Pointwise Error Estimates for Boundary Control Problems on Polygonal Domains

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To my parents.

Abstract

This thesis deals with pointwise error estimates for finite element discretizations of boundary control problems on general polygonal domains, namely, the Neumann control problem and the Dirichlet control problem with constant control constraints.

In order to show the quasi-optimal convergence rate $h^2|\ln h|$ in the L^∞-Norm for the discretizations of the Neumann control problem, first, this rate is derived for the piecewise linear discretization of the Neumann boundary value problems. We achieve this goal by exploiting graded meshes which compensate the singular behavior of the solution in the vicinity of corner points. Best possible rates of convergence on quasi-uniform meshes are also shown.

For the numerical analysis of the Neumann optimal control problem two discretization strategies are considered, namely, the variational discretization and the postprocessing approach. In both cases the quasi-optimal rate on graded meshes and best possible rates on quasi-uniform meshes are shown.

The numerical analysis for the Dirichlet optimal control problem is performed only on quasi-uniform meshes. Best possible convergence order for the piecewise linear approximation of the control is obtained on convex domains.

All the theoretical results in this work are justified by numerical experiments.

Zusammenfassung

Diese Arbeit beschäftigt sich mit punktweisen Fehlerabschätzungen für Finite-Elemente-Diskretisierungen von Randsteuerungsproblemen auf allgemeinen polygonalen Gebieten. Insbesondere wird das Neumann-Randsteuerungsproblem sowie das Dirichlet-Randsteuerungsproblem mit konstanten Steuerbeschränkungen betrachtet.

Um die quasi-optimale Konvergenzrate $h^2|\ln h|$ in der L^∞-Norm für die Diskretisierung des Neumann-Rand- steuerungsproblems zu erhalten, zeigen wir diese Rate erst für die stückweise lineare Diskretisie- rung des Neumann-Randwertproblems. Dies erreichen wir mit Hilfe von graduell verfeinerten Netzen, die das singuläre Verhalten der Lösung in der Nähe von Eckpunkten kompensieren. Außerdem werden bestmögliche Raten auf quasi-uniformen Netzen gezeigt.

Für die numerische Analysis des Neumann-Randsteuerungsproblems betrachten wir zwei Diskretisierungsstrategien, die variationelle Diskretisierung sowie den Postprocessing-Zugang. In beiden Fällen können die quasi-optimale Rate auf graduell verfeinerten Netzen und die bestmöglichen Raten auf quasi-uniformen Netzen gezeigt werden.

Die numerische Analysis für das Dirichlet-Randsteuerungsproblem wird nur auf quasi-uniformen Netzen durchgeführt. Wir erhalten die bestmögliche Konvergenzordnung für die stückweise lineare Approximation der Steuerung auf konvexen Gebieten.

Die theoretischen Ergebnisse dieser Arbeit werden durch numerische Experimente bestätigt.

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Contents

CHAPTER 1

Introduction

The aim of this thesis is to provide sharp pointwise error estimates for the finite element (FE) approximations of boundary control problems governed by elliptic partial differential equations (PDEs) on general polygonal domains with pointwise control constraints.

Let us first give a short description of optimal control problems subject to PDEs. It is well-known that PDEs are used in order to describe a wide range of physical phenomena (processes), see e.g. [36]. Sometimes, one is interested in some optimization of a physical quantity by a certain influence on the underlying process. In other words one is interested in an optimal control problem governed by a PDE. Such problems in mathematical "language" can be formulated as

$$\min_{y\in Y, u\in U} J(y,u) \quad \text{subject to} \quad e(y,u)=0,$$

where y and u denote the state (physical quantity) and the control (influence on the physical process) being functions from the Banach spaces Y and U, respectively. The objective functional $J\colon Y\times U \to \mathbb{R}$ reflects the goals of the optimization, the state equation (PDE) $e(y,u) = 0$ describes the physical process and serves as a coupling between the state and the control. The theory of optimal control problems subject to PDEs goes back to early seventies, see the fundamental contribution by Lions [56]. The theory developed very rapidly and during the next forty years many applications of PDE constrained optimization have been considered, see e.g. [49, Chapter 4], [87, Chapter 1], [55, Part V] and [54, Part V].

In this thesis we consider linear-quadratic boundary control problems. The cost functional is a standard tracking type functional given by

$$J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}. \tag{1.1}$$

This means that we want to find the state y as close as possible to the desired state y_d in the $L^2(\Omega)$-sense taking into account the weighted control costs in the $L^2(\Gamma)$-sense. In the analytical setting the control costs can be interpreted as a regularization term. The computational domain Ω is a bounded two-dimensional domain with polygonal boundary Γ and m corner points. The weight parameter $\nu > 0$ is called regularization parameter.

We minimize the objective functional above subject to two boundary value problems. Namely, the Neumann boundary value problem

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega, \\ \partial_n y &= u \quad \text{on } \Gamma, \end{aligned} \tag{1.2}$$

and the Dirichlet boundary value problem

$$\begin{aligned} -\Delta y &= f \quad \text{in } \Omega, \\ y &= u \quad \text{on } \Gamma. \end{aligned} \tag{1.3}$$

Note that the optimal control problems we consider are called boundary control problems, since the control u is the Neumann datum in (1.2) and the Dirichlet trace in (1.3). More precisely, optimal control problems subject to (1.2) and (1.3) are called Neumann and Dirichlet control problems, respectively. We also set $f \equiv 0$ in the optimization process, since the right hand side does not play an important role in it. The set of admissible controls $U_{ad} \subset U$ in our case is given by

$$u \in U_{ad} := \left\{ u \in L^2(\Gamma) \colon u_a \le u \le u_b \text{ a.e. on } \Gamma \right\},$$

where $u_a, u_b \in \mathbb{R}$ and $u_a < u_b$. This means that the control has to fulfill so-called box constraints.

As mentioned before, our aim is to derive sharp pointwise error estimates for FE approximations of the boundary control problems described above. In particular we are mostly interested in error estimates for the control variable. In the present work we consider only piecewise linear FE approximations of the state variable and different approximation strategies for the boundary control. Moreover, we emphasize that numerical approaches to Neumann and Dirichlet boundary control problems are distinctly different. Hence, we consider these problems separately.

In order to get sharp error estimates for Neumann control problems, first, one has to derive error estimates of the same quality for state equation (1.2). Since the boundary Γ is polygonal, there occur singularities in the solution, which result in a reduced regularity. In case of two-dimensional boundary value problems these singularities are explicitly known, see e.g [50, 42, 29, 43, 68, 51, 52, 63]. In the neighborhood of each corner point $x^{(j)}$, $j = 1, \ldots, m$, the solution of (1.2) and (1.3) behaves like

$$\begin{aligned} &r_j^{\lambda_{j,k}} \cos(\lambda_{j,k}\varphi_j), \\ &r_j^{\lambda_{j,k}} \sin(\lambda_{j,k}\varphi_j), \end{aligned} \qquad \lambda_{j,k} := \frac{k\pi}{\omega_j}, \quad k \in \mathbb{N}, \tag{1.4}$$

respectively, where (r_j, φ_j) are polar coordinates centered at the corner point $x^{(j)}$, and λ_j is the singular exponent depending on the opening angle ω_j. It is easy to see that the regularity assumption $y \in W^{2,\infty}(\Omega)$ used in many contributions in general does not hold if the maximal interior angle of the domain is equal to or greater than $90°$, even if the input data are regular. However, the $W^{2,\infty}(\Omega)$-regularity is required to obtain a quasi-optimal convergence rate in the

$L^\infty(\Omega)$-norm on quasi-uniform meshes. In order to achieve the best possible convergence rate in arbitrary polygonal domains, we use locally refined meshes, see e.g. [74, 16, 78, 17, 75]. For a moment let us assume that we have only one opening angle ω grater than $90°$ with the corresponding singular exponent $\lambda := \pi/\omega$. We choose the triangulation $\mathcal{T}_h$ with maximal element diameter h of the underlying domain Ω such that each element T from this triangulation satisfies the mesh grading condition

$$h_T := \operatorname{diam} T \sim \begin{cases} h^{1/\mu} & \text{if } r_T = 0, \\ h r_T^{1-\mu} & \text{if } 0 < r_T < R, \\ h & \text{if } r_T > R, \end{cases}$$

where r_T denotes the distance to the corner point, R is the radius of refinement and $\mu \in (0, 1]$ is the refinement parameter depending on the singular exponent λ. We point out that for $\mu = 1$ the mesh is quasi-uniform.

Let us give a brief overview of some fundamental contributions to pointwise error estimates for elliptic problems, where convergence rates for piecewise linear FE approximations are considered. Most of those papers deal with approximations on quasi-uniform meshes. In [69] Nitsche showed the convergence rate of h for the homogeneous Dirichlet problem in convex polygonal domains for a right hand side function from $L^2(\Omega)$. Under the assumption that the solution belongs to $W^{2,\infty}(\Omega)$, Natterer [67] showed the convergence rate of $h^{2-\varepsilon}$ with arbitrary $\varepsilon > 0$. This result is improved by Nitsche [70] who showed the approximation order $h^2|\ln h|^{3/2}$. The sharp convergence rate $h^2|\ln h|$ has been finally shown by Frehse and Rannacher [38] and by Scott [83] for the homogeneous Dirichlet problem and the Neumann problem, respectively. In genereal polygonal domains, where the regularity might be reduced, Schatz and Wahlbin [80] showed the convergence rate $h^{\min(2,\lambda)-\varepsilon}$ for the homogeneous Dirichlet problem. In a further paper [81] they improved the convergence rate to $h^{2-\varepsilon}$ by refining the mesh with $\mu < \lambda/2$ towards the corners, which have opening angles grater than $90°$. However, in that reference the dependence on the input data is hidden in the generic constant. A further improvement for locally refined meshes under the same assumption is shown by Sirch [84], who obtained the rate $h^2|\ln h|^{3/2}$. Moreover, some norm of the input data appears explicitly on the right-hand side of the error estimate, which is required to derive error estimates for optimal control problems.

Our first main result reads as follows. Under the assumption that the mesh is refined according to $\mu < \lambda/2$ near the corner with $\omega \geq 90°$, we show the estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq c h^2 |\ln h| \tag{1.5}$$

for the Neumann problem, where $c > 0$ is a generic constant independent of the mesh size, and may have different values at each appearance. This estimate contains two improvements in comparison to the results known from the literature. First, even for less regular solutions but on locally refined meshes, we show that the exponent of the logarithmic term is equal to one. This exponent is known to be sharp for piecewise linear elements [45]. Second, we can specify the sufficient regularity of the input data on the right-hand side of the estimate. Therefore, we show that the constant c depends linearly on some (weighted) Hölder norm of f and u.

With slight modifications our result can be applied to the homogeneous Dirichlet problem as well. Although [11] claims an error estimate for the homogeneous Dirichlet boundary value problem with the rate $h^2|\ln h|$, there is a mistake in the proof of [11, Lemma 2.13] fixed in [84], which led to the error rate $h^2|\ln h|^{3/2}$. Using the techniques from Chapter 4, one can guarantee the reduced exponent of the logarithmic term for the homogeneous Dirichlet problem as well.

Now, let us go back to the optimal control problems, and first, discuss related papers on the numerical approximation of Neumann boundary control problems. In this thesis we are going to investigate two discretization strategies for Neumann control problems. The first one is called the variational discretization approach proposed by Hinze in [47]. The main feature of this approach is that the space of admissible controls is not discretized. Instead, by means of the first order optimality condition, the discretization of the state and the adjoint state induces a discretization of the control. Error estimates for the control variable in the natural $L^2(\Gamma)$-norm on quasi-uniform meshes have been derived in [48, 60, 7, 76, 8]. In the last reference Apel, Pfefferer and Rösch investigated error estimates for the control variable on graded triangulations of arbitrary polygonal domains, and showed the sharp rate $h^2|\ln h|^{3/2}$ provided that the mesh is refined according to $\mu < 1/4 + \lambda/4$. The previous result can also be found in the doctoral thesis of Pfefferer [76], where the quasi-optimal convergence rate $h^{\min(2,1/2+\lambda-\varepsilon)}|\ln h|^{3/2}$ on quasi-uniform meshes is also obtained. To the best of our knowledge, pointwise boundary estimates have been investigated only by Hinze and Matthes [48]. From this reference one can deduce the rate $h^{\min(2,\lambda-\varepsilon)}|\ln h|$ on convex domains only. Using the pointwise error estimate (1.5) as well as the $L^2(\Omega)$-error estimate for the state equation from [76] and the $L^2(\Gamma)$-error estimate for the control, we show the second main result of this thesis, namely, the quasi-optimal error estimate for the control variable

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq ch^2|\ln h|,$$

where $\bar{u}$ and $\bar{u}_h$ are the solutions of the continuous and discrete optimal control problems, respectively.

A further technique for treating optimal control problems is the postprocessing approach introduced by Meyer and Rösch for distributed control problems in [65]. The idea of this approach is to compute a fully discrete solution first, see e.g. [37, 40], using piecewise constant elements for the control, and afterwards introduce the piecewise linear postprocessed control $\tilde{u}_h$ which possesses better convergence properties. Extensions to Neumann boundary control problems can be found in [60, 7, 76, 8]. In [8] Apel, Pfefferer and Rösch derived estimates for the control of the same quality as in case of the variational discretization approach. To the best of our knowledge there are no contributions dealing with maximum norm estimates for the postprocessing approach applied to Neumann control problems. Using a similar idea of the proof as in case of variational discretization approach, we show the quasi-optimal rate of convergence for the postprocessed control as well

$$\|\bar{u} - \tilde{u}_h\|_{L^\infty(\Gamma)} \leq ch^2|\ln h|,$$

which is the third main result of this thesis.

It is also worth mentioning the following contributions dealing with local mesh refinement for optimal control problems. Apel, Rösch and G. Winkler [12] considered distributed control problems in arbitrary polygonal domains, and obtained optimal rates using graded trinagulations. G. Winkler in [90], using anisotropic finite elements, showed error estimates in general polyhedral domains. Sirch in [84] contributed with several applications of mesh grading to two- and three-dimensional problems. In particular, he obtained quasi-optimal pointwise error estimates for distributed control problems. Finally, M. Winkler [91] studied refinement strategies for Neumann control problems on polyhedral domains, see also [9, 10].

Now, we move to Dirichlet control problems, and the very first thing to emphasize is that the solution of the state equation (1.3) is from $H^{1/2}(\Omega)$, since we are looking for the optimal control from $L^2(\Gamma)$. Hence, it can not be sought in the weak sense. As a remedy, one can use the method of transposition also known as the very weak formulation, which to the best of our knowledge, is first introduced in [57]. However, it can be shown that in convex domains in the unconstrained case and on general polygonal domains in the constrained case the state is at least from $H^1(\Omega)$, which guarantees the existence of a weak solution.

In order to discretize Dirichlet control problems one uses piecewise linear finite elements for the state and for the control, since it is the Dirichlet trace of the state. In this thesis we consider only quasi-uniform triangulations due to a couple of reasons. First, there are some auxiliary results needed for our proofs, which so far are available on quasi-uniform meshes only. Second, despite the fact that error estimates on quasi-uniform meshes are in general easier to prove, with some minor modifications one can apply our proof techniques on graded triangulations as well.

In comparison to the Neumann control case, there are not that many scientific papers dealing with Dirichlet control problems, see [24, 30, 61, 59, 4]. All of them deal with estimates on quasi-uniform triangulations and none of them deal with pointwise estimates. The numerical investigation of Dirichlet control problems was launched by Casas and Raymond [24], where control constained problems subject to semilinear elliptic equations on convex polygonal domains were considered, and the convergence rate of $h^{\min(1,\lambda/2)-\varepsilon}$ for the control error in the $L^2(\Gamma)$-norm was shown. May, Rannacher and Vexler [61] considered unconstrained problems on convex domains, and showed optimal rates in a weaker norm, however, the rate shown in [24] remained unimproved. Finally, Apel, Mateos, Pfefferer and Rösch [4] obtained sharp error estimates for both constrained and unconstrained problems on arbitrary polygonal domains.

In this thesis we deal with Dirichlet control problems on convex domains only. Via the first order optimality conditions one can deduce that in the vicinity of convex corners the optimal control behaves like the normal derivative of the optimal adjoint state. The adjoint state in turn is a solution of the homogeneous Dirichlet problem, and possesses the singular behavior given in (1.4). This means that in case of unconstrained problems the optimal control is a bounded function only on convex domains. Hence, the pointwise error estimate is measurable only on such domains. The constrained case is more complex. In this case we can expect better rates on non-convex domains. However, due to some technical issues, this case is out of the scope of this thesis. Here we show sharp error estimates on quasi-uniform meshes for constrained problems on convex polygonal domains

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq ch^{\min(1,\lambda-1-\varepsilon)}.$$

We also want to mention several other contributions. Of, Phan and Steinbach [71, 72] investigated Dirichlet control problems with the energy regularization, i.e., $H^{1/2}(\Gamma)$ instead of $L^2(\Gamma)$ in (1.1), which helped to gain more regular solutions, and hence, better convergence rates. M. Winkler [91] developed numerical analysis of Neumann control problems with the energy regularization $H^{-1/2}(\Gamma)$, see also [15]. This approach, however, led to less regular solutions, and as a consequence worse rates of convergence in the $L^2(\Gamma)$-norm than in case of the standard $L^2(\Gamma)$-regularization. Here, we point out that the control from Neumann control problems with the energy regularization possesses similar singular behavior as in case of Dirichlet control problems with the standard regularization.

This thesis is outlined as follows. In Chapter 2 we give some basic definitions and notations. Moreover, we give definitions of function spaces that we use in this thesis and some relations among them. In Chapter 3 we discuss possible discretizations of the underlying domains, and discuss different interpolation approaches dictated by the smoothness of interpolated functions. Chapter 4 is devoted to the pointwise error estimates for the Neumann boundary value problem on graded and quasi-uniform meshes. In this chapter we also discuss the regularity of the solution and give an application of proven error estimates. In Chapter 5 we collect known results regarding homogeneous and inhomogeneous Dirichlet problems, improve the existing pointwise error estimate for the homogeneous Dirichlet problem on graded triangulations, and show an estimate on quasi-uniform triangulations. Chapter 6 deals with maximum norm error estimates for Neumann boundary control problems on graded and quasi-uniform meshes. Therein, we show sharp convergence results for two discretization approaches introduced above. In Chapter 7 we investigate $L^\infty(\Gamma)$-norm error estimates for Dirichlet control problems on quasi-uniform meshes. We show sharp estimates forr constrained problems on convex domains. Finally we want to emphasize that theoretical results shown in Chapters 4, 6 and 7 are confirmed by numerical experiments in Matlab concluding each of these chapters.

CHAPTER 2

Preliminaries

In this chapter we give some basic definitions and notations and collect information about function spaces that we use in this thesis. We begin with the definition of polygonal domains. In Section 2.1 we give definitions of classical continuous and Hölder continuous as well as classical Sobolev and Sobolev-Slobodetskij function spaces, and show relations between these spaces via several embedding results. In Section 2.2 we introduce the concept of weighted Sobolev and Hölder spaces concluding the section with some embedding theorems.

Definition 2.0.1. *A two-dimensional and bounded domain Ω is called polygonal, if its boundary Γ is a finite chain of m straight non-intersecting and non-overlapping line segments closing in a loop.*

Throughout this thesis we denote by

- $\mathcal{C} := \{1, \ldots, m\}$ the index set corresponding to the number of corners in the domain,
- $\{\Gamma_j\}_{j\in\mathcal{C}}$ the set of edges enumerated counter-clockwise,
- $\left\{x^{(j)}\right\}_{j\in\mathcal{C}}$ the set of corner points such that $x^{(j)} = \Gamma_j \cap \Gamma_{j+1}$ with $\Gamma_{m+1} = \Gamma_1$,
- $\{\omega_j\}_{j\in\mathcal{C}}$ the set of interior angles between Γ_j and Γ_{j+1} with $\Gamma_{m+1} = \Gamma_1$,
- (r_j, φ_j), polar coordinates centered at $x^{(j)}$ such that the points $(\cdot, \varphi_j = 0)$ lie on the edge Γ_{j+1}.

Moreover, we always assume that

- c is a positive generic constant,
- $\varepsilon, \varepsilon_1$ are positive and arbitrarily small numbers,
- $\vec{\varepsilon} = (\varepsilon, \ldots, \varepsilon)^T$, $\vec{\varepsilon}_1 = (\varepsilon_1, \ldots, \varepsilon_1)^T \in \mathbb{R}^m$.

2.1 Classical function spaces

We start with the definition of continuous and Hölder continuous function spaces, which describe the so-called classical solutions of partial differential equations. From now on we assume that Ω is a polygonal domain with boundary Γ, and $\overline{\Omega}$ denotes the closure of Ω.

Definition 2.1.1. *Let $k \in \mathbb{N}_0$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ be a multi-index and*

$$D^\alpha v(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} v(x)$$

denote the α-th derivative of v. The space of k-times continuously differentiable functions is denoted by $C^k(\overline{\Omega})$. The corresponding norm is defined by

$$\|v\|_{C^k(\overline{\Omega})} := \max_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha v(x)|.$$

Furthermore, let $\sigma \in (0, 1]$ denote the so-called Hölder exponent. The space of Hölder continuous functions is denoted by $C^{k,\sigma}(\overline{\Omega})$ and the corresponding norm is given by

$$\|v\|_{C^{k,\sigma}(\overline{\Omega})} := \|v\|_{C^k(\overline{\Omega})} + \max_{|\alpha| = k} \sup_{x,y \in \Omega} \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x - y|^\sigma}. \tag{2.1}$$

If $\sigma = 1$ the derivatives $D^\alpha v$ of order $|\alpha| = k$ are called Lipschitz continuous.

We proceed with the definition of Lebesgue, classical Sobolev and Sobolev-Slobodetskij spaces, which are used to describe weak solutions of partial differential equations.

Definition 2.1.2. *Let $p \in [1, \infty]$, the Lebesgue space $L^p(\Omega)$ is defined as the space of all Lebesgue-measurable functions v with the finite norm*

$$\|v\|_{L^p(\Omega)} := \begin{cases} \left(\displaystyle\int_\Omega |v(x)|^p dx \right)^{1/p}, & p \in [1, \infty), \\ \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|, & p = \infty. \end{cases}$$

Definition 2.1.3. *Let $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\alpha \in \mathbb{N}_0^2$. The Sobolev space $W^{k,p}(\Omega)$ is defined as the space of all function $v \in L^p(\Omega)$ with $D^\alpha v \in L^p(\Omega)$ for $|\alpha| \leq k$. The corresponding norms and seminorms are given by*

$$\|v\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p}, & p \in [1, \infty), \\ \max_{|\alpha| \leq k} \|D^\alpha v\|_{L^\infty(\Omega)}, & p = \infty \end{cases}$$

and

$$|v|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|=k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p}, & p \in [1,\infty), \\ \max_{|\alpha|=k} \|D^\alpha v\|_{L^\infty(\Omega)}, & p = \infty, \end{cases}$$

respectively.

Remark 2.1.4. *In the sequel we use the equivalence $W^{k,2}(\Omega) = H^k(\Omega)$, where the space $H^k(\Omega)$ is defined via the Fourier transform, see e.g. [92, 64, 85] for the definition and the equivalence result.*

Definition 2.1.5. *Let $k \in \mathbb{N}_0$, $\sigma \in (0,1)$, $p \in [1,\infty)$ and $\alpha \in \mathbb{N}_0^2$. The Sobolev-Slobodetskij space $W^{k+\sigma,p}(\Omega)$ is induced by the norm*

$$\|v\|_{W^{k+\sigma,p}(\Omega)} := \left(\|v\|^p_{W^{k,p}(\Omega)} + |v|^p_{W^{k+\sigma,p}(\Omega)} \right)^{1/p},$$

where the seminorm is given by

$$|v|_{W^{k+\sigma,p}(\Omega)} := \left(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|(D^\alpha v)(x_1) - (D^\alpha v)(x_2)|^p}{|x_1 - x_2|^{2+\sigma p}} dx_1 dx_2 \right)^{1/p}.$$

In the following lemmas we collect some classical embedding results which can be found in [1, Chapters 5, 6 and 7], [42, Section 1.4] or [2, Chapter 8].

