

has no root in B.

Proof. Let α be a root of (13). Equivalently, by (8),

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\alpha}}}}}. \tag{14}$$

Proposition 3.11 ensures that there exists a floor function s such that $[\overline{a_0, \ldots, a_n}]$ is a continued fraction of type $\tau = (B, R, p, s)$. The p-adic limit of $[\overline{a_0, \ldots, a_n}]$, say α', satisfies (14) and does not lie in B by Theorem 5.2.

Furthermore, following a well-known result for the classical case [7], we remark that (14) can be rewritten as follows:

$$(\alpha - a_0)^{-1} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\alpha}}}},$$

which gives

$$((\alpha - a_0)^{-1} - a_1)^{-1} = a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\alpha}}},$$

and so by induction

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\ddots + \cfrac{1}{a_0 + \cfrac{1}{-1/\alpha}}}}} = -\frac{1}{\alpha}.$$

Therefore, another root of (13), say α'', is the inverse of the opposite of the p-adic limit of $[\overline{a_n, \ldots, a_0}]$, which is a continued fraction of type τ by Proposition 3.11. Moreover, α'' does not lie in B by Theorem 5.2.

In order to conclude, we need to show that (13) has no root in B_p other than α' and α''. Define $A_{n+1}, A_{n+2}, \ldots$ and $B_{n+1}, B_{n+2}, \ldots$ the sequences associated to $[\overline{a_0, \ldots, a_n}]$. Since $a_{n+1} = a_0$, the recursive formulas defining A_i and B_i yield

$$\begin{aligned} A_{k(n+1)+\ell} &= A_{k(n+1)+\ell-1}a_\ell + A_{k(n+1)+\ell-2}, \\ B_{k(n+1)+\ell} &= B_{k(n+1)+\ell-1}a_\ell + B_{k(n+1)+\ell-2}, \end{aligned} \tag{15}$$

for every $k \geq 1$ and $\ell \in \{0, \ldots, n\}$, and α is a root of (13) if and only if

$$\begin{aligned} \alpha B_{k(n+1)+n}\alpha + \alpha B_{k(n+1)+n-1} &= A_{k(n+1)+n}\alpha + A_{k(n+1)+n-1} \\ \alpha B_{k(n+1)+n}\left(\alpha + B^{-1}_{k(n+1)+n}B_{k(n+1)+n-1}\right) &= A_{k(n+1)+n}\left(\alpha + A^{-1}_{k(n+1)+n}A_{k(n+1)+n-1}\right) \end{aligned} \tag{16}$$

for every $k \geq 0$. We remark that (15) allows rewriting $A^{-1}_{k(n+1)+n}A_{k(n+1)+n-1}$ as follows:

$$\begin{aligned} A^{-1}_{k(n+1)+n}A_{k(n+1)+n-1} &= \frac{1}{A^{-1}_{k(n+1)+n-1}A_{k(n+1)+n}} \\ &= \frac{1}{a_n + A^{-1}_{k(n+1)+n-1}A_{k(n+1)+n-2}} \\ &= \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + A^{-1}_{k(n+1)+n-2}A_{k(n+1)+n-3}}} \\ &= \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_0 + A^{-1}_{(k-1)(n+1)+n}A_{(k-1)(n+1)+n-1}}}}} \\ &= \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_0}}}} \\ &= \frac{1}{\tilde{Q}_{k(n+1)}} \end{aligned}$$

for every $k \geq 1$, where $\{\tilde{Q}_i\}_{i \in \mathbb{N}}$ is the sequence of convergents of $[\overline{a_n, \ldots, a_0}]$. The same argument yields $B^{-1}_{k(n+1)+n}B_{k(n+1)+n-1} = 1/\tilde{Q}_{k(n+1)-1}$. Denote by $\{Q_i\}_{i \in \mathbb{N}}$ the convergents of $[\overline{a_0, \ldots, a_n}]$. By Proposition 3.10, for any $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that

$$\max\left\{ \left|Q_{k_\epsilon(n+1)+n} - \alpha'\right|_p, \left|\alpha'' + \frac{1}{\tilde{Q}_{k_\epsilon(n+1)-1}}\right|_p, \left|\alpha'' + \frac{1}{\tilde{Q}_{k_\epsilon(n+1)}}\right|_p \right\} < \epsilon.$$