Lemma 2.1.6. *Let $\mathcal{G}$ be a domain $\Omega \subset \mathbb{R}^2$ or a boundary part Γ_j, $j \in \mathcal{C}$. Moreover, let $\dim(\mathcal{G}) = n$.*

(i) Let $s, t \in \mathbb{R}$ with $s, t \geq 0$ and $p, q \in [1,\infty)$. Furthermore, let $s \geq t$ and $s - n/p = t - n/q$ hold. Then the continuous embedding

$$W^{s,p}(\mathcal{G}) \hookrightarrow W^{t,q}(\mathcal{G})$$

holds.

(ii) Let $s \in \mathbb{R}$ with $s \geq 0$, $k \in \mathbb{N}$, $p \in [1,\infty)$ and $\sigma \in (0,1)$. Furthermore, let $s - n/p = k + \sigma$ holds. Then the continuous embedding

$$W^{s,p}(\mathcal{G}) \hookrightarrow C^{k,\sigma}(\overline{\mathcal{G}})$$

is valid.

(iii) Let $s \in \mathbb{R}$ with $s \geq 0$, $k \in \mathbb{N}$, $p \in [1,\infty)$ and $\sigma \in [0,1]$. Furthermore, let $s - n/p > k + \sigma$ holds. Then the compact embedding

$$W^{s,p}(\mathcal{G}) \overset{c}{\hookrightarrow} C^{k,\sigma}(\overline{\mathcal{G}})$$

holds.

2.2 Weighted function spaces

In this section we introduce weighted Sobolev and Hölder spaces which serve to incorporate the singular behavior of solutions to boundary value problems (1.2) and (1.3). For definitions of these spaces we need to introduce some partitioning of the computational domain Ω.

For each $j \in \mathcal{C}$ we define the subdomains $\Omega_{R_j/i}$ by

$$\Omega_{R_j/i} := \{x \in \Omega \colon |x - x^{(j)}| < R_j/i\}$$

with radii $R_j/i > 0$, $i \in \{1, 2, 4, 8, 16, 32, 64\}$, centered at the corner point $x^{(j)}$. The radii R_j can be chosen arbitrarily with the only restriction that the circular sectors Ω_{R_j} do not overlap. Furthermore, for some technical reasons we require several subsets excluding the corners that we denoted by

$$\tilde{\Omega}^0 := \Omega \setminus \bigcup_{j=1}^{m} \Omega_{R_j/16}, \qquad \check{\Omega}^0 := \Omega \setminus \bigcup_{j=1}^{m} \Omega_{R_j/32}, \qquad \Omega^0 := \Omega \setminus \bigcup_{j=1}^{m} \Omega_{R_j/64}.$$

Definition 2.2.1. *Let $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\vec{\beta} \in \mathbb{R}^m$. The weighted Sobolev spaces $V^{k,p}_{\vec{\beta}}(\Omega)$ (spaces with homogeneous weights) and $W^{k,p}_{\vec{\beta}}(\Omega)$ (spaces with inhomogeneous weights) are defined as the set of all functions defined in Ω with the finite norms*

$$\|v\|_{V^{k,p}_{\vec{\beta}}(\Omega)} := \|v\|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} \|v\|_{V^{k,p}_{\vec{\beta}}(\Omega_{R_j})},$$
$$\|v\|_{W^{k,p}_{\vec{\beta}}(\Omega)} := \|v\|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} \|v\|_{W^{k,p}_{\vec{\beta}}(\Omega_{R_j})},$$

respectively. The weighted parts in the norms are defined by

$$\|v\|_{V^{k,p}_{\beta_j}(\Omega_{R_j})} := \begin{cases} \left(\sum\limits_{|\alpha| \le k} \|r_j(x)^{\beta_j - k + |\alpha|} D^\alpha v\|^p_{L^p(\Omega_{R_j})} \right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \max\limits_{|\alpha| \le k} \|r_j(x)^{\beta_j - k + |\alpha|} D^\alpha v\|_{L^\infty(\Omega_{R_j})}, & \text{if } p = \infty, \end{cases}$$

$$\|v\|_{W^{k,p}_{\beta_j}(\Omega_{R_j})} := \begin{cases} \left(\sum\limits_{|\alpha| \le k} \|r_j(x)^{\beta_j} D^\alpha v\|^p_{L^p(\Omega_{R_j})} \right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \max\limits_{|\alpha| \le k} \|r_j(x)^{\beta_j} D^\alpha v\|_{L^\infty(\Omega_{R_j})}, & \text{if } p = \infty. \end{cases}$$

The corresponding seminorms are given by

$$|v|_{V^{k,p}_{\vec\beta}(\Omega)} := |v|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} |v|_{V^{k,p}_{\vec\beta}(\Omega_{R_j})},$$
$$|v|_{W^{k,p}_{\vec\beta}(\Omega)} := |v|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} |v|_{W^{k,p}_{\vec\beta}(\Omega_{R_j})},$$

where the weighted seminorms are defined by setting $|\alpha| = k$ in the definition of the corresponding norms.

Definition 2.2.2. *Let $k \in \mathbb{N}_0$, $\sigma \in (0,1)$ and $\vec\beta \in \mathbb{R}^m$. The weighted Hölder spaces $N^{k,\sigma}_{\vec\beta}(\Omega)$ (spaces with homogeneous weights) and $C^{k,\sigma}_{\vec\beta}(\Omega)$ (spaces with inhomogeneous weights) are defined as the set of all functions defined in $\overline{\Omega} \setminus \{x^{(j)}\}_{j\in\mathcal{C}}$ with the finite norms*

$$\|v\|_{N^{k,\sigma}_{\vec\beta}(\Omega)} := \|v\|_{C^{k,\sigma}(\overline{\Omega}^0)} + \sum_{j=1}^{m} \|v\|_{N^{k,\sigma}_{\beta_j}(\Omega_{R_j})}, \tag{2.2}$$
$$\|v\|_{C^{k,\sigma}_{\vec\beta}(\Omega)} := \|v\|_{C^{k,\sigma}(\overline{\Omega}^0)} + \sum_{j=1}^{m} \|v\|_{C^{k,\sigma}_{\beta_j}(\Omega_{R_j})}, \tag{2.3}$$

where $C^{k,\sigma}(\overline{\Omega})$ are the classical Hölder spaces, and

$$\|v\|_{N^{k,\sigma}_{\beta_j}(\Omega_{R_j})} := \sum_{|\alpha|\le k} \|r_j(x)^{\beta_j-\sigma-k+|\alpha|} D^\alpha v\|_{C^0(\overline{\Omega}_{R_j})} + \langle v\rangle_{k,\sigma,\beta_j,\Omega_{R_j}},$$
$$\|v\|_{C^{k,\sigma}_{\beta_j}(\Omega_{R_j})} := \sum_{|\alpha|\le k} \|r_j(x)^{\max(0,\beta_j-\sigma-k+|\alpha|)} D^\alpha v\|_{C^0(\overline{\Omega}_{R_j})} + \langle v\rangle_{k,\sigma,\beta_j,\Omega_{R_j}}$$

with

$$\langle v\rangle_{k,\sigma,\beta_j,\Omega_{R_j}} := \sum_{|\alpha|=k} \sup_{x_1,x_2\in\Omega_{R_j}} \frac{|r_j(x_1)^{\beta_j}(D^\alpha v)(x_1) - r_j(x_2)^{\beta_j}(D^\alpha v)(x_2)|}{|x_1-x_2|^\sigma}.$$

We define the trace spaces of $V^{k,p}_{\vec\beta}(\Omega)$ and $W^{k,p}_{\vec\beta}(\Omega)$ according to [51, Section 6.2.1 and 7.3.1].

Definition 2.2.3. *Let $k \in \mathbb{N}$, $p \in [1,\infty]$ and $\vec\beta \in \mathbb{R}^m$. The traces of weighted Sobolev spaces $V^{k-1/p,p}_{\vec\beta}(\Gamma)$ (spaces with homogeneous weights) and $W^{k-1/p,p}_{\vec\beta}(\Gamma)$ (spaces with inhomogeneous weights) are defined as the set of all functions defined on Γ with finite norms*

$$\|v\|_{V^{k-1/p,p}_{\vec\beta}(\Gamma)} := \inf\left\{ \|u\|_{V^{k,p}_{\vec\beta}(\Omega)} : u \in V^{k,p}_{\vec\beta}(\Omega) \text{ and } u|_{\Gamma\setminus\{x^{(j)}\}_{j\in\mathcal{C}}} = v \right\},$$
$$\|v\|_{W^{k-1/p,p}_{\vec\beta}(\Gamma)} := \inf\left\{ \|u\|_{W^{k,p}_{\vec\beta}(\Omega)} : u \in W^{k,p}_{\vec\beta}(\Omega) \text{ and } u|_{\Gamma\setminus\{x^{(j)}\}_{j\in\mathcal{C}}} = v \right\}.$$

Definition 2.2.4. *Let $k \in \mathbb{N}_0$, $\sigma \in (0,1)$ and $\vec{\beta} \in \mathbb{R}^m$. The spaces $N^{k,\sigma}_{\vec{\beta}}(\Gamma)$ and $C^{k,\sigma}_{\vec{\beta}}(\Gamma)$ denote the trace spaces of $N^{k,\sigma}_{\vec{\beta}}(\Omega)$ and $C^{k,\sigma}_{\vec{\beta}}(\Omega)$, respectively, and are given by*

$$N^{k,\sigma}_{\vec{\beta}}(\Gamma) := \{v|_{\Gamma\setminus\{x^{(j)}\}_{j\in\mathcal{C}}} : v \in N^{k,\sigma}_{\vec{\beta}}(\Omega)\},$$
$$C^{k,\sigma}_{\vec{\beta}}(\Gamma) := \{v|_{\Gamma\setminus\{x^{(j)}\}_{j\in\mathcal{C}}} : v \in C^{k,\sigma}_{\vec{\beta}}(\Omega)\}.$$

In the sequel we collect some embedding results in the weighted Sobolev and Hölder spaces defined above. In the following $\mathcal{G}$ denotes either the domain Ω or its boundary Γ. The following embedding results for the weighed Sobolev spaces are given in [76, Lemma 2.30 and 2.29].

Lemma 2.2.5.

(i) Let $k \in \mathbb{N}_0$ and $1 \leq p < q \leq \infty$. Furthermore, let $\vec{\beta}, \vec{\beta}' \in \mathbb{R}^m$ satisfy

$$\beta_j + \frac{2}{p} > \beta_j' + \frac{2}{q} \qquad \forall j \in \mathcal{C}.$$

Then the continuous embedding

$$V^{k,p}_{\vec{\beta}'}(\mathcal{G}) \hookrightarrow V^{k,q}_{\vec{\beta}}(\mathcal{G})$$

holds.

(ii) Let $k \in \mathbb{N}_0$, $p \in [1,\infty]$. Furthermore, let $\vec{\beta}, \vec{\beta}' \in \mathbb{R}^m$ satisfy

$$1 + \beta_j \geq \beta_j' \qquad \forall j \in \mathcal{C}.$$

Then the continuous embedding

$$V^{k+1,p}_{\vec{\beta}'}(\mathcal{G}) \hookrightarrow V^{k,p}_{\vec{\beta}}(\mathcal{G})$$

holds.

Lemma 2.2.6.

(i) Let $k \in \mathbb{N}_0$ and $1 \leq p < q \leq \infty$. Furthermore, let $\vec{\beta}, \vec{\beta}' \in \mathbb{R}^m$ satisfy

$$\beta_j + \frac{2}{p} > \beta_j' + \frac{2}{q} \qquad \forall j \in \mathcal{C}.$$

Then the continuous embedding

$$W^{k,p}_{\vec{\beta}'}(\mathcal{G}) \hookrightarrow W^{k,q}_{\vec{\beta}}(\mathcal{G})$$

holds.

(ii) Let $l \in \mathbb{N}_0$, $p \in [1,\infty)$. Furthermore, let $\vec{\beta}, \vec{\beta}' \in \mathbb{R}^m$ satisfy

$$1 + \beta_j \geq \beta_j' \quad \text{and} \quad \beta_j' > -\frac{2}{p} \qquad \forall j \in \mathcal{C}.$$

Then the continuous embedding

$$W^{k+1,p}_{\vec{\beta}'}(\mathcal{G}) \hookrightarrow W^{k,p}_{\vec{\beta}}(\mathcal{G})$$

holds.

CHAPTER 3

Discretization

With this section we begin our investigation of pointwise estimates for finite element discretizations of the boundary value problems and the optimal control problems discussed in the Introduction. In the following sections we introduce the mesh discretization, finite element spaces and interpolation operators we use in this thesis.

3.1 Graded meshes and finite element discretization

We discretize the computational domain Ω by a family of graded triangulations $\{\mathcal{T}_h\}_{h>0}$ admissible in the sense of Ciarlet [25], i.e.,

- $\overline{\Omega} = \bigcup_{T\in\mathcal{T}} \overline{T}$,
- $T_1 \cap T_2 = \emptyset$ for all $T_1, T_2 \in \mathcal{T}_h$ with $T_1 \neq T_2$,
- $\overline{T}_1 \cap \overline{T}_2$ for all $T_1, T_2 \in \mathcal{T}_h$ with $T_1 \neq T_2$ is either the empty set or has a common edge or a common node.

The global mesh parameter is denoted by h, $h \le h_0 < 1$. Furthermore, throughout this thesis we consider only shape regular triangulations, i.e., there exists some constant $\kappa > 0$ such that

$$\frac{\rho_T}{h_T} \ge \kappa \qquad \forall T \in \mathcal{T}_h$$

holds for all $h \in (0, h_0]$, where h_T denotes the diameter of the smallest ball containing T and ρ_T the diameter of the largest ball contained in T, respectively.

We denote by

$$\mu_j \in (0,1], \qquad j \in \mathcal{C},$$

the mesh grading parameters which are collected in the vector

$$\vec{\mu} \in (0,1]^m.$$

The distance between a triangle $T \in \mathcal{T}_h$ and any corner point $x^{(j)}$ is defined by

$$r_{T,j} := \inf_{x \in T} |x - x^{(j)}|.$$

We assume that for $j = 1, \dots, m$ the element size $h_T := \operatorname{diam} T$ satisfies

$$\begin{aligned} c_1 h^{1/\mu_j} \leq h_T &\leq c_2 h^{1/\mu_j} && \text{if } r_{T,j} = 0, \\ c_1 h r_{T,j}^{1-\mu_j} \leq h_T &\leq c_2 h r_{T,j}^{1-\mu_j} && \text{if } 0 < r_{T,j} < R_j, \\ c_1 h \leq h_T &\leq c_2 h && \text{if } r_{T,j} > R_j, \end{aligned} \tag{3.1}$$

with some constants $c_1, c_2 > 0$ independent of h.

The triangulation $\mathcal{T}_h$ naturally induces a segmentation $\mathcal{E}_h$ of the boundary Γ. We define the distance between an element $E \in \mathcal{E}_h$ and a corner point $x^{(j)}$ by

$$r_{E,j} := \inf_{x \in E} |x - x^{(j)}|.$$

Due to conditions (3.1), the element size $h_E := \operatorname{diam} E$ satisfies

$$\begin{aligned} c_1 h^{1/\mu_j} \leq h_E &\leq c_2 h^{1/\mu_j} && \text{if } r_{E,j} = 0, \\ c_1 h r_{E,j}^{1-\mu_j} \leq h_E &\leq c_2 h r_{E,j}^{1-\mu_j} && \text{if } 0 < r_{E,j} < R_j, \\ c_1 h \leq h_E &\leq c_2 h && \text{if } r_{E,j} > R_j. \end{aligned} \tag{3.2}$$

Remark 3.1.1. *In case of $\mu_j = 1$ for some $j \in \mathcal{C}$ we end up with a quasi-uniform triangulation near the corresponding corner. If $\vec{\mu} = \vec{1}$ the whole mesh is quasi-unform. We also emphasize that the number of elements of $\mathcal{T}_h$ and $\mathcal{E}_h$ is of order h^{-2} and h^{-1}, respectively, which is independent of the choise of $\vec{\mu}$, see e. g. [13].*

For the finite element discretization of boundary value problems in Ω we use the space of piecewise linear and globally continuous ansatz functions in $\overline{\Omega}$, this is

$$V_h := \left\{ v_h \in C(\overline{\Omega}) : v_h|_T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}_h \right\}, \tag{3.3}$$

where $\mathcal{P}_1(T)$ denotes the space of linear polynomials on the element T.

In this thesis we denote by

- $\mathcal{I}_h$ the index set of nodal points of $\mathcal{T}_h$
- $\mathcal{I}_h^T$ the index set of nodal points of $T \in \mathcal{T}_h$
- $\{x_i\}_{i \in \mathcal{I}_h}$ the set of nodal points

- $\{\varphi_i\}_{i\in\mathcal{I}_h}$ the set of nodal basis functions — the so-called hat-functions — of V_h

For the finite element discretization of the control variable from Dirichlet control problems, see Chapter 7, we define the restriction of V_h to the boundary by

$$U_h := \{u_h \in C(\Gamma) : u_h|_E \in \mathcal{P}_1(E) \text{ for all } E \in \mathcal{E}_h\} \tag{3.4}$$

with $\mathcal{P}_1(E)$ denoting the space of linear polynomials on E.

We denote by

- $\mathcal{I}_h^\partial$ the index set of nodal points of $\mathcal{E}_h$
- $\mathcal{I}_h^{\partial,E}$ the index set of nodal points of $E \in \mathcal{E}_h$
- $\{x_i\}_{i\in\mathcal{I}_h^\partial}$ the set of boundary nodal points
- $\{\psi_i\}_{i\in\mathcal{I}_h^\partial}$ the set of nodal basis functions — hat-functions on the boundary — of U_h

Furthermore, for the finite element discretization of the control variable from Neumann control problems, see Chapter 6, we need the piecewise constant discretization of the boundary

$$U_h^0 := \{u_h \in L^\infty(\Gamma) : u_h|_E \in \mathcal{P}_0(T) \text{ for all } E \in \mathcal{E}_h\}\,, \tag{3.5}$$

where $\mathcal{P}_0(E)$ denotes the space of piecewise constant functions on the segment E.

For some technical reasons we need the following estimates concerning functions from discrete spaces.

Lemma 3.1.2. (**Inverse inequality**) *Let the underlying mesh be quasi-uniform ($\vec{\mu} = \vec{1}$) and $s \in [0,1]$, then the estimate*

$$|v_h|_{H^s(\Omega)} \le ch^{-s}\|v_h\|_{L^2(\Omega)} \qquad \forall v_h \in V_h \tag{3.6}$$

is valid.

Proof. The desired assertion can be deduced from [25, Theorem 17.2], [22, Theorem 4.5.11] or [85, Lemma 9.8] and a standard interpolation argument. □

Lemma 3.1.3. (**Discrete Sobolev inequality**) *Let $v_h \in V_h$ and $\vec{\mu} \in [0,1]$, then the estimate*

$$\|v_h\|_{L^\infty(\Omega)} \le c(1 + |\ln h|)^{1/2}\|v_h\|_{H^1(\Omega)}$$

holds.

The proof can be found in [22, Theorem 4.9.2].

3.2 Interpolation

This section is devoted to several error estimates for interpolation and quasi-interpolation operators used in this thesis. A standard proof technique of interpolation error estimates exploits a transformation to a reference element given e.g. by

- $\hat{E} := (0,1)$ for $n = 1$,
- $\hat{T}$ with vertices $(0,0)^{\mathrm{T}}$, $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$ for $n = 2$.

This approach guarantees that all generic constants depend only on the geometry of the element. For the further considerations let K be an element either from $\mathcal{E}_h$ or $\mathcal{T}_h$. We denote by F_K the affine linear transformation from $\hat{K}$ to a world element K. Moreover, via this transformation for each function $v\colon K \to \mathbb{R}$ we obtain a function $\hat{v}\colon \hat{K} \to \mathbb{R}$ defined by

$$\hat{v}(\hat{x}) := v(F_K(\hat{x})).$$

The transformation itself can be found in e.g. [25, Theorem 15.1].

Theorem 3.2.1. *Let $v \in W^{k,p}(K)$, $k \in \mathbb{N}_0$ and $p \in [1, \infty]$, then the estimates*

$$|v|_{W^{k,p}(K)} \leq c h_K^{-k} |K|^{1/p} |\hat{v}|_{W^{k,p}(\hat{K})}$$
$$|\hat{v}|_{W^{k,p}(\hat{K})} \leq c h_K^{k} |K|^{-1/p} |\hat{v}|_{W^{k,p}(\hat{K})}$$

hold.

In this thesis we need interpolation error estimates for the nodal (Lagrange) interpolant on quasi-uniform and graded meshes as well as for the Carstensen interpolant on quasi-uniform triangulations only.

3.2.1 Nodal interpolant

The nodal interpolant

$$I_h\colon C(\overline{\Omega}) \to V_h$$

is defined by

$$(I_h v)(x) := \sum_{i \in \mathcal{I}_h} v(x_i) \varphi_i(x)$$

which is equivalent to

$$(I_h v)(x) := \sum_{i \in \mathcal{I}_h^T} v(x_i) \varphi_i(x) \qquad \text{if } x \in \overline{T}.$$

Similarly, the nodal interpolant

$$I_h^\partial\colon C(\Gamma) \to U_h$$

is defined by

$$\left(I_h^\partial v\right)(x) := \sum_{i\in\mathcal{I}_h^\partial} v(x_i)\psi_i(x), \tag{3.7}$$

equivalent to

$$\left(I_h^\partial v\right)(x) := \sum_{i\in\mathcal{I}_h^{\partial,E}} v(x_i)\psi_i(x) \qquad \text{if } x\in\overline{E}.$$

It means that the operator I_h maps any continuous function v to a function from V_h such that $I_h v(x_i) = v(x_i)$ for all nodal points $\{x_i\}_{i\in\mathcal{I}_h}$. Obviously, the same holds also for the operator I_h^∂ applied to continuous functions for all nodal points $\{x_i\}_{i\in\mathcal{I}_h^\partial}$, respectively.

Elementwise interpolation error estimates for functions from weighted Sobolev spaces on graded meshes are given in [76, Section 3.2]. However, for our considerations in Chapter 4 we need stripwise interpolation error estimates stated in Lemma 4.2.11, which are proven via the elementwise estimates. Moreover, for the numerical investigation of Dirichlet control problems in Chapter 7 we need elementwise interpolation estimates on quasi-uniform triangulations. We collect these estimates in the following lemma. These results can be deduced from [25] or [22] and the standard operator-interpolation theory, see e.g. [22, Chapter 14].

Lemma 3.2.2. *Let the underlying mesh be quasi-uniform ($\vec{\mu} = \vec{1}$), and let $0 \le s \le t < \infty$ and $p, q \in [1,\infty]$. Then the interpolation error estimates*

$$\|v - I_h v\|_{W^{s,p}(T)} \le ch^{2/p-2/q+t-s}|v|_{W^{t,q}(T)},$$
$$\|v - I_h^\partial v\|_{W^{s,p}(E)} \le ch^{1/p-1/q+t-s}|v|_{W^{t,q}(E)}$$

hold, provided that $v \in W^{t,q}(T) \hookrightarrow W^{s,p}(T)$ or $v \in W^{t,q}(E) \hookrightarrow W^{s,p}(E)$.

3.2.2 Carstensen interpolant

In order to interpolate discontinuous functions one uses the concept of quasi-interpolation operators, see e.g. [26, 82, 23]. For the purposes dictated by the situation in Chapter 7 we use the so-called Carstensen interpolant

$$C_h \colon L^1(\Omega) \to V_h$$

introduced in [23] and defined by

$$\left(C_h v\right)(x) = \sum_{i\in\mathcal{I}_h} \pi_i(v)\varphi_i(x), \tag{3.8}$$

where $\pi_i(v) \in \mathbb{R}$ is given by

$$\pi_i(v) = \frac{(v,\varphi_i)_{L^2(\Omega)}}{(1,\varphi_i)_{L^2(\Omega)}} \qquad \forall i \in \mathcal{I}_h.$$

Lemma 3.2.3. *Let the underlying mesh be quasi-uniform ($\vec{\mu} = \vec{1}$). Moreover, let C_h be the Carstensen interpolant defined by (3.8) and $v \in H^{1/2}(\Omega)$. Then for all $s \in [0, 1/2]$ the error estimate*

$$\|v - C_h v\|_{H^s(\Omega)} \leq c h^{1/2-s} \|v\|_{H^{1/2}(\Omega)}$$

holds.

Proof. From the definition of the Carstensen interpolant, [23, Theorem 3.1(3.)] and the interpolation theory we get

$$\|C_h v\|_{H^t(\Omega)} \leq c\|v\|_{H^t(\Omega)} \qquad \forall t \in [0, 1], \tag{3.9}$$

provided that $v \in H^t(\Omega)$. Now, using the triangle inequality, the fact that $C_h p = p$ for all $p \in \mathcal{P}_0(\Omega)$ and stability property (3.9), we arrive at

$$\|v - C_h v\|_{H^r(\Omega)} \leq c\|v - p\|_{H^r(\Omega)} \leq c|v|_{H^r(\Omega)}, \qquad r \in [0, 1] \tag{3.10}$$

where in the last step we used [35, Theorem 6.1]. From (3.10) with $r = 0$, [23, Theorem 3.1(2.)] and an interpolation argument we get

$$\|v - C_h v\|_{L^2(\Omega)} \leq c h^{1/2} \|v\|_{H^{1/2}(\Omega)}. \tag{3.11}$$

Finally, the previous estimate and (3.10) with $r = 1/2$ yield the desired assertion. □

CHAPTER 4

Estimates for Neumann boundary value problems

The aim of this chapter is to derive sharp pointwise error estimates for the piecewise linear FE discretization of the Neumann problem

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma, \end{aligned} \tag{4.1}$$

where Ω is a general polygonal domain with boundary Γ. In order to achieve this aim, one has to guarantee the weighted $W^{2,\infty}(\Omega)$-regularity of the solution. Therefore, we start this chapter with a discussion of the regularity of the solution in weighted Sobolev spaces from Section 2.2. This together with the mesh refinement strategy from Section 3.1 are essential tools for recovery of optimal convergence rates. The proof itself expoits the so-called dyadic decomposition, see e.g. [81], in some neighborhood of each singular corner. This approach guarantees that the underlying mesh in each stripe of the decomposistion is quasi-uniform, and hence, properties of quasi-uniform triangulations can be used. In those corners, where the $W^{2,\infty}(\Omega)$-regularity is not violated, we apply the idea of a quasi-uniform mesh extension, see e.g. [32], and a global pointwise discretization error estimate from [83]. In the inner part of the domain we apply a local L^∞-norm error estimate [88, Theorem 10.1] and, exploiting higher regularity of the solution, standard interpolation error estimates, since the mesh in this subdomain is quasi-uniform.

This chapter has the following structure. In Section 4.1 we discuss the regularity and give some a priori estimates needed for the proof of the main result. Section 4.2 starts with a list of auxiliary discretization error estimates in different norms required in the main theorem of this chapter or other chapters. Moreover, in this section we prove the main results of this chapter, namely pointwise FE discretization error estimates on graded and quasi-uniform meshes. In Section 4.2.3 we give an application of the main results to semilinear elliptic problems, which can be an important tool for the numerical investigation of the Neumann control problems subject to a semilinear state equation. And finally, in the last section of this chapter, via numerical experiments, we confirm the theoretical results.

4.1 Regularity

This section is devoted to regularity results in weighted Sobolev spaces. But first, we give the weak formulation of problem (4.1) and discuss corner singularities in polygonal domains.

The variational formulation of problem (4.1) reads as follows:

Find $y \in H^1(\Omega)$ such that

$$a(y, v) = \langle f, v\rangle_\Omega + \langle g, v\rangle_\Gamma \quad \forall v \in V := H^1(\Omega), \tag{4.2}$$

where $a\colon V \times V \to \mathbb{R}$ is the bilinear form defined by

$$a(y, v) := \int_\Omega (\nabla y \cdot \nabla v + yv). \tag{4.3}$$

The existence and uniqueness of a solution $y \in V$ follows from the Lax–Milgram theorem, see e.g. [22, Theorem 2.7.7 and Remark 2.7.11], provided that $f \in [H^1(\Omega)]^*$ and $g \in [H^{1/2}(\Gamma)]^*$, where $[\,\cdot\,]^*$ denotes the dual space.

In the theory of FE method the $H^1(\Omega)$-regularity is not sufficient to provide optimal (sub-optimal) rates of convergence using piecewise linear approximations. Therefore, one has to discuss higher regularity of the solution to (4.2), which in the context of polygonal domains leads to a discussion of corner singularities.

It is a well known fact that, due to corner singularities, the solution of (4.2) might be less regular in the vicinity of the corner points $\{x^{(j)}\}_{j\in\mathcal{C}}$. Fundamental contributions to the study of corner singularities are given by Kondrat'ev [50], Grisvard [42, 43], Dauge [29], Nazarov and Plamenevsky [68], Kozlov, Maz'ya and Rossmann [51, 52]. A good summary on the derivation of singular solutions can be found in [91, Section 2.2]. In order to give a motivation to the forthcoming regularity results, here we discuss the singular solutions itself.

It is shown in e.g. [43, Chapter 2] that the solution of (4.2) in the neighborhood of each corner point $x^{(j)}$, $j \in \mathcal{C}$ possesses the following structure

$$y = \sum_{k\in\mathbb{Z}} c_k S_j^k(r_j, \varphi_j)$$

with the stress intensity factors $c_k \in \mathbb{R}$ and the singular functions

$$S_j^k(r_j, \varphi_j) := \begin{cases} r_j^{\lambda_{j,k}} \cos(\lambda_{j,k}\varphi_j), & \text{if } \lambda_{j,k} \notin \mathbb{Z}, \\ r_j^{\lambda_{j,k}} \left(\ln r_j \cos(\lambda_{j,k}\varphi_j) + \varphi_j \sin(\lambda_{j,k}\varphi_j)\right), & \text{if } \lambda_{j,k} \in \mathbb{Z}, \end{cases}$$

where $k \in \mathbb{N}$ and the singular exponents are defined by

$$\lambda_{j,k} := \frac{k\pi}{\omega_j}. \tag{4.4}$$

For our further considerations we define by

$$\lambda := \min_{j\in\mathcal{C}} \lambda_{j,1}$$

the strongest singularity of the solution. This definition is needed for error estimates, since we are always interested in worst-case estimates, and these are determined by λ for boundary value problems.

First, let us discuss the $H^2(\Omega)$-regularity. It is known from the classical FE theory that convergence rates of the error in the $L^2(\Omega)$- and $H^1(\Omega)$-norms are optimal for piecewise linear approximations if the solution is $H^2(\Omega)$-regular. However, it is easy to check that $S_j^1(r_j,\varphi_j) \notin H^2(\Omega)$ if $\omega_j > \pi$, and therefore, $y \notin H^2(\Omega)$ in general. As a remedy one can use the weighted Sobolev spaces, which guarantee the weighted $H^2(\Omega)$-regularity of the solution. This consideration motivates Lemma 4.1.1. Moreover, using the discretization strategy from Section 3.1, one can recover the optimal convergence rate on general polygonal domains, see Lemma 4.2.3 for $L^2(\Omega)$-norm error estimates or [76, Lemma 3.41] for both $L^2(\Omega)$- and $H^1(\Omega)$-norm error estimates.

Now, let us study the $W^{2,\infty}(\Omega)$-regularity of the solution. On the one hand, from Theorem 4.2.1 we can see that in order to get the quasi-optimal convergence rate for piecewise linear elements in the $L^\infty(\Omega)$-norm, the $W^{2,\infty}(\Omega)$-regularity of the solution has to be guaranteed or the mesh grading has to be exploited. On the other hand, it is easy to see that $S_j^1(r_j,\varphi_j) \notin W^{2,\infty}(\Omega)$ if $\omega_j \geq \pi/2$. It means that also in this case the weighted Sobolev spaces have to be exploited and the corresponding weights have to be greater than in case of the weighted $H^2(\Omega)$-regularity. A priori estimates concerning the weighted $W^{2,\infty}(\Omega)$-regularity are collected in Remark 4.1.2. As in case of $L^2(\Omega)$- and $H^1(\Omega)$-norm error estimates, using the mesh grading strategy from Section 3.1, we can show the quasi-optimal convergence rate in the maximum norm on general polygonal domains, which is the main result of this chapter, see Theorem 4.2.7.

Now, we recall an a priori estimate in the weighted $H^2(\Omega)$-norm. Comparable results can be found in e.g. [62], [93], [68, Section 4.5], [51, Section 7]. However, due to similarities of the considered problems as well as the notation we cite the result from [76, Lemma 3.11].

Lemma 4.1.1. *Let $\vec{\beta} \in [0,1)^m$ satisfy the condition*

$$1-\lambda_j < \beta_j, \quad \forall j \in \mathcal{C}.$$

For every $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$, the solution of problem (4.1) belongs to $W^{2,2}_{\vec{\beta}}(\Omega)$, and satisfies the a priori estimate

$$\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}\right).$$

In the following remark we discuss the regularity results in the weighted $W^{2,\infty}(\Omega)$-norm required for the forthcoming numerical analysis.

Remark 4.1.2. *There are several contributions dealing with regularity results in weighted Hölder spaces, which can be used to conclude the regularity result $y \in W^{2,\infty}_{\vec{\gamma}}(\Omega)$ with some $\vec{\gamma} \in [0,2)^m$.*

Using the weighted N-spaces introduced in Section 2.2, one can deduce from [68, Chapter 4, Section 5.5], [52, Theorem 1.4.5]

$$\|y\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} \leq c\left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}\right) \tag{4.5}$$

with the suitable weights $\delta \in [\sigma, 2+\sigma)^m$. As a conclusion, in [76, Lemma 3.13] it is shown that $y \in W^{2,\infty}_{\vec{\gamma}}(\Omega)$ under the assumption

$$\begin{cases} \vec{\gamma} \in [0,2)^m & \text{with} \quad \gamma_j > 2 - \lambda_j, \\ \vec{\delta} \in [\sigma, 2+\sigma)^m & \text{with} \quad \delta_j := \gamma_j + \sigma, \end{cases} \qquad j \in \mathcal{C}. \tag{4.6}$$

A further possibility is to use the weighted C-spaces, see Section 2.2. These spaces are more suitable for the Neumann problem, since $N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ does not even contain constant functions if $\delta_j < 1+\sigma$ for some $j \in \mathcal{C}$. However, to the best of our knowledge a direct proof of an estimate like (4.5) using weighted C-spaces is not available in the literature, but can be deduced with similar arguments as used in the proof of [76, Lemma 3.13]. Related results are also shown in [63, Theorem 8.3.1] for polyhedral domains ($n = 3$). Therein, the estimate

$$\|y\|_{C^{2,\sigma}_{\vec{\delta}}(\Omega)} \leq c\|f\|_{C^{0,\sigma}_{\vec{\delta}}(\Omega)} \tag{4.7}$$

under the assumption $g \equiv 0$ is proven, and with a trivial embedding we can conclude that $y \in W^{2,\infty}_{\vec{\gamma}}(\Omega)$ with $\vec{\gamma}$, $\vec{\delta}$ as in (4.6).

4.2 Discretization error estimates

The finite element solution of problem (4.2) is the element $y_h \in V_h$, which satisfies

$$a(y_h, v_h) = \langle f, v_h\rangle_\Omega + \langle g, v_h\rangle_\Gamma \quad \forall v_h \in V_h, \tag{4.8}$$

where the discrete test space is given by (3.3). The existence and uniqueness follows from the Lax-Milgram theorem.

4.2.1 Existing results

This section is devoted to existing a priori discretization error estimates for problem (4.1) which are used in this thesis. We begin with the global pointwise estimate of Scott [83]. We use this result in the proof of Theorem 4.2.7, locally, near corners with opening angles less than 90°.

Theorem 4.2.1. *Assume that the solution y of (4.2) belongs to $W^{2,\infty}(\Omega)$ and the domain Ω is convex. Let $y_h \in V_h$ be the solution of (4.8). Then, the finite element error can be estimated by*

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2|\ln h|\|y\|_{W^{2,\infty}(\Omega)}. \tag{4.9}$$

Remark 4.2.2. *Note that for* $\omega_j < 90°$, $j = 1, \dots, m$, *the condition* $y \in W^{2,\infty}(\Omega)$ *always holds, provided that* $f \in C^{0,\sigma}(\overline{\Omega})$ *and* $g \in C^{1,\sigma}_{pw}(\Gamma)$ *with* $\sigma \in (0, 1]$.

In what follows we give error estimates in the $L^2(\Omega)$-norm on both graded and quasi-uniform triangulations. These results are essential in the proof of the main results of this chapter and in the proof of one of the main results in Chapter 6. These results are given in [76, Lemma 3.41 and Corollary 3.42].

Lemma 4.2.3. *Let* y *and* y_h *be the solutions of* (4.2) *and* (4.8), *respectively. Moreover, let* $y \in W^{2,2}_{\vec{\beta}}(\Omega)$ *with*

$$\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1} - \vec{\mu}, \quad \vec{\beta} \ge \vec{0}.$$

Then the discretization error estimate

$$\|y - y_h\|_{L^2(\Omega)} \le ch^2 \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}$$

holds.

Lemma 4.2.4. *Let* y *and* y_h *be the solutions of* (4.2) *and* (4.8), *respectively, and* $\vec{\mu} = \vec{1}$ *(quasi-uniform meshes). Moreover, let* $y \in W^{2,2}_{\vec{\beta}}(\Omega)$ *with*

$$\vec{\beta} = \vec{1} - \vec{\lambda} + \vec{\varepsilon}, \quad \vec{\beta} \ge \vec{0} \quad \text{and} \quad 0 < \varepsilon < \lambda.$$

Then the discretization error can be estimated by

$$\|y - y_h\|_{L^2(\Omega)} \le ch^{2\min(1,\lambda-\varepsilon)} \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}.$$

The last error estimates that we recall in this section are the $L^2(\Gamma)$-norm estimates, which we apply in Chapter 6. The results can be found in [76, Theorem 3.48 and Corollary 3.49].

Theorem 4.2.5. *Let* y *and* y_h *be the solutions of* (4.2) *and* (4.8), *respectively. Moreover, let* $\varrho \in [0, 1/2]$, $\vec{\mu} \in (\varrho/2, 1]^m$ *and* $y \in W^{2,2}_{\vec{\beta}}(\Omega) \cap W^{2,\infty}_{\vec{\gamma}}(\Omega)$ *with*

$$\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1} - \vec{\mu}, \quad \vec{\beta} > \vec{0},$$

and let $\vec{\gamma}$ *fulfill one of the following conditions*

$$(i) \quad \vec{0} \le \vec{\gamma} < \vec{2} + \vec{\varrho} - 2\vec{\mu}, \quad \text{or} \quad (ii) \quad \vec{\gamma} = \vec{0} \quad \text{and} \quad \vec{\mu} \le \vec{1} \text{ if } \varrho = 0.$$

Then, the estimate

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{1+\varrho} \left(\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \right)$$

is valid.

Lemma 4.2.6. *Let y and y_h be the solutions of (4.2) and (4.8), respectively, and $\vec{\mu} = \vec{1}$ (quasi-uniform meshes). Moreover, let $\varrho \in [0, 1/2]$, $y \in W^{2,\infty}_{\vec{\gamma}}(\Omega)$ with*

$$\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\varepsilon}, \quad \vec{\gamma} \geq 0 \quad \text{and} \quad 0 < \varepsilon < \lambda.$$

Then, the estimate

$$\|y - y_h\|_{L^2(\Gamma)} \leq ch^{\min(2,\varrho+\lambda-\varepsilon)} |\ln h|^{1+\varrho} \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)}$$

holds.

4.2.2 Pointwise estimates for the Neumann problem

In this section we present the main results of this chapter. Namely, we show the quasi-optimal FE discretization error estimate in the maximum norm on graded meshes and the best possible error estimate in the same norm on quasi-uniform meshes.

Theorem 4.2.7. *Assume that y, the solution of (4.2), belongs to $W^{2,2}_{\vec{\beta}}(\Omega) \cap W^{2,\infty}_{\vec{\gamma}}(\Omega)$ with*

$$\vec{1} - \vec{\lambda} < \vec{\beta} \leq \vec{1} - \vec{\mu}, \quad \vec{\beta} \geq \vec{0},$$

and $\vec{\gamma}$ satisfying one of the following conditions

$$(i) \quad \vec{0} \leq \vec{\gamma} < \vec{2} - 2\vec{\mu} \quad \text{or} \quad (ii) \quad \vec{\gamma} = \vec{0} \text{ and } \vec{\mu} \leq \vec{1}$$

Then, the solution y_h of (4.8) fulfills the error estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \left(\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \right).$$

Corollary 4.2.8. *Let $\vec{\mu} = \vec{1}$ (quasi-uniform meshes). Assume that y, the solution of (4.2), belongs to $W^{2,\infty}_{\vec{\gamma}}(\Omega)$ with*

$$\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\varepsilon}_1, \quad \varepsilon_1 = \varepsilon/2 \quad \text{and} \quad 0 < \varepsilon < \lambda.$$

Then, the solution y_h of (4.8) fulfills the error estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^{\min(2,\lambda-\varepsilon)} |\ln h| \left(\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \right).$$

Remark 4.2.9. *The weighted $H^2(\Omega)$-regularity can be concluded from Lemma 4.1.1. Exploiting also a trivial embedding we obtain*

$$\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right) \leq c \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} \right).$$

Obviously, if the grading condition $\vec{\mu} < \vec{\lambda}$ is fulfilled, the range of feasible $\vec{\beta}$ is non-empty. In Remark 4.1.2 we discussed several ways how the $W^{2,\infty}_{\vec{\gamma}}(\Omega)$-regularity can be guaranteed. Assuming that f and g possess the regularity demanded by the right-hand side of (4.5), or (4.7) for $g \equiv 0$, we can conclude that $y \in W^{2,\infty}_{\vec{\gamma}}(\Omega)$ under the assumption $\gamma_j > 2 - \lambda_j$, compare also (4.6). Moreover, taking into account the assumptions on $\vec{\gamma}$, $\vec{\mu}$ used in Theorem 4.2.7, we arrive at the stronger grading condition $\vec{\mu} < \vec{\lambda}/2$.

The remainder of this section is devoted to the proof of Theorem 4.2.7 and Corollary 4.2.8. To prove Theorem 4.2.7 we distinguish among three cases, where the maximum can be attained:

1. If the maximum is attained in $\Omega_{R_j/16}$, with $\omega_j \geq 90°$, we can restore the best-possible convergence rate using weighted regularity results, i.e., $y \in W^{2,\infty}_{\gamma}(\Omega_{R_j})$ with $\gamma > 0$, and locally refined meshes, see Section 3.1. To prove the desired estimate, we apply a technique of Schatz and Wahlbin [81], this is, we introduce a dyadic decomposition of $\Omega_{R_j/16}$ around the singular point, and apply local estimates on each subset, where the meshes are locally quasi-uniform.

2. If the maximum is attained in $\Omega_{R_j/16}$ with $\omega_j < 90°$, we apply Theorem 4.2.1 for a localized problem near the corner. Local refinement in $\Omega_{R_j/16}$ is not needed in this case.

3. If the maximum is attained in $\tilde{\Omega}^0$, we use an interior maximum norm estimate, e.g. from [88, Theorem 10.1], and exploit higher regularity in the interior of the domain.

To prove Corollary 4.2.8 we follow the same steps as for the proof of Theorem 4.2.7, however the rates that we get are sub-optimal due to the quasi-uniformity of the underlying meshes.

In the following $x_0 \in \Omega$ denotes the point where $|y - y_h|(x)$ attains its maximum.

Case 1: $x_0 \in \Omega_{R_j/16}$ with $\omega_j \geq 90°$.

For the further analysis we assume that $x^{(j)}$ is located at the origin and $R_j = 1$. Furthermore, we suppress the subscript j such that $\Omega_R = \Omega_{R_j}$, $\mu = \mu_j$, etc. Analogous to [81] we introduce a dyadic decomposition of Ω_R,

$$\Omega_J := \{x \in \Omega : d_{J+1} \leq |x| \leq d_J\}, \quad J = 0, \dots, I,$$

with $d_J := 2^{-J}$ for $J = 0, \dots, I$ and $d_{I+1} = 0$. Obviously, there holds

$$\Omega_R = \bigcup_{J=0}^{I} \Omega_J, \tag{4.10}$$

see also Figure 4.1. The largest index I is chosen such that

$$d_I = c_I h^{1/\mu}$$

with a mesh-independent constant $c_I \geq 1$. This constant is specified in the proof of Lemma 4.2.15 where a kick-back argument is applied, which holds for sufficiently large c_I only. We hide this constant in the generic constant if there is no need in it.

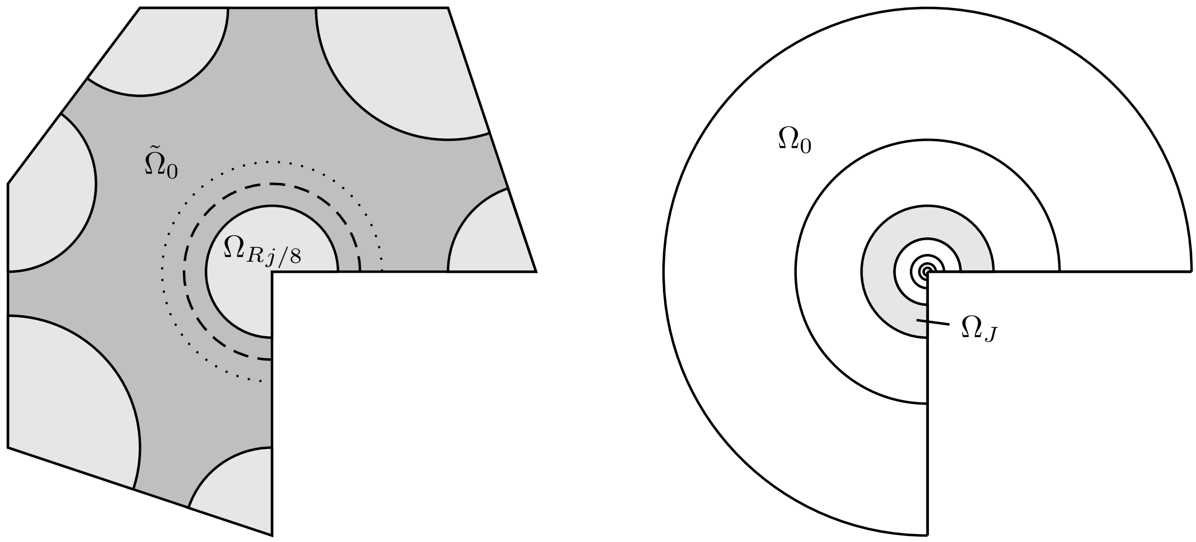

Figure 4.1: Partition of Ω in subdomains $\tilde{\Omega}^0$ and $\Omega_{R_j/8}$ (left) and partition of Ω_R in subdomains Ω_J (right)

We also introduce the extended domains Ω'_J for $J \geq 1$ and Ω''_J for $J \geq 2$ by

$$\Omega'_J := \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1},$$
$$\Omega''_J := \Omega'_{J-1} \cup \Omega'_J \cup \Omega'_{J+1}$$

with the obvious modifications for $J = I - 1, I$. Obviously, the meshes $\mathcal{T}_h$ are locally quasi-uniform with mesh sizes

$$h_T \sim h_J := hd_J^{1-\mu} \quad \text{if} \quad T \cap \Omega_J \neq \emptyset$$

for $J = 0, \dots, I$. This allows us to deduce local error estimates presented in the sequel.