We ease the notation by setting $A_\epsilon = A_{k_\epsilon(n+1)+n}$ and $B_\epsilon = B_{k_\epsilon(n+1)+n}$, and rewrite (16) as

$$\begin{aligned} \alpha B_\epsilon\left(\alpha + \frac{1}{\tilde{Q}_{k_\epsilon(n+1)-1}}\right) &= A_\epsilon\left(\alpha + \frac{1}{\tilde{Q}_{k_\epsilon(n+1)}}\right) \\ (\alpha B_\epsilon - A_\epsilon)\left(\alpha - \alpha''\right) &= -\alpha B_\epsilon\left(\alpha'' + \frac{1}{\tilde{Q}_{k_\epsilon(n+1)-1}}\right) + A_\epsilon\left(\alpha'' + \frac{1}{\tilde{Q}_{k_\epsilon(n+1)}}\right). \end{aligned}$$

Thus, if $\alpha \neq \alpha''$ the following estimates hold:

$$\left|\alpha - A_\epsilon B_\epsilon^{-1}\right|_p \leq \epsilon \cdot \frac{1}{|\alpha - \alpha''|_p} \cdot \max\left\{|\alpha|_p, \left|A_\epsilon B_\epsilon^{-1}\right|_p\right\}$$
$$\left|\alpha - A_\epsilon B_\epsilon{}^{-1}\right|_p \leq \epsilon \cdot \frac{1}{|\alpha - \alpha''|_p} \cdot \max\left\{|\alpha|_p, |\alpha'|_p, \epsilon\right\}.$$

This proves that α converges p-adically to α', i.e. $\alpha = \alpha'$. □

Remark 7.5. We note that (13) can be rewritten in the form Z^2+tZ+u. Indeed, if we multiply polynomial (13) by B_n and put $T = B_nX$, we get

$$T^2 + TB_{n-1} - B_nA_nB_n^{-1}T - B_nA_{n-1}.$$

Setting $Z = T + B_{n-1}$, this becomes

$$Z^2 - (B_nA_nB_n^{-1} + B_{n-1})Z + B_nA_nB_n^{-1}B_{n-1} - B_nA_{n-1},$$

$$Z^2 - \underbrace{(B_nA_nB_n^{-1} + B_{n-1})}_{t}Z + \underbrace{B_n(A_nB_n^{-1} - A_{n-1}B_{n-1}^{-1})B_{n-1}}_{u}.$$

We stress that $|u|_p = 1$ by (6). Moreover, it is possible to prove (using (7)) that

$$A_nB_n^{-1} - A_{n-1}B_{n-1}^{-1} = (-1)^{n+1}B_1^{-1}(B_0B_2^{-1})(B_1B_3^{-1})\dots(B_{n-2}B_n^{-1}).$$

Given a polynomial

$$XAX + XB + CX + D, \tag{17}$$

there is no terminating algorithm – as far as we know – to check whether it has form (13) or not. Namely, there is no effective way to check at once whether there exist or not a nonnegative n and a suitable sequence of elements $a_0, a_1, \dots, a_n$ satisfying the hypotheses of Theorem 7.4 and such that $A = B_n, B = B_{n-1}, C = -A_n, D = -A_{n-1}$. If we limit the check to $n \leq 2$, nevertheless, we obtain a slightly more general version of Proposition 7.2.

Corollary 7.6. *Let f be the polynomial in (17). Then each of the following conditions is sufficient for f to have no root in B:*

i) $A = 1 = -D, B = 0$, *and the hypotheses of Proposition 7.2 are satisfied for* $a = C$,
ii) $B = 1, C = DA - 1$, *and the hypotheses of Theorem 7.4 are satisfied for* $a_0 = -D$ *and* $a_1 = A$.
iii) $C = (A-1)B^{-1}D + (D+1)B^{-1}$ *and the hypotheses of Theorem 7.4 are satisfied for* $a_0 = -(D+1)B^{-1}, a_1 = B$ *and* $a_2 = (A-1)B^{-1}$.

Proof. One can directly check that f has the form (13) for $n \in \{0, 1, 2\}$ if and only if its coefficients satisfy the constraints given in (i), (ii), or (iii). □

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