For the convenience of the reader, we briefly summarize the forthcoming considerations. In Lemma 4.2.14 we show local L^∞-norm error estimates in the subsets Ω_J where the meshes are locally quasi-uniform. We distinguish between two cases. In subdomains Ω_J for $J > I - 2$, we can use a local pointwise error estimate from [88, Theorem 10.1], and for $J = I - 2, I - 1, I$ we use a different approach based on an inverse inequality which we prove in Lemma 4.2.10. Both techniques allow a local decomposition of the finite element error into a best-approximation term, for which we apply interpolation error estimates that we have in Lemma 4.2.11, and a pollution term. The pollution term arises as a weighted L^2-error which we discuss in Lemma 4.2.15. For the proof of this estimate we also require local error estimates in $H^1(\Omega_J)$ stated in Lemma 4.2.13 for the primal problem as well as the interpolation error estimates from Lemma 4.2.11 for the dual problem.

Before showing local interpolation error estimates, we show a stripwise inverse inequality.

Lemma 4.2.10. *For every $v_h \in V_h$ and every $J = I - 2, I - 1, I$ the estimate*

$$\|v_h\|_{L^\infty(\Omega_J)} \leq cd_J^{-1}\|v_h\|_{L^2(\Omega'_J)}$$

is valid.

Proof. We denote by T_* the element where $|v_h|$ attains its maximum within Ω_J and by $\hat{T}$ the reference element. By Theorem 3.2.1 and norm equivalences in finite-dimensional spaces we have

$$\|v_h\|_{L^\infty(\Omega_J)} \leq \|v_h\|_{L^\infty(T_*)} = \|\hat{v}_h\|_{L^\infty(\hat{T})} \leq c\|\hat{v}_h\|_{L^2(\hat{T})} \leq ch_{T_*}^{-1}\|v_h\|_{L^2(\Omega'_J)} \leq cd_J^{-1}\|v_h\|_{L^2(\Omega'_J)},$$

which proves the desired result, since $h_{T_*} \geq ch^{1/\mu} \sim d_I \sim d_J$ for $J = I-2, I-1, I$. □

In the following lemma stripwise interpolation error estimates on graded meshes are collected. These estimates can be found in Lemma [76, Lemma 3.58].

Lemma 4.2.11. *Let $p \in [2,\infty]$ and $l \in \{0,1\}$.*

(i) For $1 \leq J \leq I-2$ the estimates

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \leq ch^{2-l} d_J^{(2-l)(1-\mu)+1-2/p-\beta} |v|_{W^{2,p}_\beta(\Omega'_J)}, \tag{4.11}$$

$$\|v - I_h v\|_{L^\infty(\Omega_J)} \leq ch^{2-2/p} d_J^{(2-2/p)(1-\mu)-\beta} |v|_{W^{2,p}_\beta(\Omega'_J)} \tag{4.12}$$

are valid if $v \in W^{2,p}_\beta(\Omega'_J)$ with $\beta \in \mathbb{R}$.

(ii) Let $\theta_l := \max\{0, (3-l-2/p)(1-\mu)-\beta\}$ and $\theta_\infty := \max\{0, (2-2/p)(1-\mu)-\beta\}$. For $J = I-1, I$ the inequalities

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \leq cc_I^{\theta_l+1-2/p} h^{(3-l-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)}, \tag{4.13}$$

$$\|v - I_h v\|_{L^\infty(\Omega_J)} \leq cc_I^{\theta_\infty} h^{(2-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)} \tag{4.14}$$

hold if $v \in W^{2,p}_\beta(\Omega'_J)$ with $2/p - 2 < \beta < 2 - 2/p$.

Remark 4.2.12. *Lemma 4.2.11 remains valid when replacing Ω_J by Ω'_J and Ω'_J by Ω''_J, respectively. In this case the index range in part (i) is $J = 2,\dots,I-3$, and in part (ii) $J = I-2,\dots,I$.*

The next result is needed in the proof of Lemma 4.2.15, and is given in [76, Lemma 3.60] or [8, Lemma 3.9].

Lemma 4.2.13. *The following assertions hold:*

(i) For $2 \leq J < I-2$ the estimate

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c\Big(hd_J^{2-\mu-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y - y_h\|_{L^2(\Omega'_J)}\Big)$$

is valid for $y \in W^{2,\infty}_\beta(\Omega_R)$ with $\beta \in \mathbb{R}$.

(ii) For $J \geq I-2$ *the inequality*

$$\|y-y_h\|_{H^1(\Omega_J)} \leq c\Big(c_I^5 h^{(2-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big)$$

holds for $y \in W^{2,\infty}_\beta(\Omega_R)$ *with* $-2<\beta<2$.

In the next lemma we show local pointwise error estimates.

Lemma 4.2.14. *The following estimates hold:*

(i) For $2 \leq J < I-2$ *the estimate*

$$\|y-y_h\|_{L^\infty(\Omega_J)} \leq c\Big(h^2|\ln h|d_J^{2-2\mu-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big)$$

is valid for $y \in W^{2,\infty}_\beta(\Omega_R)$ *with* $\beta \in \mathbb{R}$.

(ii) For $J \geq I-2$ *the inequality*

$$\|y-y_h\|_{L^\infty(\Omega_J)} \leq c\Big(h^{(2-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big)$$

holds for $y \in W^{2,\infty}_\beta(\Omega_R)$ *with* $-2<\beta<2$.

Proof. Let us first consider the case $J < I-2$. In Theorem 10.1 and Example 10.1 from [88] the estimate

$$\|y-y_h\|_{L^\infty(\Omega_J)} \leq c\Big(|\ln h| \inf_{\chi\in V_h}\|y-\chi\|_{L^\infty(\Omega'_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big) \tag{4.15}$$

is stated. The desired result in case $2 \leq J < I-2$ follows from (4.15) and (4.12)

$$\|y-I_hy\|_{L^\infty(\Omega'_J)} \leq ch^2 d_J^{2-2\mu-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)}.$$

For the case $J = I, I-1, I-2$ we use the triangle inequality

$$\|y-y_h\|_{L^\infty(\Omega_J)} \leq \|y-I_hy\|_{L^\infty(\Omega_J)} + \|I_hy-y_h\|_{L^\infty(\Omega_J)}, \tag{4.16}$$

where the first term on the right hand side of (4.16) can be estimated using inequality (4.14)

$$\|y-I_hy\|_{L^\infty(\Omega_J)} \leq ch^{(2-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega'_J)}.$$

We estimate the second term on the right hand side of (4.16) by applying the inverse inequality from Lemma 4.2.10 and introducing the function y. We get

$$\|I_hy-y_h\|_{L^\infty(\Omega_J)} \leq cd_J^{-1}\|I_hy-y_h\|_{L^2(\Omega'_J)} \leq cd_J^{-1}\Big(\|y-I_hy\|_{L^2(\Omega'_J)} + \|y-y_h\|_{L^2(\Omega'_J)}\Big).$$

Finally, using (4.13) with $p=\infty$ we obtain

$$d_J^{-1}\|y-I_hy\|_{L^2(\Omega'_J)} \leq cd_J^{-1}h^{(3-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)} \leq ch^{2-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)},$$

where we used $d_J^{-1}h^{1/\mu} \leq d_I^{-1}h^{1/\mu} = c_I^{-1} \leq c$. □

The next lemma gives an estimate for the second terms on the right-hand sides of the estimates from Lemma 4.2.14, the so-called pollution terms. To cover all cases $J = 2, \dots, I$, we introduce the weight function $\sigma(x) := r(x) + d_I$, and easily confirm that these pollution terms are bounded by $\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})}$. To estimate this term, we can basically use the Aubin-Nitsche method involving a kick back argument. Similar results can be found in [8, Lemma 3.10], where $\|\sigma^{-\tau}(y - y_h)\|_{L^2(\Omega_{R/8})}$ with $\tau = 1/2$ is considered, or in [76, Lemma 3.61], where the previous estimate is generalized to exponents satisfying $1 - \lambda < \tau < 1$. However, for $\tau = 1$ some modifications have to be made.

Lemma 4.2.15. *Let $0 < \gamma \le 2 - 2\mu - 2\varepsilon$ with $\varepsilon > 0$ sufficiently small, then for $y \in W^{2,\infty}_\gamma(\Omega_R)$ and for the mesh size h small enough the estimate*

$$\|(r + d_I)^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \le c\Big(h^2|\ln h|\|y\|_{W^{2,\infty}_\gamma(\Omega_R)} + |\ln h|\|y - y_h\|_{L^2(\Omega_R)}\Big)$$

is valid.

Remark 4.2.16. *Note that the generic constant c depends on ε and tends to infinity for $\varepsilon \to 0$, which follows from estimate (4.31) in the proof below. This means that the grading parameter μ has to be chosen strictly smaller than $\lambda/2$, otherwise the generic constant will dominate in numerical experiments. From the experiments presented in Section 4.3 it can be observed that for $\lambda = 2/3$ it is enough to choose $\mu = 0.3$ in order to get proven convergence rate for arbitrary h values.*

Proof of Lemma 4.2.15. We define by χ the characteristic function, which is equal to one in $\Omega_{R/8}$ and equal to zero in $\Omega \setminus \mathrm{cl}(\Omega_{R/8})$. Next, we introduce the dual boundary value problem

$$\begin{aligned} -\Delta w + w &= \sigma^{-2}(y - y_h)\chi \quad && \text{in } \Omega, \\ \partial_n w &= 0 && \text{on } \Gamma \end{aligned} \tag{4.17}$$

with its weak formulation:

Find $w \in V$ such that

$$a(\varphi, w) = (\sigma^{-2}(y - y_h)\chi, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in V. \tag{4.18}$$

Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function which is equal to one in $\Omega_{R/8}$, $\operatorname{supp} \eta \subset \overline{\Omega}_{R/4}$, and $\partial_n \eta = 0$ on $\partial\Omega_R$, with $\|\eta\|_{W^{k,\infty}} \le c$ for $k \in \mathbb{N}_0$. By setting $\varphi = \eta v$ in (4.18) with some $v \in V$ one can show that $\tilde{w} = \eta w$ fulfills the equation

$$a_{\Omega_R}(v, \tilde{w}) = (\eta\sigma^{-2}(y - y_h)\chi - (\Delta\eta)w - 2\nabla\eta \cdot \nabla w, v)_{L^2(\Omega_R)} \quad \forall v \in V, \tag{4.19}$$

where the bilinear form $a_{\Omega_R} : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{R}$ is defined by

$$a_{\Omega_R}(\varphi, w) := \int_{\Omega_R} (\nabla\varphi \cdot \nabla w + \varphi w).$$

By this we get

$$\begin{aligned}\|\sigma^{-1}(y-y_h)\|^2_{L^2(\Omega_{R/8})} &= (\eta\sigma^{-2}(y-y_h)\chi, y-y_h)_{L^2(\Omega_R)} \\ &= a_{\Omega_R}(y-y_h,\tilde{w}) + ((\Delta\eta)w, y-y_h)_{L^2(\Omega_R)} + 2(\nabla\eta\cdot\nabla w, y-y_h)_{L^2(\Omega_R)} \\ &\le a_{\Omega_R}(y-y_h,\tilde{w}) + \Big(\|(\Delta\eta)w\|_{L^2(\Omega_R)} + 2\|\nabla\eta\cdot\nabla w\|_{L^2(\Omega_R)}\Big)\|y-y_h\|_{L^2(\Omega_R)} \\ &\le a_{\Omega_R}(y-y_h,\tilde{w}) + c\|w\|_{H^1(\Omega_R)}\|y-y_h\|_{L^2(\Omega_R)}. \end{aligned} \tag{4.20}$$

In the next step we are going to estimate the first term on the right-hand side of the previous inequality. Since $\tilde{w}$ is equal to zero in $\Omega_R\setminus\overline{\Omega}_{R/4}$ we can use the Galerkin orthogonality of $y-y_h$, i.e.,

$$a_{\Omega_R}(y-y_h, I_h\tilde{w}) = a(y-y_h, I_h\tilde{w}) = 0.$$

By this and an application of the Cauchy-Schwarz inequality we get

$$a_{\Omega_R}(y-y_h,\tilde{w}) = a_{\Omega_R}(y-y_h,\tilde{w}-I_h\tilde{w}) \le c\sum_{J=2}^{I}\|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)}. \tag{4.21}$$

Due to $\operatorname{supp}\eta\subset\overline{\Omega}_{R/4}$ there holds $\tilde{w}-I_h\tilde{w}\equiv 0$ in Ω_0 and Ω_1 provided that h is sufficiently small. Now, using the results from the previous lemmas and distinguishing between $2\le J\le I-3$ and $J=I-2, I-1, I$ we can estimate the terms on the right hand side of (4.21).

Let us discuss the case $2\le J\le I-3$ first. For the dual interpolation error we get from (4.11) with $\beta = 1+\varepsilon$ and $\beta = 1-\varepsilon$ the estimates

$$\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \le chd_J^{-\mu-\varepsilon}|\tilde{w}|_{W^{2,2}_{1+\varepsilon}(\Omega'_J)}, \tag{4.22}$$

$$\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \le chd_J^{-\mu+\varepsilon}|\tilde{w}|_{W^{2,2}_{1-\varepsilon}(\Omega'_J)}. \tag{4.23}$$

Both estimates are needed in the sequel. For the primal error using Lemma 4.2.13 we get

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\Big(hd_J^{\mu+2\varepsilon}|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big), \tag{4.24}$$

where we used the assumption on the weight γ. To get an estimate for (4.21) in case of $2\le J\le I-3$ we multiply the first term on the right hand side of (4.24) with the right hand side of (4.22), and the second term on the right hand side of (4.24) with (4.23). This leads to

$$\begin{aligned}&\|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \\ &\qquad\le ch^2d_J^{\varepsilon}|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}|\tilde{w}|_{W^{2,2}_{1+\varepsilon}(\Omega'_J)} + chd_J^{-1-\mu+\varepsilon}\|y-y_h\|_{L^2(\Omega'_J)}|\tilde{w}|_{W^{2,2}_{1-\varepsilon}(\Omega'_J)}.\end{aligned} \tag{4.25}$$

Now, we recall the local a priori estimates from [76, Lemma 3.9, (3.25)–(3.27)], which yield in our case

$$|\tilde{w}|_{W^{2,2}_{1+\varepsilon}(\Omega'_J)} \le \|F\|_{W^{0,2}_{1+\varepsilon}(\Omega''_J)} + \|\tilde{w}\|_{V^{1,2}_{\varepsilon}(\Omega''_J)} \tag{4.26}$$

with the right hand side of (4.19)

$$F := \eta\sigma^{-2}(y-y_h)\chi - (\Delta\eta)w - 2\nabla\eta\cdot\nabla w.$$

Inserting estimate (4.26) into (4.25) yields

$$\begin{aligned}\|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \le ch^2 d_J^{\varepsilon}|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}\Big(\|F\|_{W^{0,2}_{1+\varepsilon}(\Omega''_J)}+\|\tilde{w}\|_{V^{1,2}_{\varepsilon}(\Omega''_J)}\Big)\\ +chd_J^{-\mu+\varepsilon}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega'_J)}|\tilde{w}|_{W^{2,2}_{1-\varepsilon}(\Omega'_J)}\end{aligned} \tag{4.27}$$

for $J=2,\dots,I-3$, where we also used the fact that $d_J^{-1}\le c\sigma^{-1}(x)$ for $x\in\Omega'_J$.
For the sets Ω_J with $J=I-2,I-1,I$ we apply Lemma 4.2.13 for the primal problem to get

$$\|y-y_h\|_{H^1(\Omega_J)}\le c\Big(h^2|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}+d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big),$$

where we used that $h^{(2-\gamma)/\mu}<h^2$, and Lemma 4.2.11 for the dual problem to get

$$\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)}\le cc_I^{\max\{0,-\mu+\varepsilon\}}h^{\varepsilon/\mu}|\tilde{w}|_{W^{2,2}_{1-\varepsilon}(\Omega'_J)}.$$

Moreover, the Leibniz rule using $\|\eta\|_{W^{k,\infty}(\Omega_R)}\le c$, $k=0,1,2$, and the global a priori estimate of Lemma 4.1.1 with $\beta=1-\varepsilon$ yield the estimate

$$|\tilde{w}|_{W^{2,2}_{1-\varepsilon}(\Omega_R)}\le c\|w\|_{W^{2,2}_{1-\varepsilon}(\Omega_R)}\le c\|\sigma^{-2}(y-y_h)\|_{W^{0,2}_{1-\varepsilon}(\Omega_{R/8})}\le c\|\sigma^{-1-\varepsilon}(y-y_h)\|_{L^2(\Omega_{R/8})}. \tag{4.28}$$

Combining the last three estimates leads to

$$\begin{aligned}&\|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)}\\ &\le c\Big(h^{2+\varepsilon/\mu}|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}+c_I^{\max\{0,-\mu+\varepsilon\}}h^{\varepsilon/\mu}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega'_J)}\Big)\|\sigma^{-1-\varepsilon}(y-y_h)\|_{L^2(\Omega_{R/8})}\\ &\le c\Big(h^2|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}+c_I^{\max\{-\varepsilon,-\mu\}}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega'_J)}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})},\end{aligned} \tag{4.29}$$

where we exploited the property $\sigma^{-\varepsilon}\le d_I^{-\varepsilon}=c_I^{-\varepsilon}h^{-\varepsilon/\mu}$. Inserting inequalities (4.27) and (4.29) into (4.21) yields

$$\begin{aligned}&a_{\Omega_R}(y-y_h,\tilde{w})\\ &\le c\sum_{J=2}^{I-3}h^2d_J^{\varepsilon}|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}\Big(\|F\|_{W^{0,2}_{1+\varepsilon}(\Omega''_J)}+\|\tilde{w}\|_{V^{1,2}_{\varepsilon}(\Omega''_J)}\Big)\\ &+c\sum_{J=2}^{I-3}hd_I^{-\mu+\varepsilon}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega'_J)}|\tilde{w}|_{W^{2,2}_{1-\varepsilon}(\Omega'_J)}\\ &+c\sum_{J=I-2}^{I}\Big(h^2|y|_{W^{2,\infty}_{\gamma}(\Omega''_J)}+c_I^{-\varepsilon}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega'_J)}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})},\end{aligned} \tag{4.30}$$

where we used $d_J^{-\mu+\varepsilon}\le d_I^{-\mu+\varepsilon}$ and $\mu>\varepsilon$. The three sums in (4.30) are treated in a different way. For the first two terms we apply the discrete Cauchy-Schwarz inequality. Moreover, for the first sum we use a basic property of geometric series

$$\sum_{J=2}^{I-3}d_J^{2\varepsilon}\le\sum_{J=0}^{I-1}\Big(2^{-2\varepsilon}\Big)^J=\frac{1-2^{-2\varepsilon I}}{1-2^{-2\varepsilon}}\le c(1-d_I^{2\varepsilon})\le c, \tag{4.31}$$

with $c = (1-2^{-2\varepsilon})^{-1}$, which implies $\left(\sum_{J=2}^{I-3} d_J^{2\varepsilon}\right)^{1/2} \le c$. To treat the second sum in (4.30) we insert estimate (4.28) as well as use the properties $\sigma^{-\varepsilon} \le d_I^{-\varepsilon}$ and $hd_I^{-\mu} = c_I^{-\mu}$. This leads to

$$\begin{aligned} &a_{\Omega_R}(y-y_h,\tilde{w}) \\ &\le ch^2|y|_{W^{2,\infty}_\gamma(\Omega_R)}\Big(\|F\|_{W^{0,2}_{1+\varepsilon}(\Omega_R)} + \|\tilde{w}\|_{V^{1,2}_\varepsilon(\Omega_R)}\Big) \\ &+ cc_I^{-\mu}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})} \\ &+ c\Big(h^2|y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{-\varepsilon}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}, \end{aligned} \tag{4.32}$$

Due to the properties of the cut-off function η and $\|r^\varepsilon\|_{L^\infty(\Omega)} + \|r^{1+\varepsilon}\|_{L^\infty(\Omega)} \le c$, one can show that

$$\|F\|_{W^{0,2}_{1+\varepsilon}(\Omega_R)} \le c\Big(\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})} + \|w\|_{H^1(\Omega_R)}\Big).$$

To estimate the $V^{1,2}_\varepsilon(\Omega_R)$-norm of $\tilde{w}$ we use the trivial embedding

$$H^1(\Omega_R) \simeq W^{1,2}_0(\Omega_R) \hookrightarrow W^{1,2}_\varepsilon(\Omega_R),$$

and exploit that the norms in $W^{1,2}_\varepsilon(\Omega_R)$ and $V^{1,2}_\varepsilon(\Omega_R)$ are equivalent for $\varepsilon > 0$ [51, Theorem 7.1.1]. Taking also into account the Leibniz rule with $\|\eta\|_{W^{k,\infty}(\Omega_R)} \le c$, we obtain

$$\|\tilde{w}\|_{V^{1,2}_\varepsilon(\Omega_R)} \le c\|\tilde{w}\|_{H^1(\Omega_R)} \le c\|w\|_{H^1(\Omega_R)} \le c|\ln h|\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}. \tag{4.33}$$

The last step is confirmed at the end of this proof. Using the previous results, inequality (4.32) can be rewritten in the following way

$$\begin{aligned} &a_{\Omega_R}(y-y_h,\tilde{w}) \\ &\le c\Big(h^2|\ln h||y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{-\varepsilon}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}. \end{aligned} \tag{4.34}$$

By inserting (4.34) and the last step of (4.33) into (4.20), and dividing by $\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}$, we obtain

$$\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})} \le c\Big(h^2|\ln h||y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{-\varepsilon}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})} + |\ln h|\|y-y_h\|_{L^2(\Omega_R)}\Big).$$

Here, we used also $\sigma^{-1} = (r+d_I)^{-1} \le r^{-1} \le (R/8)^{-1} \le c$ if $r \ge R/8$. Finally, we get

$$\Big(1 - cc_I^{-\varepsilon}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})} \le c\Big(h^2|\ln h||y|_{W^{2,\infty}_\gamma(\Omega_R)} + |\ln h|\|y-y_h\|_{L^2(\Omega_R)}\Big).$$

By choosing the constant c_I large enough, such that $cc_I^{-\varepsilon} < 1$ holds, the desired result follows.

It remains to prove the last step in (4.33). A similar proof was already given in [84, Lemma 4.13]. There holds

$$\begin{aligned} \|w\|^2_{H^1(\Omega_R)} &\le a(w,w) = (\sigma^{-2}(y-y_h)\chi, w) = (\sigma^{-1}(y-y_h), \sigma^{-1}w)_{L^2(\Omega_{R/8})} \\ &\le \|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}\|\sigma^{-1}w\|_{L^2(\Omega_R)} \\ &\le c|\ln h|\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}\|w\|_{H^1(\Omega_R)}, \end{aligned} \tag{4.35}$$

where in the last step we used estimate (4.36) from [84, Lemma 4.13], which is also valid for the Neumann boundary value problem. □

From Lemma 4.2.14 with $0 < \gamma \le 2 - 2\mu - 2\varepsilon$ and Lemma 4.2.15 we conclude the local estimate

$$\|y - y_h\|_{L^\infty(\Omega_{R/8})} \le ch^2|\ln h|\|y\|_{W^{2,\infty}_\gamma(\Omega_R)} + c|\ln h|\|y - y_h\|_{L^2(\Omega_R)}. \tag{4.36}$$

Corollary 4.2.17. *Let $0 < \gamma < 2$ and $\mu = 1$, then for $y \in W^{2,\infty}_\gamma(\Omega_R)$ the inequality*

$$\|(r + d_I)^{-1}(y - y_h)\|_{L^2(\Omega_{R/8})} \le c\Big(h^{2-\gamma-\varepsilon_1}\|y\|_{W^{2,\infty}_\gamma(\Omega_R)} + |\ln h|\|y - y_h\|_{L^2(\Omega_R)}\Big)$$

holds.

Proof. The proof for this inequality is structured similarly to the proof of the previous lemma, but is simpler. For the sake of completeness we give it here, however, we skip the identical steps.

We want to estimate inequality (4.21) in the case $\mu = 1$. As in the proof of Lemma 4.2.15 we start with the case $2 \le J \le I - 3$. For the dual interpolation error we use (4.11) only with $\beta = 1 - \varepsilon_1$

$$\|\tilde w - I_h\tilde w\|_{H^1(\Omega_J)} \le chd_J^{-1+\varepsilon_1}|\tilde w|_{W^{2,2}_{1-\varepsilon_1}(\Omega'_J)}, \tag{4.37}$$

and for the primal interpolation error we get from Lemma 4.2.13

$$\|y - y_h\|_{H^1(\Omega_J)} \le c\Big(hd_J^{1-\gamma}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y - y_h\|_{L^2(\Omega'_J)}\Big). \tag{4.38}$$

By means of the previous two estimates we conclude for $2 \le J \le I - 3$

$$\begin{aligned}
&\|y - y_h\|_{H^1(\Omega_J)}\|\tilde w - I_h\tilde w\|_{H^1(\Omega_J)} \\
&\qquad \le c\Big(h^2 d_J^{-\gamma+\varepsilon_1}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + hd_J^{-1+\varepsilon_1}\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)}\Big)|\tilde w|_{W^{2,2}_{1-\varepsilon_1}(\Omega'_J)} \\
&\qquad \le c\Big(h^{2-\gamma+\varepsilon_1}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + hd_J^{-1+\varepsilon_1}\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)}\Big)|\tilde w|_{W^{2,2}_{1-\varepsilon_1}(\Omega'_J)}
\end{aligned} \tag{4.39}$$

where we used that $d_J^{-1} \le c\sigma^{-1}(x)$ for $x \in \Omega'_J$ and $d_J^{-\gamma+\varepsilon_1} \le d_I^{-\gamma+\varepsilon_1} \le ch^{-\gamma+\varepsilon_1}$ for $\varepsilon_1 > 0$ small enough, since $0 < \gamma < 2$ and $d_I = c_I h$.

For $J = I, I-1, I-2$ we get from Lemma 4.2.13 for the primal variables

$$\|y - y_h\|_{H^1(\Omega_J)} \le c\Big(h^{2-\gamma}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y - y_h\|_{L^2(\Omega'_J)}\Big),$$

and from Lemma 4.2.11 for the dual ones

$$\|\tilde w - I_h\tilde w\|_{H^1(\Omega_J)} \le ch^{\varepsilon_1}|\tilde w|_{W^{2,2}_{1-\varepsilon_1}(\Omega'_J)}.$$

Combining the last two estimates together we obtain

$$\begin{aligned}
\|y - y_h\|_{H^1(\Omega_J)}\|\tilde w - I_h\tilde w\|_{H^1(\Omega_J)} \le c\Big(&h^{2-\gamma+\varepsilon_1}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} \\
&+ h^{\varepsilon_1}\|\sigma^{-1}(y - y_h)\|_{L^2(\Omega'_J)}\Big)|\tilde w|_{W^{2,2}_{1-\varepsilon_1}(\Omega'_J)},
\end{aligned} \tag{4.40}$$

where we used that $d_J^{-1} \le c\sigma^{-1}(x)$ for $x \in \Omega'_J$.

Inequalities (4.39) and (4.40) together yield

$$a_{\Omega_R}(y-y_h,\tilde{w}) \le c\sum_{J=2}^{I}\Big(h^{2-\gamma+\varepsilon_1}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + h^{\varepsilon_1}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega'_J)}\Big)|\tilde{w}|_{W^{2,2}_{1-\varepsilon_1}(\Omega'_J)},$$

where we used that $hd_J^{-1+\varepsilon_1} \le hd_I^{-1+\varepsilon_1} \le ch^{\varepsilon_1}$. Now, using the fact that $\sum_{J=2}^{I} 1 \sim |\ln h|$ and the discrete Cauchy-Schwarz inequality, we obtain

$$a_{\Omega_R}(y-y_h,\tilde{w}) \le c\Big(h^{2-\gamma+\varepsilon_1}|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + h^{\varepsilon_1}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\Big)|\tilde{w}|_{W^{2,2}_{1-\varepsilon_1}(\Omega_R)}.$$

By applying estimate (4.28) we conclude

$$\begin{aligned}
a_{\Omega_R}(y-y_h,\tilde{w}) &\\
&\le c\Big(h^{2-\gamma+\varepsilon_1}|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + h^{\varepsilon_1}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\Big)\|\sigma^{-1-\varepsilon_1}(y-y_h)\|_{L^2(\Omega_{R/8})}\\
&\le c\Big(h^{2-\gamma}|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{-\varepsilon_1}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})},
\end{aligned}$$

where we used that $\sigma^{-\varepsilon_1} \le d_I^{-\varepsilon_1} \le c_I^{-\varepsilon_1}h^{-\varepsilon_1}$. Finally, using the fact that $|\ln h|^{1/2} \le ch^{-\varepsilon_1}$ for h small enough we arrive at

$$a_{\Omega_R}(y-y_h,\tilde{w}) \le \Big(h^{2-\gamma-\varepsilon_1}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^{-\varepsilon_1}\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_R)}\Big)\|\sigma^{-1}(y-y_h)\|_{L^2(\Omega_{R/8})}. \tag{4.41}$$

The remaining steps of the proof are identical with the part of the proof of Lemma 4.2.15 starting with estimate (4.34). □

Lemma 4.2.14 together with the estimate $|\ln h| d_J^{-\gamma} \le ch^{-\gamma-\varepsilon_1}$ and Corollary 4.2.17 imply the local estimate

$$\|y-y_h\|_{L^\infty(\Omega_{R/8})} \le ch^{2-\gamma-\varepsilon_1}\|y\|_{W^{2,\infty}_\gamma(\Omega_R)} + c|\ln h|\|y-y_h\|_{L^2(\Omega_R)}, \tag{4.42}$$

which holds on quasi-uniform meshes ($\vec{\mu} = \vec{1}$).

Case 2: $x_0 \in \Omega_{R_j/16}$ with $\omega_j < 90°$.

We prove the result under the assumption that the mesh is quasi-uniform within Ω_{R_j}. Note that the convergence rate is not reduced if mesh refinement is still used. We assume that the corner $x^{(j)}$ is located in the origin, and drop the subscript j as in the previous case. The basic idea is to apply Theorem 4.2.1 in a local fashion, which can be realized with a technique from e.g. [32, Theorem 1]. First, we introduce a polygonal domain $\hat{\Omega}_R$ (see Figure 4.2). Note that the distance l is positive for h small enough, which allows us to extend the mesh

$$\mathcal{T}_h|_{\Omega_{R/2}} := \{T \in \mathcal{T}_h : T \cap \Omega_{R/2} \neq \emptyset\}$$

quasi-uniformly to an exact triangulation $\hat{\mathcal{T}}_h$ of $\hat{\Omega}_R$. We also introduce a smooth cut-off function η_1 such that $\eta_1 = 1$ in $\Omega_{R/2}$ and $\mathrm{dist}(\operatorname{supp}\eta_1, \partial\hat{\Omega}_R \setminus \Gamma) \ge c > 0$. For our further considerations we define the Ritz projection of $\tilde{y} = \eta_1 y$ as follows. Let

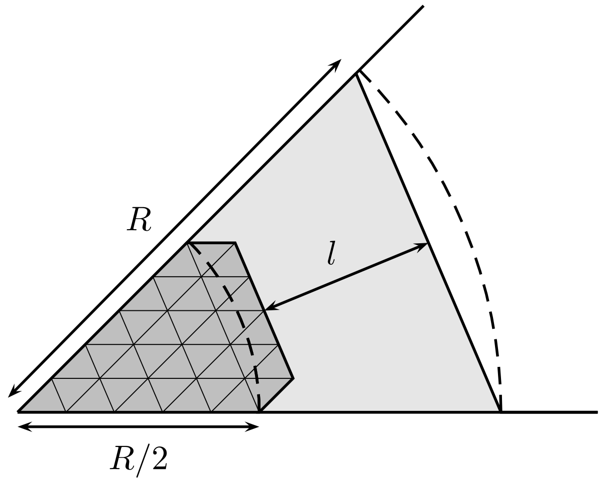

Figure 4.2: $\mathcal{T}_h|_{\Omega_{R/2}}$ - dark gray domain, $\hat{\Omega}_R$ - dark gray and light gray domains

$$V_h(\hat{\mathcal{T}}_h) := \{v_h \in C(\mathrm{cl}\,\hat{\Omega}_R) : v_h|_T \in \mathcal{P}_1 \text{ for all } T \in \hat{\mathcal{T}}_h\}$$

denote the space of ansatz functions with respect to the new triangulation $\hat{\mathcal{T}}_h$. The function $\tilde{y}_h \in V_h(\hat{\mathcal{T}}_h)$ is the unique solution of

$$a(\tilde{y} - \tilde{y}_h, v_h) = 0 \quad \forall v_h \in V_h(\hat{\mathcal{T}}_h). \tag{4.43}$$

As $y = \tilde{y}$ in $\Omega_{R/8}$, we get from the triangle inequality

$$\|y - y_h\|_{L^\infty(\Omega_{R/8})} \leq \|\tilde{y} - \tilde{y}_h\|_{L^\infty(\Omega_{R/8})} + \|\tilde{y}_h - y_h\|_{L^\infty(\Omega_{R/8})}. \tag{4.44}$$

Since $\tilde{y} \in W^{2,\infty}(\hat{\Omega}_R)$, in order to estimate the first term in (4.44) we apply Theorem 4.2.1

$$\|\tilde{y} - \tilde{y}_h\|_{L^\infty(\Omega_{R/8})} \leq ch^2|\ln h|\|\tilde{y}\|_{W^{2,\infty}(\Omega_R)} \leq ch^2|\ln h|\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)}, \tag{4.45}$$

where we used the Leibniz rule in the last step. Note that it is possible to construct η_1 such that $\|\eta_1\|_{W^{k,\infty}(\Omega)} \leq c$ for $k = 0, 1, 2$. Next, we confirm that the function $\tilde{y}_h - y_h$ is discrete harmonic in $\Omega_{R/2}$, this is, for every $v_h \in V_h$ with $\operatorname{supp} v_h \subset \overline{\Omega}_{R/2}$ there holds

$$a(\tilde{y}_h - y_h, v_h) = a(\tilde{y} - y, v_h) = 0.$$

This is a consequence of $\eta_1 \equiv 1$ (and hence $y = \tilde{y}$) on $\Omega_{R/2}$, as well as $v_h \equiv 0$ in $\Omega \setminus \Omega_{R/2}$. An application of the discrete Sobolev inequality from Lemma 3.1.3 and the discrete Caccioppoli type estimate from [31, Lemma 3.3] then yield

$$\|\tilde{y}_h - y_h\|_{L^\infty(\Omega_{R/8})} \leq c|\ln h|^{1/2}\|\tilde{y}_h - y_h\|_{H^1(\Omega_{R/4})} \leq cd^{-1}|\ln h|^{1/2}\|\tilde{y}_h - y_h\|_{L^2(\Omega_{R/2})},$$

where $d = \mathrm{dist}(\partial\Omega_{R/2} \setminus \Gamma, \partial\Omega_{R/4} \setminus \Gamma)$ and by construction $d = 1/4$ (remember $R = 1$). Next, we use the triangle inequality and the fact that $y = \tilde{y}$ in $\Omega_{R/2}$

$$\begin{aligned}\|\tilde{y}_h - y_h\|_{L^\infty(\Omega_{R/8})} &\leq c|\ln h|^{1/2}\Big(\|\tilde{y} - \tilde{y}_h\|_{L^2(\hat{\Omega}_R)} + \|y - y_h\|_{L^2(\Omega)}\Big) \\ &\leq ch^2|\ln h|^{1/2}\|\tilde{y}\|_{H^2(\hat{\Omega}_R)} + c|\ln h|^{1/2}\|y - y_h\|_{L^2(\Omega)}, \end{aligned} \tag{4.46}$$

where we used a standard L^2-error estimate in the last step. Estimates (4.45) and (4.46) finally yield the local estimate

$$\|y - y_h\|_{L^\infty(\Omega_{R/8})} \le ch^2|\ln h|\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + |\ln h|\|y - y_h\|_{L^2(\Omega)}. \tag{4.47}$$

Case 3: $x_0 \in \tilde{\Omega}^0$.

It remains to obtain an estimate if $|y - y_h|$ attains its maximum at a point $x_0 \in \tilde{\Omega}^0$. Note that the auxiliary domains used in the following steps are defined in Section 2.2. We use [88, Theorem 10.1] with $s = 0$ to get

$$\|y - y_h\|_{L^\infty(\tilde{\Omega}^0)} \le c\Big(|\ln h|\|y - I_h y\|_{L^\infty(\check{\Omega}^0)} + \|y - y_h\|_{L^2(\check{\Omega}^0)}\Big).$$

Since the domain $\check{\Omega}^0 \subset \Omega^0$ has a constant and positive distance to the corners of Ω, we conclude with standard interpolation error estimates from Lemma 3.2.2

$$\begin{aligned}\|y - y_h\|_{L^\infty(\tilde{\Omega}^0)} &\le c\Big(h^2|\ln h|\|y\|_{W^{2,\infty}(\Omega^0)} + \|y - y_h\|_{L^2(\check{\Omega}^0)}\Big)\\ &\le c\Big(h^2|\ln h|\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y - y_h\|_{L^2(\check{\Omega}^0)}\Big).\end{aligned} \tag{4.48}$$

Finally, we are in a position to prove the main results.

Proof of Theorem 4.2.7. Estimates (4.36), (4.47) and (4.48) result in

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^2|\ln h|\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + |\ln h|\|y - y_h\|_{L^2(\Omega)}.$$

For the remaining term on the right-hand side we apply Lemma 4.2.3 and conclude the desired result. □

Proof of Corollary 4.2.8. From estimate (4.42) together with $\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\varepsilon}_1$ and $\vec{\gamma} \ge \vec{0}$, estimates (4.47) and (4.48) we get

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^{\min(2,\lambda-\varepsilon)}|\ln h|\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + |\ln h|\|y - y_h\|_{L^2(\Omega)}.$$

For the right hand side term we apply Lemma 4.2.4 and by this conclude the desired estimate. □

4.2.3 Pointwise estimates for semilinear elliptic problems

The aim of this section is to apply the results from the previous section to certain nonlinear problems. Such a pointwise error estimate can be then applied in the numerical analysis of the Neumann control problem subject to a semilinear problem. However, this is out of the scope of this thesis. Here, we investigate the semilinear problem

$$\begin{aligned}-\Delta y + d(\cdot, y) &= f \quad \text{in } \Omega,\\ \partial_n y &= g \quad \text{on } \Gamma,\end{aligned} \tag{4.49}$$

where we assume that the input data f and g are sufficiently regular such that the solution y belongs to $W^{2,2}_{\vec{\beta}}(\Omega) \cap W^{2,\infty}_{\vec{\gamma}}(\Omega)$ with $\vec{\beta} \in [0,1)^m$ and $\vec{\gamma} \in [0,2)^m$. Under the assumptions, stated below, this regularity is shown in e.g. [76, Corollary 3.26], provided that f and g belong to some weighted N-space. Note that the forthcoming theory have been already done in [76], where only sub-optimal rates are obtained.

The nonlinear part $d\colon \Omega \times \mathbb{R} \to \mathbb{R}$ fulfills the following assumptions:

(A1) The function $d = d(x,y)$ is measurable with respect to $x \in \Omega$ for all $y \in \mathbb{R}$, and differentiable with respect to y for almost all $x \in \Omega$. Furthermore, the function d is monotonically increasing with respect to y, i.e.,

$$\frac{\partial d}{\partial y}(x,y) \geq 0 \quad \text{for a. a. } x \in \Omega \text{ and } y \in \mathbb{R},$$

and fulfills locally a mixed Lipschitz/Hölder condition of the following form:
For some $\sigma \in (0,1)$ and all $M > 0$ there exists $L_{d,M} > 0$ such that

$$|d(x_1,y_1) - d(x_2,y_2)| \leq L_{d,M}(|x_1 - x_2|^\sigma + |y_1 - y_2|)$$

for all $x_i \in \Omega$ and $y_i \in \mathbb{R}$ with $|y_i| < M$, $i = 1,2$.

(A2) The function d is strictly monotonically increasing w.r.t. y in some subset $E_\Omega \subset \Omega$ of positive measure, i.e., some constant $c_\Omega > 0$ exists such that $\frac{\partial d}{\partial y}(x,y) \geq c_\Omega$ in $E_\Omega \times \mathbb{R}$.

The variational solution of (4.49) is a function $y \in H^1(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

$$a(y,v) + (d(\cdot,y),v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} + (g,v)_{L^2(\Gamma)} \quad \forall v \in V, \tag{4.50}$$

where $a : V \times V \to \mathbb{R}$ is the bilinear form defined by

$$a(y,v) := \int_\Omega \nabla y \cdot \nabla v.$$

Under the assumptions on d, this variational formulation possesses a unique solution [86, Theorem 4.10]. Its finite element approximation $y_h \in V_h$, with V_h given by (3.3), is the unique solution of the variational formulation

$$a(y_h,v_h) + (d(\cdot,y_h),v_h)_{L^2(\Omega)} = (f,v_h)_{L^2(\Omega)} + (g,v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h. \tag{4.51}$$

Next, we show sharp pointwise error estimates for this approximate solution on graded triangulations satisfying (3.1) and quasi-uniform triangulations ($\vec{\mu} = \vec{1}$). All the following considerations, except an application of the pointwise error estimates from the previous section, are taken from [76, Section 3.2.6], where the reader also can find a more detailed proof of the following result.

Theorem 4.2.18. *Let the previous assumptions hold. Assume that the weight $\vec{\beta} \in [0,1)^m$ fulfills*

$$\vec{1} - \vec{\lambda} < \vec{\beta} < \vec{1} - \vec{\mu}, \quad \vec{\beta} > 0,$$

and the weight $\vec{\gamma} \in [0,2)^m$ *satisfies*

$$(i) \quad \vec{0} \leq \vec{\gamma} < \vec{2} - 2\vec{\mu}, \quad \textit{or} \quad (ii) \quad \vec{\gamma} = \vec{0} \textit{ and } \vec{\mu} \leq \vec{1}.$$

Then the discretization error can be estimated by

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h| \left(\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \right).$$

Moreover, let $\vec{\mu} = \vec{1}$ *(quasi-uniform meshes), and the weight* $\vec{\gamma}$ *be chosen as*

$$\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\varepsilon}_1, \quad \varepsilon_1 = \varepsilon/2 \quad \textit{and} \quad 0 < \varepsilon < \lambda.$$

Then the discretization error estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^{\min(2,\lambda-\varepsilon)} |\ln h| \left(\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \right).$$

holds.

Proof. For the further analysis of problem (4.49) we consider an equivalent formulation of variational equation (4.50). We define $\alpha \in C^\infty(\overline{\Omega})$ by

$$\alpha := \eta_{E_\Omega} c_\Omega,$$

where η_{E_Ω} is an infinitely differentiable cut-off function, which is equal to one in a proper subset of E_Ω and $\operatorname{supp} \eta_{E_\Omega} \subset E_\Omega$. Variational equation (4.50) can be reformulated by means of

$$\tilde{a}(y,v) + (\tilde{d}(\cdot,y),v)_{L^2(\Omega)} = (f - d(\cdot,0),v)_{L^2(\Omega)} + (g,v)_{L^2(\Gamma)} \quad \forall v \in V, \tag{4.52}$$

where $\tilde{a} : V \times V \to \mathbb{R}$ denotes the bilinear form

$$\tilde{a}(y,v) := \int_\Omega (\nabla y \cdot \nabla v + \alpha y v), \tag{4.53}$$

and the function $\tilde{d}$ is defined by

$$\tilde{d}(x,y) := d(x,y) - d(x,0) - \alpha(x) y.$$

Note that the function $\tilde{d}$ fulfills Assumption (A1), and bilinear form (4.53) is continuous and coercive. Furthermore, we have $\tilde{d}(x,0) = 0$ for a.a. $x \in \Omega$.

We introduce an equivalent formulation of (4.51)

$$\tilde{a}(y_h,v_h) + (\tilde{d}(\cdot,y_h),v_h)_{L^2(\Omega)} = (f - d(\cdot,0),v_h)_{L^2(\Omega)} + (g,v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h. \tag{4.54}$$

We also observe that the solution $y \in V$ of (4.52) fulfills

$$\tilde{a}(y,v) = F_{d,y}(v) := (f - d(\cdot,0) - \tilde{d}(\cdot,y),v)_{L^2(\Omega)} + (g,v)_{L^2(\Gamma)} \quad \forall v \in V, \tag{4.55}$$

and we introduce a certain Ritz projection $\tilde{y}_h \in V_h$ with respect to the bilinear form $\tilde{a}$, this is

$$\tilde{a}(\tilde{y}_h, v_h) = F_{d,y}(v_h) \quad \forall v_h \in V_h. \tag{4.56}$$

Taking into account the previous considerations, we obtain

$$\begin{aligned}\|y - y_h\|_{L^\infty(\Omega)} &\le \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + \|\tilde{y}_h - y_h\|_{L^\infty(\Omega)} \\ &\le \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + c|\ln h|^{1/2}\|\tilde{y}_h - y_h\|_{H^1(\Omega)},\end{aligned} \tag{4.57}$$

where in the last step we used the discrete Sobolev inequality from Lemma 3.1.3.

In order to show a maximum norm estimate for the first term on the right-hand side of (4.57), we can not directly use the result from Theorem 4.2.7, since the underlying differential equation is a different one. To this end, we introduce a further approximation of y, namely $\breve{y}_h \in V_h$, which solves

$$a(\breve{y}_h, v_h) + (\breve{y}_h, v_h)_{L^2(\Omega)} = F_{d,y}(v_h) + ((1-\alpha)y, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h.$$

Note that the function $\tilde{y}_h$ is the solution of

$$a(\tilde{y}_h, v_h) + (\tilde{y}_h, v_h)_{L^2(\Omega)} = F_{d,y}(v_h) + ((1-\alpha)\tilde{y}_h, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h.$$

Using the triangle inequality, Theorem 4.2.7, the discrete Sobolev inequality and a Lipschitz property for solutions of linear elliptic equations, we obtain

$$\begin{aligned}&\|y - \tilde{y}_h\|_{L^\infty(\Omega)} \le c\left(\|y - \breve{y}_h\|_{L^\infty(\Omega)} + \|\breve{y}_h - \tilde{y}_h\|_{L^\infty(\Omega)}\right) \\ &\le ch^2|\ln h|\left(\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}\right) + c|\ln h|^{1/2}\|(1-\alpha)(y - \tilde{y}_h)\|_{L^2(\Omega)}.\end{aligned} \tag{4.58}$$

Using the finite element error estimate in $L^2(\Omega)$ from Lemma 4.2.3, whose proof can be easily extended to problems (4.55) and (4.56), the second term on the right-hand side can be bounded by the first one.

For the second term in (4.57) we proceed as follows. Due to the coercivity of the bilinear form $\tilde{a}$ with some coercivity constant $c_* = c_*(E_\Omega, c_\Omega)$, variational equations (4.54) and (4.56), and the monotonicity and Lipschitz continuity of $\tilde{d}$, we conclude

$$\begin{aligned}c_*\|\tilde{y}_h - y_h\|^2_{H^1(\Omega)} &\le \tilde{a}(\tilde{y}_h - y_h, \tilde{y}_h - y_h) = \int_\Omega (\tilde{d}(\cdot, y_h) - \tilde{d}(\cdot, y))(\tilde{y}_h - y_h) \\ &= \int_\Omega (\tilde{d}(\cdot, y_h) - \tilde{d}(\cdot, \tilde{y}_h))(\tilde{y}_h - y_h) + \int_\Omega (\tilde{d}(\cdot, \tilde{y}_h) - \tilde{d}(\cdot, y))(\tilde{y}_h - y_h) \\ &\le \int_\Omega (\tilde{d}(\cdot, \tilde{y}_h) - \tilde{d}(\cdot, y))(\tilde{y}_h - y_h) \\ &\le \|\tilde{y}_h - y\|_{L^2(\Omega)}\|\tilde{y}_h - y_h\|_{H^1(\Omega)}.\end{aligned} \tag{4.59}$$

Consequently, the second term on the right hand side of (4.57) can be bounded by

$$\|\tilde{y}_h - y_h\|_{H^1(\Omega)} \le c\|y - \tilde{y}_h\|_{L^2(\Omega)} \le ch^2\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}. \tag{4.60}$$

This follows from a standard finite element error estimate in the $L^2(\Omega)$-norm, which can be proven analogously to Lemma 4.2.3.

The proof for the estimate on quasi-uniform meshes is identical to the previous considerations if one uses Corollary 4.2.8 and Lemma 4.2.4 instead of Theorem 4.2.7 and Lemma 4.2.3, respectively. □

4.3 Numerical examples

In this section, we verify the theoretical results from Theorem 4.2.7 and Corollary 4.2.8 by numerical computations. To this end we use the following numerical example. The computational domain Ω_ω depending on the interior angle $\omega \in (0, 2\pi)$ is defined by

$$\Omega_\omega := (-1,1)^2 \;\cap\; \{x \in \mathbb{R}^2 : (r(x), \varphi(x)) \in (0, \sqrt{2}] \times (0, \omega)\}, \tag{4.61}$$

where the interior angle ω is chosen either $3\pi/4$ (convex domain) or $3\pi/2$ (non-convex domain), respectively.

To generate meshes satisfying the condition (3.1), we start with a coarse initial mesh and apply several uniform refinement steps. Afterwards, depending on the grading parameter μ we transform the mesh by moving all nodes x_i within a circular sector with radius R around the origin according to

$$x_{i,new} = x_i \left(\frac{r(x_i)}{R}\right)^{1/\mu - 1} \qquad \forall x_i \in \Omega_\omega \cap S_R$$

for all $i \in \mathcal{I}_h$ with $|x_i| < R$. One can show that this transformation implies mesh condition (3.1). Note that also other refinement strategies are possible. For instance, one can successively mark and refine all elements violating (3.1). The local refinement can be realized with e.g. a newest vertex bisection algorithm [18].

The benchmark problem we consider is taken from [76, Example 3.66] and reads

$$\begin{aligned} -\Delta y + y &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ \partial_n y &= \partial_n \Big(r^\lambda \cos(\lambda\varphi)\Big) && \text{on } \Gamma := \partial\Omega_\omega \end{aligned}$$

with $\lambda = \pi/\omega$. The unique solution of this problem is $y = r^\lambda \cos(\lambda\varphi)$. The experimental order of convergence $\text{eoc}(L^\infty(\Omega_\omega))$ is calculated by

$$\text{eoc}(L^\infty(\Omega_\omega)) := \frac{\ln\left(\|y - y_{h_{k-1}}\|_{L^\infty(\Omega_\omega)} / \|y - y_{h_k}\|_{L^\infty(\Omega_\omega)}\right)}{\ln(h_{k-1}/h_k)},$$

where h_{k-1} and h_k are the mesh sizes of two consecutive triangulations $\mathcal{T}_{h_{k-1}}$ and $\mathcal{T}_{h_k}$, respectively.

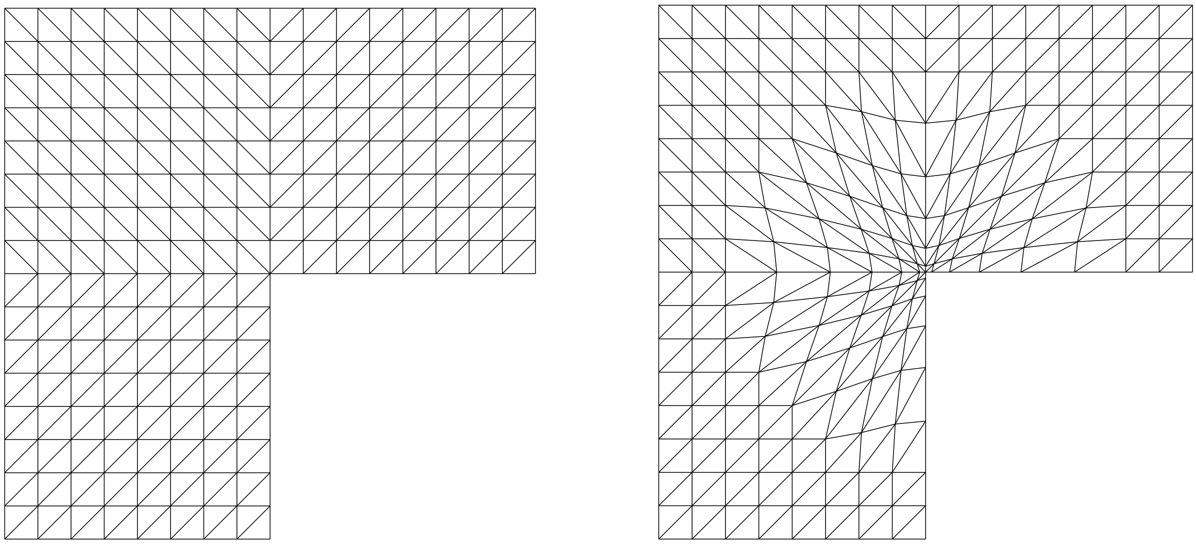

Figure 4.3: Triangulation of the domain $\Omega_{3\pi/4}$ with a quasi-uniform ($\mu = 1$) and a graded mesh ($\mu = 0.5$)

Example in a convex domain

In Table 4.1 one can find the computed errors $\|e_h\|_{L^\infty(\Omega_{3\pi/4})} := \|I_h y - y_h\|_{L^\infty(\Omega_{3\pi/4})}$ on different meshes with $\mu = 0.6 < 2/3 = \lambda/2$ and $\mu = 1$. We measure only the discrete L^∞-norm, since the initial error is dominated by this norm, due to

$$\|y - y_h\|_{L^\infty(\Omega_\omega)} \leq \|y - I_h y\|_{L^\infty(\Omega_\omega)} + \|I_h y - y_h\|_{L^\infty(\Omega_\omega)}.$$

Note that the interpolation error is bounded by ch^2 if $\mu < \lambda/2$. From Theorem 4.2.7 we expect that meshes with grading parameter $\mu < \lambda/2 = 2/3$ yield a convergence rate tending to 2 if the mesh size tends to zero. For the choice $\mu = 0.6$ this is confirmed. As predicted in Corollary 4.2.8 the convergence rate $\lambda - \varepsilon = 4/3 - \varepsilon$ for arbitrary $\varepsilon > 0$ is confirmed for quasi-uniform meshes as well.

Example in a non-convex domain

In Table 4.2 the errors $\|y - y_h\|_{L^\infty(\Omega_{3\pi/2})}$ can be found. The grading parameters are $\mu = 0.3 < 1/3 = \lambda/2$, $\mu = 0.6$ and $\mu = 1$. Meshes with $\mu = 1$ and $\mu = 0.3$ can be found in Figure 4.3. One can see that for meshes with $\mu < \lambda/2$ the convergence rate is optimal as it follows from Theorem 4.2.7. If meshes are not graded optimally ($\mu = 0.6$), the convergence order is not optimal too. The rate $2/3 - \varepsilon$ stated for quasi-uniform meshes in Corollary 4.2.8 can also be observed from the numerical results.

	$\mu = 1$		$\mu = 0.6$	
mesh size h	$\|e_h\|_{L^\infty(\Omega_\omega)}$	eoc	$\|e_h\|_{L^\infty(\Omega_\omega)}$	eoc
0.707107	3.97e-02		3.97e-02	
0.403914	1.18e-02	1.75	1.21e-02	2.12
0.233893	3.37e-03	1.82	3.87e-03	2.09
0.135498	9.43e-04	1.84	1.17e-03	2.19
0.070628	2.60e-04	1.86	3.42e-04	1.89
0.036008	1.09e-04	1.26	9.38e-05	1.92
0.018176	4.50e-05	1.27	2.48e-05	1.94
0.009131	1.83e-05	1.30	6.45e-06	1.96
0.004587	7.39e-06	1.31	1.66e-06	1.97
0.002298	2.96e-06	1.32	4.22e-07	1.98

Table 4.1: Discretization errors $e_h = y - y_h$ with $\omega = 3\pi/4$.

	$\mu = 1$		$\mu = 0.6$		$\mu = 0.3$	
mesh size h	$\|e_h\|_{L^\infty(\Omega_\omega)}$	eoc	$\|e_h\|_{L^\infty(\Omega_\omega)}$	eoc	$\|e_h\|_{L^\infty(\Omega_\omega)}$	eoc
0.707107	6.20e-02		6.20e-02		6.20e-02	
0.403914	3.72e-02	0.74	4.27e-02	0.67	4.80e-02	0.68
0.233893	3.39e-02	0.64	1.85e-02	1.53	3.30e-02	0.98
0.135498	1.52e-02	0.65	8.31e-03	1.46	1.47e-02	2.09
0.070628	9.62e-03	0.66	3.83e-03	1.19	4.79e-03	2.00
0.036008	6.07e-03	0.66	1.77e-03	1.15	1.44e-03	1.91
0.018176	3.83e-03	0.67	8.17e-04	1.13	4.07e-04	1.92
0.009131	2.41e-03	0.67	3.78e-04	1.12	1.11e-04	1.92
0.004587	1.52e-03	0.67	1.75e-04	1.12	2.96e-05	1.95
0.002298	9.57e-04	0.67	8.09e-05	1.12	7.70e-06	1.96

Table 4.2: Discretization errors $e_h = y - y_h$ with $\omega = 3\pi/2$.

CHAPTER 5

Estimates for Dirichlet boundary value problems

This chapter is devoted to descretization error estimates for two different types of Dirichlet boundary value problems. We consider $W^{1,\infty}(\Omega)$-seminorm error estimates on quasi-uniform meshes and $L^\infty(\Omega)$-error estimates on both graded and quasi-uniform triangulations for the homogeneous Dirichlet boundary value problem

$$\begin{aligned} -\Delta y &= f \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma \end{aligned} \tag{5.1}$$

with the right hand side function f being smooth enough, and $L^2(\Omega)$-error estimates on quasi-uniform triangulations for the inhomogeneous Dirichlet problem

$$\begin{aligned} -\Delta y &= 0 \quad \text{in } \Omega, \\ y &= g \quad \text{on } \Gamma, \end{aligned} \tag{5.2}$$

where the boundary datum $g \in L^2(\Gamma)$.

The declared error estimates in the $W^{1,\infty}(\Omega)$-seminorm for (5.1) and in the $L^2(\Omega)$-norm on quasi-uniform triangulations for (5.2) are needed in the forthcoming study of pointwise error estimates for Dirichlet boundary control problems. Here we recall that Dirichlet boundary control problems on graded meshes are out of the scope of this thesis. The $L^\infty(\Omega)$-error estimate on graded triangulations for (5.1) is an improvement of [11, Theorem 4.4], where the rate $h^2|\ln h|^{3/2}$ is shown, and here we get the rate $h^2|\ln h|$, using the proof techniques from the previous chapter.

5.1 Estimates for the homogeneous Dirichlet problem

Consider the homogeneous Dirichlet boundary value problem (5.1). The variational formulation for this problem reads as follows:

Find $y \in H_0^1(\Omega)$ such that

$$a(y,v) = (f,v)_{L^2(\Omega)} \quad \forall v \in V := H_0^1(\Omega), \tag{5.3}$$

where $a : V \times V \to \mathbb{R}$ is the bilinear form defined by

$$a(y,v) := \int_\Omega \nabla y \cdot \nabla v. \tag{5.4}$$

The existence and uniqueness of a solution $y \in H_0^1(\Omega)$ follows from the Lax-Milgram Theorem, provided that $f \in [H_0^1(\Omega)]^*$.

5.1.1 Regularity

Now, as for the Neumann problem, we present two a priori results in weighted Sobolev spaces for the homogeneous Dirichlet problem, namely, a priori estimates in the weighted $H^2(\Omega)$- and $W^{2,\infty}(\Omega)$-norms, which we need for Theorem 5.1.5 and Corollary 5.1.6.

Lemma 5.1.1. *Let $\vec{\beta} \in [0,1)^m$ satisfy the condition*

$$\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1}.$$

Then for every $f \in V_{\vec{\beta}}^{0,2}(\Omega)$, the solution of problem (5.1) belongs to $V_{\vec{\beta}}^{2,2}(\Omega)$ and satisfies the a priori estimate

$$\|y\|_{V_{\vec{\beta}}^{2,2}(\Omega)} \le c\|f\|_{V_{\vec{\beta}}^{0,2}(\Omega)}.$$

Proof. The proof is just an application of [68, Theorem 3.1] in a general polygonal domain. □

Lemma 5.1.2. *Let the right hand side satisfy $f \in N_{\vec{\delta}}^{0,\sigma}(\overline{\Omega})$ for arbitrary $\sigma \in (0,1)$ and weights $\vec{\gamma} \in [0,2)^m$ whose components γ_j, $j \in \mathcal{C}$, satisfy one of the conditions*

(i)	$\max(0, 2-\lambda_j) < \gamma_j \le 2,$	$\delta_j = \gamma_j + \sigma,$
(ii)	$\gamma_j = 0,\ 2 - \lambda_j < 0,$	$\delta_j = \sigma.$

Then the solution of the problem (5.1) belongs to $V_{\vec{\gamma}}^{2,\infty}(\Omega)$ and the a priori estimate

$$\|y\|_{V_{\vec{\gamma}}^{2,\infty}(\Omega)} \le c\|f\|_{N_{\vec{\delta}}^{0,2}(\Omega)}$$

holds true.

Proof. The desired estimates follows from the trivial embedding $N_{\vec{\delta}}^{2,\infty}(\Omega) \hookrightarrow V_{\vec{\gamma}}^{2,\infty}(\Omega)$, see the definitions of these spaces in Section 2.2, and [52, Theorem 8.7.1]. □

5.1.2 Finite element discretization

The FE solution $y_h \in V_{h,0}$ of problem (5.3) satisfies

$$a(y_h, v_h) = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_{h,0}, \tag{5.5}$$

where

$$V_{h,0} = \{v_h \in C(\overline{\Omega}) : v_h|_T \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \Gamma\}. \tag{5.6}$$

The existence and uniqueness of this function follows from the Lax-Milgram theorem.

$W^{1,\infty}(\Omega)$-seminorm error estimates

In Chapter 7 we consider pointwise error estimates for Dirichlet boundary control problems. An important tool for such estimates is an estimate for the adjoint state. In case of Dirichlet control problems, as one can see in Chapter 7, we need a pointwise estimate for the gradient of the discretization error for the adjoint equation, which has the same structure as (5.1). For quasi-uniform triangulations this estimate can be deduced from the main theorem of [77] or [32, Theorem 2], which holds also for graded meshes.

Theorem 5.1.3. *Let Ω be convex and $\vec{\mu} = \vec{1}$ (quasi-uniform meshes). Moreover, let y and y_h be the solutions of* (5.3) *and* (5.5)*, respectively. Then the best approximation property*

$$\|\nabla(y - y_h)\|_{L^\infty(\Omega)} \le c \min_{\chi \in V_{h,0}} \|\nabla(y - \chi)\|_{L^\infty(\Omega)}$$

holds.

Remark 5.1.4. *If the solution of* (5.1) *possesses no singularities in the neighborhood of concave corners, the best approximation property can be also applied in non-convex domains, see remarks after the main theorem in [77].*

$L^\infty(\Omega)$-norm error estimates

Now, let us sharpen the convergence rate in the $L^\infty(\Omega)$-norm on graded meshes. In [84, Theorem 4.4] it is shown that the estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^2 |\ln h|^{3/2} \|f\|_{C^{0,\sigma}(\overline{\Omega})}, \tag{5.7}$$

holds on graded meshes with $\mu < \lambda/2$, provided that $f \in C^{0,\sigma}(\overline{\Omega})$ with $\sigma \in (0,1)$. Note that in the reference only one singular corner is assumed, therefore, only one grading parameter is needed for that result.

Now, using the proof techniques from Section 4.2.2, we reduce the exponent of the logarithmic term in estimate (5.7). Moreover, we reformulate the result such that it holds for general polygonal domains and also show an error estimate on quasi-uniform triangulations.

Theorem 5.1.5. *Assume that y, the solution of (5.3), belongs to $V^{2,2}_{\vec{\beta}}(\Omega) \cap V^{2,\infty}_{\vec{\gamma}}(\Omega)$ with*

$$\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1} - \vec{\mu}, \quad \vec{\beta} \ge \vec{0}$$

and $\vec{\gamma}$ satisfying one of the following conditions

$$(i) \quad \vec{0} \le \vec{\gamma} < \vec{2} - 2\vec{\mu}, \quad or \quad (ii) \quad \vec{\gamma} = \vec{0} \text{ and } \vec{\mu} \le \vec{1}.$$

Then, the solution y_h of (5.5) fulfills the error estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^2 |\ln h| \left(\|y\|_{V^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{V^{2,\infty}_{\vec{\gamma}}(\Omega)} \right).$$

Proof. The proof is very similar to that of Theorem 4.2.7, however, there are some differences, which we point out here. First, note that the stripwise interpolation error estimates in Lemma 4.2.11 hold also in weighted V-spaces, since the corresponding seminorms are equivalent. Therefore, Lemmas 4.2.13 and 4.2.14 also hold true for weighted V-functions.

The modification of Lemma 4.2.15 needs more attention. The dual problem (4.17) in case of the Dirichlet problem reads as

$$\begin{aligned} -\Delta w &= \sigma^{-2}(y - y_h)\chi \quad && \text{in } \Omega, \\ w &= 0 && \text{on } \Gamma \end{aligned} \tag{5.8}$$

with the obvious modification of the bilinear form $a_{\Omega_R}(\cdot,\cdot)$. Estimate (4.28) holds true using the estimate from Lemma 5.1.1 instead of Lemma 4.1.1. All the other steps are either identical or are valid due to the seminorm equivalence in Lemmas 4.2.13 and 4.2.14.

Estimate (4.47) holds true following the same steps as in **Case 2** with only one exeption, namely, instead of using the a priori error estimate for the Neumann problem from Theorem 4.2.1, we have to use an equivalent one for the Dirichlet problem, e.g. the one from [38].

Inequality (4.48) in **Case 3** obviously holds for y in the corresponding weighted V-norm. The desired result follows from the prevoius considerations as well as the a priori estimate in the $L^2(\Omega)$-norm on graded meshes, see e.g. [17], [73] and [78]. □

Corollary 5.1.6. *Let $\vec{\mu} = \vec{1}$ (quasi-uniform meshes). Moreover, assume that y, the solution of (5.3), belongs to $V^{2,2}_{\vec{\beta}}(\Omega) \cap V^{2,\infty}_{\vec{\gamma}}(\Omega)$ with*

$$\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\varepsilon}_1, \quad \varepsilon_1 = \varepsilon/2 \quad \text{and} \quad 0 < \varepsilon < \lambda.$$

Then, the solution y_h of (4.8) fulfills the error estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^{\min(2,\lambda-\varepsilon)} |\ln h| \left(\|y\|_{V^{2,2}_{\vec{\beta}}(\Omega)} + \|y\|_{V^{2,\infty}_{\vec{\gamma}}(\Omega)} \right).$$

Proof. The proof follows the same steps as the proof of Corollary 4.2.8 adopting the modifications used in the proof of Theorem 5.1.5. □

Remark 5.1.7. *Similarly to the Neuman case, here we discuss the required regularity in Theorem 5.1.5 and Corollary 5.1.6. The weighted $H^2(\Omega)$-regularity follows from Lemma 5.1.1 and a trivial embedding. We have*

$$\|y\|_{V^{2,2}_{\vec{\beta}}(\Omega)} \le c\|f\|_{V^{0,2}_{\vec{\beta}}(\Omega)} \le c\|f\|_{L^2(\Omega)}.$$

Lemma 5.1.2 guarantees the required weighted $W^{2,\infty}(\Omega)$-regularity

$$\|y\|_{V^{2,\infty}_{\vec{\gamma}}(\Omega)} \le c\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} \le c\|f\|_{C^{0,\sigma}(\overline{\Omega})},$$

where the last estimate is trivially fulfilled.

Finally, we point out that in order to get the best possible rate of convergence, the assumptions $\vec{\beta} \le \vec{1} - \vec{\mu}$ and $\vec{\gamma} < \vec{2} - 2\vec{\mu}$ from Theorem 5.1.5 have to be fulfilled. This is the case if $\vec{\mu} < \vec{\lambda}/2$.

5.2 Estimates for the inhomogeneous Dirichlet problem

At the beginning of this chapter we mentioned that we are interested in $L^2(\Omega)$-norm error estimates for problem (5.2) given that the boundary datum g is only an $L^2(\Gamma)$-function. Such estimates are of interest in the numerical investigation of Dirichlet boundary control problems considered in Chapter 7. In this section we only collect existing results for the application in Chapter 7.

5.2.1 Method of transposition

Since a solution of problem (5.2) can not be expected from the space $H^1(\Omega)$, we are not allowed to look for it as a weak solution. As a remedy to this issue, we can consider it as a very weak solution of the state equation. The very weak solution can be computed via the so-called methods of transposition. This method for convex domains goes back at least to [57]. For the analysis of optimal control problems it is used in [39, 24, 30] and [61]. The very weak solution of (5.2) in convex domains is the unique element $y \in L^2(\Omega)$ which satisfies

$$(y, \Delta v)_{L^2(\Omega)} = (g, \partial_n v) \quad \forall v \in V := H^2(\Omega) \cap H^1_0(\Omega). \tag{5.9}$$

If the underlying domain is non-convex, the test space V has to be substituted by

$$V_\Delta := H^1_\Delta(\Omega) \cap H^1_0(\Omega) \quad \text{with} \quad H^1_\Delta(\Omega) := \left\{v \in H^1(\Omega)\colon \Delta v \in L^2(\Omega)\right\}.$$

In convex domains the space V_Δ coincides with V, and in non-convex domains the splitting

$$V_\Delta = V \bigoplus_{\lambda_j<1,\ j\in\mathcal{C}} \operatorname{span}\left\{\xi r_j^{\lambda_j} \sin(\lambda_j \varphi_j)\right\}$$

holds, where ξ is a smooth cut-off function with $\xi \equiv 1$ in some neighborhood of the corresponding non-convex corners, see [6, Remark 2.2].

The existence and uniqueness of a solution to the very weak formulation (5.9) is proven in [61, Lemma 2.1] for convex domains and in [6, Lemma 2.3] for non-convex domains. Note also that if a weak solution $y \in \{v \in H^1(\Omega)\colon v|_\Gamma = u\}$ of problem (5.2) exists, it also satisfies the very weak formulation (5.9), see [27].

From [6, Corollary 2.6] we can cite the following a priori estimate.

Theorem 5.2.1. *For every $g \in L^2(\Gamma)$ there exists a unique solution $y \in H^{1/2}(\Omega)$ of (5.9). Moreover, the a priori estimate*

$$\|y\|_{H^{1/2}(\Omega)} \leq c\|g\|_{L^2(\Gamma)}$$

holds.

There are two approaches for treating (5.9), namely, Berggren's approach introduced in [20] and the regularization approach proposed in [6]. The idea of Berggren's approach is to avoid the explicit use of test functions from V_Δ, and the idea of the regularization approach is an initial regularization of the boundary data and an application of the standard variational method.

5.2.2 Finite element discretization

Here we consider a finite element discretization for the regularization approach only, since this approach delivers slightly better convergence rates than Berggren's discretization approach.

Introduce a sequence of functions $g^h \in H^{1/2}(\Gamma)$ such that

$$\lim_{h\to 0} \|g - g^h\|_{L^2(\Gamma)} = 0,$$

and define the spaces

$$V_*^h := \left\{ v \in H^1(\Omega)\colon v|_\Gamma = g^h \right\},$$

as well as

$$V_{h,*} := V_*^h \cap V_h.$$

A finite element solution $y_h \in V_{h,*}$ of problem (5.9) satisfies

$$a(y_h, v_h) = 0 \quad \forall v_h \in V_{h,0}, \tag{5.10}$$

where the bilinear form $a(\cdot,\cdot)$ is given by (5.4) and the space $V_{h,0}$ is defined according to (5.6).

For the numerical solution of (5.10) we have the following convergence result from [6, Corollary 3.3].

Lemma 5.2.2. *Let y and y_h be the solutions of (5.9) and (5.10), respectively. Then the discretization error estimate*

$$\|y - y_h\|_{L^2(\Omega)} \leq ch^{\min(1/2, \lambda - 1/2 - \varepsilon)}\|g\|_{L^2(\Gamma)}$$

holds, provided that $g \in L^2(\Gamma)$.

CHAPTER 6

Estimates for Neumann boundary control problems

This chapter is devoted to maximum norm error estimates of the Neumann boundary control problem

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \tag{6.1}$$

subject to

$$\begin{aligned} -\Delta y + y &= 0 \quad \text{in } \Omega, \\ \partial_n y &= u \quad \text{on } \Gamma, \end{aligned} \tag{6.2}$$

$$u \in U_{ad} := \left\{ u \in L^2(\Gamma) \colon u_a \le u \le u_b \text{ a.e. on } \Gamma \right\}. \tag{6.3}$$

In the formulation above $J(\cdot,\cdot)$ is the so-called objective functional, y_d denotes the desired state, which is first assumed to be an $L^2(\Omega)$-function. We require more regularity for the desired state in the sequel when deriving error estimates. The function y is called the state and is coupled with the control u via state equation (6.2). The control bounds u_a, u_b are assumed to be constant and the regularization parameter ν is a positive real number.

In Section 6.1 we give preliminary definitions, and discuss optimality conditions and regularity results for the continuous optimal control problem. In the next section we consider two discretization strategies for the optimal control problems and derive sharp pointwise error estimates for the control. Finally, in Section 6.3 we confirm theoretical results via a numerical experiment for one of the considered approaches.

6.1 Analysis of the continuous problem

Before we start with the numerical analysis of optimal control problem (6.1)–(6.3), we give some preliminary definitions and discuss existence and uniqueness of a solution to the continuous problem as well as some regularity results.

Preliminaries

The variational formulation of state equation (6.2) reads:
Find $y \in H^1(\Omega)$ such that

$$a(y, v) = (u, v)_{L^2(\Gamma)} \quad \forall v \in H^1(\Omega), \tag{6.4}$$

where the bilinear form $a(\cdot, \cdot)$ is given by (4.3), We define by

$$S\colon L^2(\Gamma) \to L^2(\Omega), \quad u \mapsto Su := y$$

the solution operator of (6.4). Inserting Su into (6.1) eliminates the state variable y and gives the reduced objective functional

$$j(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \quad \text{s.t. } u \in U_{ad} \tag{6.5}$$

We also define the adjoint equation

$$\begin{aligned} -\Delta p + p &= z \quad \text{in } \Omega, \\ \partial_n p &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $z \in L^2(\Omega)$, with its weak formulation:
Find $p \in H^1(\Omega)$ such that

$$a(p, v) = (z, v)_{L^2(\Omega)} \quad \forall v \in H^1(\Omega). \tag{6.6}$$

A weak solution of (6.6) is called the adjoint state and the solution operator of the adjoint equation is given by

$$P\colon L^2(\Omega) \to H^1(\Omega), \quad z \mapsto Pz := p.$$

Optimality conditions

The existence and uniqueness of a solution to optimal control problem (6.1)–(6.3) is guaranteed by the following result.

Theorem 6.1.1. *There exists a unique element $\bar{u} \in U_{ad}$, which solves optimal control problem (6.5) associated with the optimal state $\bar{y} = S\bar{u}$ and the optimal adjoint state $\bar{p} = P(S\bar{u} - y_d)$. Moreover, the variational inequality*

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad} \tag{6.7}$$

holds, and is equivalent to the projection formula

$$\bar{u}(x) = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}(x)\right) \text{ for a.a. } x \in \Gamma. \tag{6.8}$$

where $\Pi_{[u_a,u_b]}$ is the $L^2(\Gamma)$-projection onto U_{ad} and possesses the pointwise representation

$$\left(\Pi_{[u_a,u_b]}v\right)(x) := \max(u_a, \min(u_b, v(x))). \tag{6.9}$$

Proof. The existence and uniqueness of a solution $\bar{u} \in U_{ad}$ can be deduced from [86, Theorem 2.14]. The variational inequality (6.7) follows from the Fréchet differentiability of functional (6.5) and [86, Theorem 2.22]. The equivalence of variational inequality (6.7) and projection formula (6.8) can be found in e.g. [49, Sections 1.5.1 and 1.7.2]. □

Regularity

In the following theorem we collect regularity results needed for the further numerical analysis of the optimal control problem. This result can be found in [76, Theorem 4.4].

Theorem 6.1.2. *Let $\bar{y} = S\bar{u}$ and $\bar{p} = P(S\bar{u} - y_d)$ be the optimal state and the optimal adjoint state, respectively, associated with the optimal control $\bar{u} \in U_{ad}$. Then the inequality*

$$\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \leq c\Big(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\overline{\Omega})}\Big)$$

holds provided that

$$y_d \in C^{0,\sigma}(\overline{\Omega})$$

and

$$\begin{aligned} &\max(0, 1-\lambda_j) < \beta_j < 1 \quad \text{or} \quad \beta_j = 0 \text{ and } 1-\lambda_j < 0,\\ &\max(0, 2-\lambda_j) < \gamma_j < 2 \quad \text{or} \quad \gamma_j = 0 \text{ and } 2-\lambda_j < 0 \end{aligned}$$

is fulfilled for each $j \in \mathcal{C}$.

6.2 Discretization error estimates

In this section, we investigate two different types of discretization of optimal control problem (6.1)–(6.3), and derive sharp pointwise error estimates for the control variable on both graded and quasi-uniform meshes.

6.2.1 Variational discretization

The first possible discretization strategy is the approach of variational discretization. This approach is introduced by Hinze in [47] and applied to Neumann control problems in [48]. The underlying feature of this approach is that the state and the adjoint state are approximated by piecewise linear and globally continuous elements, and the control is not discretized, however, it inherits its discretization via the discrete projection formula (6.14).

The variational formulation (6.4) now reads as:

Find $y_h \in V_h$

$$a(y_h, v_h) = (u, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h, \tag{6.10}$$

where V_h is given by (3.3), and we define by

$$S_h\colon L^2(\Gamma) \to V_h, \quad u \mapsto S_h u := y_h$$

the discrete counterpart of the solution operator S. The discretized optimal control problem in the reduced form is given by

$$j_h(u) := \frac{1}{2}\|S_h u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \quad \text{s.t. } u \in U_{ad}. \tag{6.11}$$

The weak formulation of the adjoint equation reads:
Find $p_h \in V_h$ such that

$$a(p_h, v_h) = (z, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h, \tag{6.12}$$

where $z \in L^2(\Omega)$. The corresponding discrete counterpart of the continuous adjoint operator P is given by

$$P_h\colon L^2(\Omega) \to V_h, \quad z \mapsto P_h z := p_h.$$

Analogous to the continuous case one can show, see e.g. [48], that discrete optimal control problem (6.11) has a unique solution $\bar{u}_h \in U_{ad}$.

Lemma 6.2.1. *There exists a unique element $\bar{u}_h \in U_{ad}$, which solves optimal control problem (6.11) associated with the discrete optimal state $\bar{y}_h = S_h\bar{u}$ and the discrete optimal adjoint state $\bar{p}_h = P_h(S_h\bar{u} - y_d)$. Moreover, the variational inequality*

$$(\bar{p}_h + \nu\bar{u}_h, u - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad} \tag{6.13}$$

holds and is equivalent to the projection formula

$$\bar{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h|_\Gamma\right). \tag{6.14}$$

The following lemma gives us a tool for a compact proof of the desired error estimates for the control.

Lemma 6.2.2. *Let $z \in L^2(\Omega)$ and $v \in L^2(\Gamma)$. Moreover let P_h and S_h be the discrete adjoint operator and the discrete solution operator, respectively. Then for $\vec{0} < \vec{\mu} \leq \vec{1}$ the estimates*

$$\|P_h z\|_{L^\infty(\Omega)} \leq c\|z\|_{L^2(\Omega)},$$
$$\|S_h v\|_{L^\infty(\Omega)} \leq c\|v\|_{L^2(\Gamma)}$$

hold.

Proof. Both estimates follow from [76, Corollary 3.47] with $r = 2$ and $s = 2$, respectively. □

6.2.2 Error estimates for the variational discretization approach

We start this section with recalling error estimates for the control in the $L^2(\Gamma)$-norm. Such estimates are essential in the forthcoming proof of pointwise estimates. Error estimates for the control variable in the $L^2(\Gamma)$-norm on graded meshes have already been studied in [7, 76, 8]. In [7] the authors showed a suboptimal convergence rate and in [76, 8] a quasi-optimal one. In [76] also an estimate on quasi-uniform meshes is given.

Although the rate of convergence in [76, Theorem 4.10] is $h^2|\ln h|^{3/2}$, it is possible to obtain $h^2|\ln h|$ if one uses Theorem 4.2.5 with $\varrho = 0$ instead of $\varrho = 1/2$ in the proof of [76, Theorem 4.10]. This, however, leads to the stronger grading condition $\vec{\mu} < \vec{\lambda}/2$.

In order to reduce the exponent of the logarithmic term in the $L^2(\Gamma)$-estimate on quasi-uniform meshes given in [76, Corollary 4.11], one can use Corollary 4.2.6 with $\varrho = 0$ instead of $\varrho = 1/2$ in the proof of [76, Corollary 4.11].

We collect the above considerations in the following theorem, and from now on we assume that

$$y_d \in C^{0,\sigma}(\overline{\Omega}),$$

where $\sigma \in (0, 1]$. Moreover, we recall that

$$\lambda := \min_{j \in \mathcal{C}} \lambda_{j,1}, \quad \lambda_{j,1} := \frac{\pi}{\omega_j}.$$

In the following considerations, the factors $\|\bar{u}\|_{L^2(\Gamma)}$, $\|y_d\|_{C^{0,\sigma}(\overline{\Omega})}$, u_a, u_b and ν are hidden in the generic constant c.

Theorem 6.2.3. *Let $\bar{u}$ and $\bar{u}_h$ be given by projection formulas (6.8) and (6.14), respectively.*

(i) If $\vec{\mu} < \vec{\lambda}/2$, then the dicretization error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch^2|\ln h|$$

holds.

(ii) If $\vec{\mu} = \vec{1}$ and $0 < \varepsilon < \lambda$, then the error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch^{\min(2,\lambda-\varepsilon)}|\ln h|$$

is valid.

In order to show the main result of this section, first, we have to derive quasi-optimal error estimates for the adjoint state.

Lemma 6.2.4. *Let $\bar{p} = P(S\bar{u} - y_d)$ be the optimal adjoint state and $\bar{p}_h = P_h(S_h\bar{u}_h - y_d)$ the discrete optimal adjoint state associated with the optimal control $\bar{u}$ and the discrete optimal control $\bar{u}_h$ be given by projection formulas (6.8) and (6.14), respectively.*

(i) If $\vec{\mu} < \vec{\lambda}/2$, then the dicretization error estimate

$$\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \leq ch^2|\ln h|$$

holds.

(ii) If $\vec{\mu} = \vec{1}$ and $0 < \varepsilon < \lambda$, then the error estimate

$$\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \leq ch^{\min(2,\lambda-\varepsilon)}|\ln h|$$

is valid.

Proof. We start with the first estimate. The proof is similar to [11, Theorem 3.8], which is given for a distributed control problem. We introduce auxiliary functions and use the triangle inequality

$$\begin{aligned}\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} &= \|P(S\bar{u} - y_d) - P_h(S_h\bar{u}_h - y_d)\|_{L^\infty(\Omega)} \\ &\leq \|(P - P_h)(S\bar{u} - y_d)\|_{L^\infty(\Omega)} + \|P_h(S\bar{u} - S_h\bar{u})\|_{L^\infty(\Omega)} \\ &+ \|P_hS_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)}. \end{aligned} \tag{6.15}$$

The first term in (6.15) can be estimated using Theorem 4.2.7 together with Theorem 6.1.2

$$\|(P - P_h)(S\bar{u} - y_d)\|_{L^\infty(\Omega)} \leq ch^2|\ln h|. \tag{6.16}$$

Now, applying Lemma 6.2.2 for the second term in (6.15), we obtain

$$\|P_h(S\bar{u} - S_h\bar{u})\|_{L^\infty(\Omega)} \leq c\|S\bar{u} - S_h\bar{u}\|_{L^2(\Omega)} \leq ch^2, \tag{6.17}$$

where the last inequality follows from Lemma 4.2.3 and Theorem 6.1.2. To estimate the last term, we apply the first estimate from Lemma 6.2.2 together with the trivial embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ and the second estimate from the same lemma

$$\|P_hS_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)} \leq c\|S_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)} \leq c\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|, \tag{6.18}$$

where the last step is valid due to Theorem 6.2.3 (i). The desired assertion follows from inequalities (6.16)–(6.18).

The proof for the estimate on quasi-uniform meshes is similar to the proof for the first one. The desired result can be shown by using Corollary 4.2.8, Lemma 4.2.4 and Theorem 6.2.3 (ii) instead of Theorem 4.2.7, Lemma 4.2.3 and Theorem 6.2.3 (i), respectively, in the proof of the estimate on graded meshes. □

Finally, we are able to show the main estimates of this section.

Theorem 6.2.5. *Let $\bar{u}$ and $\bar{u}_h$ be given by projection formulas (6.8) and (6.14), respectively.*

(i) If $\vec{\mu} < \vec{\lambda}/2$, then the dicretization error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq ch^2|\ln h|$$

holds.

(ii) If $\vec{\mu} = \vec{1}$ and $0 < \varepsilon < \lambda$, then the error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq ch^{\min(2,\lambda-\varepsilon)}|\ln h|$$

is valid.

Proof. Projection formulas (6.8) and (6.14), its Lipschitz continuity and the continuity of $\bar{p} - \bar{p}_h$ yield

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} = \frac{1}{\nu}\|\bar{p} - \bar{p}_h\|_{L^\infty(\Gamma)} \leq \frac{1}{\nu}\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)}.$$

The desired result follows from Lemma 6.2.4. □

Remark 6.2.6. *It is also possible to derive pointwise error estimates for the optimal state. However, due to kinks of the optimal control caused by projection formula (6.8), the optimal control belongs to some weighted $W^{1,\infty}(\Gamma)$-space but not to some weighted $C^{1,\sigma}(\Gamma)$-space. This, in turn, means that $\bar{y}$ belongs to some weighted $W^{2,p}(\Omega)$-space with some arbitrary $1 \leq p < \infty$ but not to desired weighted $W^{2,\infty}(\Omega)$-space. One can show that the lack of the weighted $W^{2,\infty}(\Omega)$-regularity in the proofs of Theorem 4.2.7 and Corollary 4.2.8 leads to*

$$\|(S - S_h)\bar{u}\|_{L^\infty(\Omega)} \leq ch^{2-\varepsilon}$$

on graded meshes and

$$\|(S - S_h)\bar{u}\|_{L^\infty(\Omega)} \leq ch^{\min(2,\lambda)-\varepsilon}$$

on quasi-uniform meshes, respectively. The triangle inequality together with the estimates above, Lemma 6.2.2 and Theorem 6.2.3 imply the same rates for the error of the optimal state. There holds

$$\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq \|(S - S_h)\bar{u}\|_{L^\infty(\Omega)} + \|S_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)} \leq c \begin{cases} h^{2-\varepsilon}, \\ h^{\min(2,\lambda)-\varepsilon} \end{cases}$$

on graded and quasi-uniform meshes, respectively.

Note that if no kinks of the optimal control occur — which is the case, for instance, if control constraints are not present — one can recover the desired weighted $W^{2,\infty}(\Omega)$-regularity of the optimal state, and hence, the quasi-optimal convergence rate ($h^2|\ln h|$) on graded meshes and on quasi-uniform meshes for $\lambda > 2$.

6.2.3 Postprocessing approach

The second discretization strategy that we consider is the postprocessing approach introduced by Meyer and Rösch in [65] with an extension to Neumann control problems given in [60]. This approach is based on a full discretization of the optimal control problem (6.1)–(6.3). It means that the state and its adjoint are discretized according to the previous section, and the control u is discretized by piecewise constant functions

$$\begin{aligned} U_h^0 &:= \{u_h \in L^\infty(\Gamma) : u_h|_E \in \mathcal{P}_0 \text{ for all } E \in \mathcal{E}_h\}, \\ U_{h,ad} &:= U_h^0 \cap U_{ad}. \end{aligned}$$

However, in the postpocessing approach the optimal control $\tilde{u} \notin U_{h,ad}$ is computed in the postprocessing step, which essentially gives the name to the discretization strategy, and allows to gain better convergence rates. The fully discretized optimal control problem reads as

$$j_h(u_h) := \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Gamma)} \to \min! \quad \text{s.t. } u_h \in U_{h,ad}. \tag{6.19}$$

Using the same argumentation as in the continuous case, one can show the following existence and uniqueness result.

Lemma 6.2.7. *There exists a unique element $\bar{u}_h \in U_{h,ad}$, which solves optimal control problem (6.19) associated with the optimal discrete optimal state $\bar{y}_h = S_h\bar{u}_h$ and the optimal discrete adjoint state $\bar{p}_h = P_h(S_h\bar{u}_h - y_d)$. Moreover, the variational inequality*

$$(\bar{p}_h + \nu\bar{u}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{h,ad} \tag{6.20}$$

holds.

6.2.4 Error estimates for the postprocessing approach

In order to obtain results of the same quality as in Lemma 6.2.4 and Theorem 6.2.5 we need an assumption upon the active set which is, to the best of our knowledge, always used in derivation of error estimates for the state in the $L^2(\Omega)$-norm. As already discussed in the previous section, the control variable (in general) has kinks at transition points between active and inactive sets, and hence, is less regular. However, we assume that these kinks can occur only at a finite number of points, which motives the definition of the subsets

$$K_1 := \bigcup_{E\in\mathcal{E}_h :\, \bar{u}\notin W^{2,2}_{\vec{2}-2\vec{\mu}}(E)} E, \qquad K_2 := \bigcup_{E\in\mathcal{E}_h :\, \bar{u}\in W^{2,2}_{\vec{2}-2\vec{\mu}}(E)} E.$$

Remark 6.2.8. *Note, that K_1 contains the elements where kinks occur, and K_2 contains the elements that are strictly active or inactive, since $p|_\Gamma, u_a, u_b \in W^{2,2}_{\vec{2}-2\vec{\mu}}(E)$ for $E \subset K_2$.*

Assumption 1. *There exists a positive constant c independent of h such that $|K_1| \leq ch$.*

We do not use the previous assumption explicitly, however, it is essential for Theorem 6.2.9.

As mentioned before, the postprocessing approach is first considered in [65]. The idea is to compute a fully discrete solution of problem (6.19) and to get an improved control by applying projection formula (6.8) to the discrete optimal adjoint state $\bar{p}_h$ to obtain

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h|_\Gamma\right). \tag{6.21}$$

The standard approach in the analysis of $L^2(\Omega)$-norm error estimates for the postprocessing approach is as follows. First, one shows an estimate for the state, which is usually the most difficult part of the numerical analysis. Having shown this estimate and using standard arguments, one can show the same convergence rate (as for the state) for the adjoint state, and in the last step — the postprocessing step itself — the rate for the error of the postprocessed control.

Here we argue in a different way, namely, as in the previous section by showing the quasi-optimal rate for the adjoint state, and as a consequence of it, for the control. However, in comparison to the previous section, here we show the estimate for the adjoint state via an $L^2(\Omega)$-norm estimate for the state, which make the proof a bit shorter. The pointwise error estimate for the state can be obtained separately from the first two estimates.

We start our discussion with the existing estimates for the state in the $L^2(\Omega)$-norm. Note that these results have been studied in the same references and of the same quality as the corresponding estimates for the control in case of the approach of variational discretization.

The forthcoming results are already shown in [76, Theorem 4.20, Corollary 4.21], however, here we deal with the same issue as before, namely, the exponent of the logarithmic term in the corresponding estimates is not equal to one. In order to overcome this issue, the corresponding proofs require only one modification. Namely, one can apply Theorem 4.2.5 with $\varrho = 0$ in the proof of [76, Lemma 4.18], where a FE error in the L^2-norm on the boundary for the adjoint state is estimated. By the same idea, using Corollary 4.2.6 with $\varrho = 0$ in the proof of [76, Corollary 4.21], we can show the corresponding estimate on quasi-uniform triangulations.

As in the case of the variational discretization approach, we assume that

$$y_d \in C^{0,\sigma}(\overline{\Omega}),$$

where $\sigma \in (0, 1]$, and recall that

$$\lambda := \min_{j\in\mathcal{C}} \lambda_{j,1}, \quad \lambda_{j,1} := \frac{\pi}{\omega_j}.$$

Moreover, we assume that in the following considerations the factors $\|\bar{u}\|_{L^2(\Gamma)}$, $\|y_d\|_{C^{0,\sigma}(\overline{\Omega})}$, u_a, u_b and ν are hidden in the generic constant c.

Theorem 6.2.9. *Let Assumption 1 hold. Furthermore, let $\bar{y} = S\bar{u}$ be the optimal state and $\bar{y}_h = S_h\bar{u}_h$ be the optimal discrete state associated with the optimal control $\bar{u}$ and the discrete optimal control $\bar{u}_h$ be given by projection formulas (6.8) and (6.14), respectively.*

(i) If $\vec{\mu} < \vec{\lambda}/2$*, then the discretization error estimate*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^2 |\ln h|$$

holds.

(ii) If $\vec{\mu} = \vec{1}$ *and* $0 < \varepsilon < \lambda$*, then the error estimate*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^{\min(2,\lambda-\varepsilon)} |\ln h|$$

is valid.

Lemma 6.2.10. *Let Assumption 1 be satisfied. Furthermore, let* $\bar{p} = P(\bar{y} - y_d)$ *be the optimal adjoint state and* $\bar{p}_h = P_h(\bar{y}_h - y_d)$ *be the discrete optimal adjoint state associated with the optimal control* $\bar{u}$ *and the discrete optimal control* $\bar{u}_h$ *be given by projection formulas* (6.8) *and* (6.14)*, respectively.*

(i) If $\vec{\mu} < \vec{\lambda}/2$*, then the discretization error estimate*

$$\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \leq ch^2 |\ln h|,$$

holds.

(ii) If $\vec{\mu} = \vec{1}$ *and* $0 < \varepsilon < \lambda$*, then the error estimate*

$$\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \leq ch^{\min(2,\lambda-\varepsilon)} |\ln h|$$

is valid.

Proof. Introducing the auxiliary function $P_h(\bar{y} - y_d)$ and using the triangle inequality, there holds

$$\begin{aligned} \|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} &= \|P(\bar{y} - y_d) - P_h(\bar{y}_h - y_d)\|_{L^\infty(\Omega)} \\ &\leq \|(P - P_h)(\bar{y} - y_d)\|_{L^\infty(\Omega)} + \|P_h(\bar{y} - \bar{y}_h)\|_{L^\infty(\Omega)}. \end{aligned}$$

The estimate for the first term on the right hand side of the previous estimate follows Theorem 4.2.7 together with Theorem 6.1.2

$$\|(P - P_h)(\bar{y} - y_d)\|_{L^\infty(\Omega)} \leq ch^2 |\ln h|, \tag{6.22}$$

and the estimate for the second term follows from Lemma 6.2.2 and Theorem 6.2.9 (i)

$$\|P_h(\bar{y} - \bar{y}_h)\|_{L^\infty(\Omega)} \leq \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^2 |\ln h|. \tag{6.23}$$

The desired estimate follows from (6.22) and (6.23).

To show the estimate on quasi-uniform meshes one can use Corollary 4.2.8 instead of Theorem 4.2.7 in (6.22) and Theorem 6.2.3 (ii) instead of its first part in (6.23). □

The main result of this section is stated in the following theorem.

Theorem 6.2.11. *Let Assumption 1 be fulfilled. Moreover, let $\bar{u}$ and $\tilde{u}_h$ be given by projection formulas (6.8) and (6.21), respectively.*

(i) If $\vec{\mu} < \vec{\lambda}/2$, then the discretization error estimate

$$\|\bar{u} - \tilde{u}_h\|_{L^\infty(\Gamma)} \le ch^2 |\ln h|,$$

holds

(ii) If $\vec{\mu} = \vec{1}$ and $0 < \varepsilon < \lambda$, then the error estimate

$$\|\bar{u} - \tilde{u}_h\|_{L^\infty(\Gamma)} \le ch^{\min(2,\lambda-\varepsilon)} |\ln h|$$

is valid.

Proof. Taking into account projection formula (6.21), the proof is identical to the proof of Theorem 6.2.5. □

Remark 6.2.12. *In case of the postprocessing approach it is possible to show the same convergence rates discussed in Remark 6.2.6 for the error of the optimal state. However, the proof is not as trivial as the one in case of variational discretization. It requires some auxiliary results involving estimates for the so-called regularized Green's function.*

Note also that, if the control possesses no kinks, the quasi-optimal convergence rates for the state can be recovered on graded meshes.

6.3 Numerical example for the postprocessing approach

In this section we illustrate the theoretical estimates for the postprocessing approach. For this reason, we compute the maximum norm error for the example from [76, Example 4.22].

The optimal control problem we consider reads as

$$\frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_\omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma_\omega)} + \int_{\Gamma_\omega} g_1 y \to \min!$$

subject to

$$\begin{aligned} -\Delta y + y &= f \qquad && \text{in } \Omega_\omega, \\ \partial_n y &= u + g_2 && \text{on } \Gamma_\omega, \end{aligned}$$

$$u \in U_{ad} := \{u \in L^2(\Gamma_\omega) : u_a \le u \le u_b \text{ a.e. on } \Gamma_\omega\},$$

where the computational Ω_ω is given by (4.61).

The control problem from above has the additional term $\int_{\Gamma_\omega} g_1 y$ in the objective functional and the additional functions f and g_2 on the in the state equation, which makes it possible to choose the data such that the exact solution has exactly the proven regularity. Nevertheless, one can analyze this problem analogously to the initial one. The optimality system of the problem reads as

$$\begin{aligned} -\Delta y + y &= f && \text{in } \Omega_\omega, \\ \partial_n y &= u + g_2 && \text{on } \Gamma_\omega, \\ -\Delta p + p &= y - y_d && \text{in } \Omega_\omega, \\ \partial_n p &= g_1 && \text{on } \Gamma_\omega, \\ u &= \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}p\right) && \text{on } \Gamma_\omega. \end{aligned}$$

We set $\nu = 1$, $u_a = -0.5$ and $u_b = 0.5$. Moreover, the data f, y_d, g_1 and g_2 are chosen as follows

$$\begin{aligned} f &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ y_d &= 2r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g_1 &= -\partial_n \left(r^\lambda \cos(\lambda\varphi)\right) && \text{on } \Gamma_\omega, \\ g_2 &= \partial_n \left(r^\lambda \cos(\lambda\varphi)\right) - \Pi_{[u_a,u_b]}\left(r^\lambda \cos(\lambda\varphi)\right) && \text{on } \Gamma_\omega. \end{aligned}$$

The the unique solution of this problem is given by

$$\begin{aligned} \bar y &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ \bar p &= -r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ \bar u &= \Pi_{[u_a,u_b]}\left(r^\lambda \cos(\lambda\varphi)\right) && \text{on } \Gamma_\omega, \end{aligned}$$

which has exactly the regularity shown in Theorem 6.1.2.

We solve the discrete optimality system above using a primal-dual active set strategy described in [87, Section 2.12.4], see also [21, 53, 46], and calculate the experimental order of convergence $\mathrm{eoc}(L^\infty(\Gamma_\omega))$ by

$$\mathrm{eoc}(L^\infty(\Gamma_\omega)) := \frac{\ln\left(\|\bar u - u_{h_{k-1}}\|_{L^\infty(\Gamma_\omega)} / \|\bar u - u_{h_k}\|_{L^\infty(\Gamma_\omega)}\right)}{\ln(h_{k-1}/h_k)}. \tag{6.24}$$

Example in a convex domain

The first domain we consider is $\Omega_{3\pi/4}$. In Table 6.1 one can find the computed errors for the control variable on a sequence of quasi-uniform meshes ($\mu = 1$) and a sequence of graded meshes ($\mu = 0.6$). In the first case the calculated convergence order is equal to $\lambda - \varepsilon = 4/3 - \varepsilon$. In the case of graded meshes the calculated convergence rate tends to 2. In both cases the calculated convergence orders agree with the theoretical results from Theorem 6.2.11.

	$\mu = 1$		$\mu = 0.6$	
mesh size h	$\|e_h\|_{L^\infty(\Gamma_\omega)}$	eoc	$\|e_h\|_{L^\infty(\Gamma_\omega)}$	eoc
0.403914	5.04e-03		6.00e-03	
0.233893	1.82e-03	1.47	2.04e-03	1.98
0.135498	7.53e-04	1.27	9.11e-04	1.47
0.070628	3.01e-04	1.32	2.73e-04	1.85
0.036008	1.20e-04	1.32	7.63e-05	1.89
0.018176	4.80e-05	1.33	2.04e-05	1.93
0.009131	1.91e-05	1.33	5.37e-06	1.94
0.004587	7.58e-06	1.33	1.40e-06	1.95

Table 6.1: Discretization errors $e_h = u - u_h$ with $\omega = 3\pi/4$.

	$\mu = 1$		$\mu = 0.6$		$\mu = 0.3$	
mesh size h	$\|e_h\|_{L^\infty(\Gamma_\omega)}$	eoc	$\|e_h\|_{L^\infty(\Gamma_\omega)}$	eoc	$\|e_h\|_{L^\infty(\Gamma_\omega)}$	eoc
0.403914	3.23e-02		3.81e-02		4.52e-02	
0.233893	2.25e-02	0.52	1.77e-02	1.40	2.50e-02	1.53
0.135498	1.48e-02	0.60	8.21e-03	1.40	1.21e-02	1.89
0.070628	9.53e-03	0.64	3.82e-03	1.17	3.60e-03	2.16
0.036008	6.05e-03	0.66	1.77e-03	1.14	9.47e-04	2.12
0.018176	3.82e-03	0.66	8.18e-04	1.13	2.41e-04	2.08
0.009131	2.41e-03	0.67	3.78e-04	1.12	6.05e-05	2.05
0.004587	1.52e-03	0.67	1.75e-04	1.12	1.52e-05	2.03

Table 6.2: Discretization errors $e_h = u - u_h$ with $\omega = 3\pi/2$.

Example in a non-convex domain

As an example for a non-convex domain we consider $\Omega_{3\pi/2}$. We calculated convergence rates for the control variable on three different meshes and collected them in Table 6.2. We observe that on the sequence of quasi-uniform meshes the convergence rate is equal to $\lambda - \varepsilon = 2/3 - \varepsilon$. In case of the non-optimal grading parameter $\mu = 0.6$, the condition $\mu < \lambda/2$ is not fulfilled, and this is why the convergence rates are also sub-optimal. If the mesh is graded optimally (in this case $\mu = 0.3 < \lambda/2$), the calculated rates of convergence are also optimal. Hence, the theoretical results from Theorem 6.2.5 are confirmed for the example on the non-convex domain.

CHAPTER 7

Estimates for Dirichlet boundary control problems

In this chapter we derive pointwise error estimates for another boundary control problem, namely, the Dirichlet boundary control problem given by

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \tag{7.1}$$

subject to

$$\begin{aligned} -\Delta y &= 0 \quad \text{in } \Omega, \\ y &= u \quad \text{on } \Gamma, \end{aligned} \tag{7.2}$$

$$U_{ad} := \{u \in L^2(\Gamma) : u_a \leq u \leq u_b \text{ a.e. on } \Gamma\}. \tag{7.3}$$

First of all, we point out that we derive pointwise error estimates on quasi-uniform triangulations of convex domains only.

As in Chapter 6, we denote by $J(\cdot,\cdot)$ the objective functional and by y_d the desired state, which initially is a function from $L^2(\Omega)$, and later on is assumed to be as regular as we need. The state variable $y \in H^{1/2}(\Omega)$ is the very weak solution of state equation (7.2) and the control variable $u \in L^2(\Gamma)$ is the natural boundary value in the corresponding very weak formulation. We also assume that the control bounds u_a, u_b are constants and $0 \in [u_a, u_b]$.

This chapter has the following structure. In the first section we collect some preliminary definitions, state optimality conditions for the continuous problem, discuss the singular behavior of the optimal control, and derive regularity results. In the second section we discuss the FE discretization of the optimal control problem and derive sharp pointwise error estimates for the control on quasi-uniform meshes. In the last section we confirm the theoretical results via numerical experiments.

7.1 Analysis of the continuous problem

We start the investigation of the optimal control problem above with the analysis of the state equation. We also state the corresponding adjoint equation and give the first order optimality conditions for the continuous problem. The latter parts of this section deal with the singular behavior of the optimal control in the vicinity of corner points and regularity results.

Preliminaries

Remember, in Chapter 5 we discussed that for the initial choice of $u \in L^2(\Gamma)$ we are allowed to consider only very weak solutions of (7.2). The corresponding very weak formulation of the state equation reads as:

Find $y \in L^2(\Omega)$ such that

$$(y, \Delta v)_{L^2(\Omega)} = (u, \partial_n v) \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega). \tag{7.4}$$

We define by

$$S \colon L^2(\Gamma) \to L^2(\Omega), \quad u \mapsto Su := y$$

the solution operator of (7.4). As before, inserting Su into (7.1), we eliminate the state variable and get the reduced objective functional

$$j(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \quad \text{s.t. } u \in U_{ad} \tag{7.5}$$

The adjoint equation is given by

$$\begin{aligned} -\Delta p &= z \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $z \in L^2(\Omega)$, and the corresponding weak formulation read as:

Find $p \in H^1_0(\Omega)$ such that

$$a(p, v) = (z, v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega), \tag{7.6}$$

where the bilinear form is given by (5.4). We define the solution operator of adjoint equation by

$$P \colon L^2(\Omega) \to H^1_0(\Omega), \quad z \mapsto Pz := p.$$

From now on we define

$$\lambda := \min_{j \in \mathcal{C}} \lambda_{j,1}, \quad \lambda_{j,1} := \frac{\pi}{\omega_j}.$$

For solutions of (7.6) the following a priori result holds.

Lemma 7.1.1. *Let $1/2 < s < \min(5/2, \lambda)$. Then for every $z \in H^{s-1}(\Omega)$ there exists a unique solution $p \in H_0^1(\Omega) \cap H^{s+1}(\Omega)$ of (7.6), and the a priori estimate*

$$\|p\|_{H^{s+1}(\Omega)} \leq c\|z\|_{H^{s-1}(\Omega)}$$

holds.

Proof. The desired a priori estimate is a direct consequence of [29, Corollary 18.18]. □

Optimality conditions

Next, we give the first order optimality conditions for the optimal control problem (7.5). This result for convex domains can be found in [24], see also [3, Lemma 3.1] for general polygonal domains.

Lemma 7.1.2. *The optimal control problem (7.5) possesses a unique solution $\bar{u} \in U_{ad}$ associated with the optimal state $\bar{y} = S\bar{u}$ and the optimal adjoint state $\bar{p} = P(S\bar{u} - y_d)$. Moreover, the variational inequality*

$$(\nu\bar{u} - \partial_n\bar{p}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad} \tag{7.7}$$

holds, and is equivalent to the projection formula

$$\bar{u}(x) = \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n\bar{p}(x)\right) \text{ for a.a. } x \in \Gamma. \tag{7.8}$$

where the operator $\Pi_{[u_a,u_b]}$ is defined according to (6.9).

Remark 7.1.3. *From [27, Lemma 3.4] we can deduce that the normal derivative of the optimal adjoint state is given by*

$$(\partial_n\bar{p}, z)_{L^2(\Gamma)} = -(\bar{y} - y_d, z)_{L^2(\Omega)} + a(\bar{p}, z)_{L^2(\Omega)} \quad \forall z \in H^1(\Omega), \tag{7.9}$$

which is nothing else but the first Green formula.

Singular behavior of the optimal control

In this section we discuss the singular behavior of the optimal adjoint state and the optimal control near corners. This is essential for the study of the regularity of the corresponding quantities. For a better comprehension of the forthcoming discussion we recall the following regularity result from [3, Theorem 3.2].

Lemma 7.1.4. *Let $y_d \in L^\infty(\Omega)$, $m \in \mathbb{Z}$ and $p \in (1, \infty]$. Define*

$$\mathcal{J}_{t,p}^m = \{j \in \mathcal{C} \text{ s.t. } 0 < m\lambda_j < 2 + t - 2/p \text{ and } m\lambda_j \notin \mathbb{Z}\}.$$

Moreover, let $\bar{p} = P(S\bar{u} - y_d)$ *be the optimal adjoint state, then there exists a unique function* $\bar{p}_{reg} \in W^{2,p}(\Omega)$*, with* $p < +\infty$ *for the constrained problem and*

$$p < p_D := \frac{2}{1 - \min(1, \lambda)},$$

for the unconstrained problem, and unique real numbers $\{c_{j,m}\}_{j \in \mathcal{J}^m_{0,p}}$ *such that*

$$\bar{p} = \bar{p}_{reg} + \sum_{m=1}^{3} \sum_{j \in \mathcal{J}^m_{0,p}} c_{j,m} \xi_j r_j^{m\lambda_j} \sin(m\lambda_j \varphi), \tag{7.10}$$

where $\{\xi_j\}_{j \in \mathcal{C}}$ *are cut-off functions, which are equal to one in some neighborhood of the corner points and decay smoothly to zero.*

First, from (7.10) we deduce that

$$\lim_{r_j \to 0} \partial_n \bar{p}(r, \varphi) = \begin{cases} 0, & \text{if } \omega_j < \pi, \\ \pm\infty, & \text{if } \omega_j > \pi, \end{cases} \tag{7.11}$$

where $\varphi \in \{0, \omega_j\}$. Second, we also know that projection formula (7.8) holds, and therefore, in case of unconstrained problems ($-u_a = u_b = +\infty$) the pointwise control error is measurable on convex domains only. However, in case of constrained problems the optimal control is flattened in the neighborhood of concave corners, which allows to measure the pointwise control error on general polygonal domains, see the numerical example in Section 7.3. In this chapter, however, due to some technical difficulties, we consider only constrained problems on convex domains. Nevertheless, our results with some modifications can be extended to arbitrary polygonal domains.

We also recall a result from [3, Corollary 4.4] which we use to show the regularity of the optimal adjoint state.

Lemma 7.1.5. *Let* $y_d \in W^{1,q}(\Omega)$ *with some* $q > 2$, $m \in \mathbb{Z}$ *and* $p \in (1, \infty]$. *Define*

$$\mathcal{L}^m_p = \{j \in \mathcal{C} \text{ s.t. } 0 < m\lambda_j < 3 - 2/p \text{ and } m\lambda_j \in \mathbb{Z}\}.$$

Moreover, let $\bar{p} = P(S\bar{u} - y_d)$ *be the optimal adjoint state. Then, for* $p > 2$ *with*

$$\frac{2p-2}{\lambda_j p} \notin \mathbb{Z} \quad \forall j \in \mathcal{C},$$

and

$$p \le q, \quad p < \min\left(p_D, \frac{2}{2 - \min(\lambda, 2)}\right)$$

there exists a unique function $\bar{p}_{reg} \in W^{3,p}(\Omega)$ *and unique real numbers* $(c_{j,m})_{j \in \mathcal{J}^m_{1,p}}$ *and* $(d_{j,m})_{j \in \mathcal{L}^m_p}$ *such that*

$$\begin{aligned} \bar{p} = \bar{p}_{reg} &+ \sum_{m=1}^{5} \sum_{j \in \mathcal{J}^m_{1,p}} c_{j,m} \xi_j r_j^{m\lambda_j} \sin(m\lambda_j \varphi) \\ &+ \sum_{m=1,3} \sum_{j \in \mathcal{L}^m_p} d_{j,m} \xi_j r_j^2 \left(\log(r_j) \sin(2\varphi_j) + \varphi_j \cos(2\varphi_j)\right), \end{aligned} \tag{7.12}$$

where the cut-off functions $\{\xi_j\}_{j\in\mathcal{C}}$ are defined as in Lemma 7.1.4.

Regularity

From now on we assume that $y_d \in W^{1,q}(\Omega)$ with some $q > 2$.

Lemma 7.1.6. *Let $\bar{p} = P(S\bar{u} - y_d)$ be the optimal adjoint state associated with the optimal control $\bar{u} \in U_{ad}$. Then*

$$\begin{aligned} \bar{p} &\in H^{\min(3,\lambda+1-\varepsilon)}(\Omega), \\ \partial_n \bar{p} &\in H^{\min(3/2,\lambda-1/2-\varepsilon)}(\Gamma_j), \\ \partial_n \bar{p} &\in C^{0,\min(1,\lambda-1-\varepsilon)}(\Gamma_j), \quad \forall j \in \mathcal{C}. \end{aligned}$$

Proof. The first assertion is a direct consequence of Lemma 7.1.5 together with Lemma 2.1.6 (i). The second one can be deduced from Lemma 7.1.5, [42, Theorem 1.5.1.2] and Lemma 2.1.6 (i). The third assertion follows from the second one and Lemma 2.1.6 (ii) if $\lambda \leq 2$ or from Lemma 2.1.6 (iii) if $\lambda > 2$. □

Remark 7.1.7. *Note that in the previous lemma it is not possible to get the desired regularity on the whole boundary, since it is only Lipschitz regular. However, the piecewise regularity stated above is sufficient for the forthcoming FE error analysis.*

The following regularity result for the optimal state can be deduced from [3, Corollary 4.2].

Lemma 7.1.8. *Let $\bar{y} = S\bar{u}$ be the optimal state associated with the optimal control $\bar{u} \in U_{ad}$. Moreover, let the optimal control has a finite number of kink points. Then there holds*

$$\bar{y} \in H^{\min(2,\lambda)-\varepsilon}(\Omega).$$

7.2 Discretization error estimates

In this section we derive sharp pointwise error estimates on quasi-uniform triangulations for the Dirichlet control problem (7.1)–(7.3).

We discretize the optimal control problem according to [4]. The control is discretized by piecewise linear ansatz functions on the boundary

$$U_h := \left\{ u_h \in C(\overline{\Omega}) : u_h \in \mathcal{P}_1(E) \text{ for all } E \in \mathcal{E}_h \right\},$$

and the set of admissible controls is given by

$$U_{h,ad} := U_h \cap U_{ad}.$$

The state is discretized by piecewise linear functions in the domain and the corresponding variational formulation reads as:

Find $y_h \in V_h$ such that

$$a(y_h, v_h) = 0 \quad \forall v_h \in V_{h,0} \quad \text{and} \quad y_h|_\Gamma = u_h, \tag{7.13}$$

where $V_{h,0}$ is given by (5.6), and the discrete solution operator $S_h \colon L^2(\Gamma) \to V_h$ is defined according to (3.1) in [4]

$$a(S_h u, v_h) = 0 \quad \forall v_h \in V_{h,0} \quad \text{and} \quad (S_h u - u, u_h)_{L^2(\Gamma)} = 0 \quad \forall u_h \in U_h. \tag{7.14}$$

The applicability of the nonconforming discretization (7.14) is guaranteed by [6].

The discretized optimal control problem in the reduced form is given by

$$j_h(u_h) := \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Gamma)} \to \min! \quad \text{s.t. } u_h \in U_{h,ad}. \tag{7.15}$$

The adjoint state is also discretized by piecewise linears and the corresponding variational equation reads as:

Find $p_h \in V_{h,0}$ such that

$$a(p_h, v_h) = (z, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_{h,0}, \tag{7.16}$$

where $z \in L^2(\Omega)$. The discrete adjoint operator P_h is given by

$$P_h \colon L^2(\Omega) \to V_{h,0}, \quad z \mapsto P_h z := p_h.$$

The following first order optimality conditions for the discretized optimal control problem are given in [24].

Lemma 7.2.1. *There exists a unique element $\bar{u}_h \in U_{h,ad}$, which solves the optimal control problem (7.15) associated with the discrete optimal state $\bar{y}_h = S_h \bar{u}_h$ and the discrete optimal adjoint state $\bar{p}_h = P_h(S_h \bar{u}_h - y_d)$. Moreover, the variational inequality*

$$(\nu \bar{u}_h - \partial_n^h \bar{p}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \quad \forall u_h \in U_{h,ad} \tag{7.17}$$

holds, where the discrete normal derivative $\partial_n^h \bar{p}_h \in U_h$ is defined as the unique solution of

$$(\partial_n^h \bar{p}_h, v_h)_{L^2(\Gamma)} = -(\bar{y}_h - y_d, v_h)_{L^2(\Omega)} + a(\bar{p}_h, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h. \tag{7.18}$$

The next lemma is devoted to the stability of the discrete solution operators.

Lemma 7.2.2. *Let $z \in H^\varepsilon(\Omega)$ with some $\varepsilon > 0$ and $v \in L^2(\Gamma)$. Moreover let P_h and S_h be the discrete adjoint operator and the discrete solution operator, respectively. Then the estimates*

$$|P_h z|_{W^{1,\infty}(\Omega)} \leq c\|z\|_{H^\varepsilon(\Omega)},$$
$$\|S_h v\|_{H^s(\Omega)} \leq c\|v\|_{L^2(\Gamma)}$$

hold, where $0 < s \leq 1/2$.

Proof. The first estimate follows from Theorem 5.1.3 with $\chi = 0$, the embedding $H^{2+\varepsilon}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ from Lemma 2.1.6 (ii) and Lemma 7.1.1

$$|P_h z|_{W^{1,\infty}(\Omega)} \leq c\|Pz\|_{W^{1,\infty}(\Omega)} \leq c\|Pz\|_{H^{2+\varepsilon}(\Omega)} \leq c\|z\|_{H^\varepsilon(\Omega)}.$$

We start the proof for the second estimate by introducing the auxiliary function Sv and using the triangle inequality

$$\|S_h v\|_{H^s(\Omega)} \leq \|(S - S_h)v\|_{H^s(\Omega)} + \|Sv\|_{H^s(\Omega)}. \tag{7.19}$$

The second term can be estimated using the trivial embedding $H^{1/2}(\Omega) \hookrightarrow H^s(\Omega)$ and Theorem 5.2.1

$$\|Sv\|_{H^s(\Omega)} \leq c\|Sv\|_{H^{1/2}(\Omega)} \leq c\|v\|_{L^2(\Gamma)}. \tag{7.20}$$

In order to estimate the first term in (7.19), we use the Carstensen interpolant C_h defined by (3.8), and apply the triangle inequality

$$\|Sv - S_h v\|_{H^s(\Omega)} \leq \|Sv - C_h(Sv)\|_{H^s(\Omega)} + \|C_h(Sv) - S_h v\|_{H^s(\Omega)}. \tag{7.21}$$

For the first term in (7.21) we can apply Lemma 3.2.3 and Theorem 5.2.1

$$\|Sv - C_h(Sv)\|_{H^s(\Omega)} \leq ch^{1/2-s}\|v\|_{L^2(\Gamma)}.$$

For the second term in (7.21) we use an inverse inequality from Lemma 3.1.2 and the triangle inequality with the additional term Sv. We have

$$\|C_h(Sv) - S_h v\|_{H^s(\Omega)} \leq ch^{-s}\left(\|Sv - C_h(Sv)\|_{L^2(\Omega)} + \|Sv - S_h v\|_{L^2(\Omega)}\right). \tag{7.22}$$

Once again, we use Lemma 3.2.3 and Theorem 5.2.1 to estimate the first term in (7.22) and the estimate for the second term in (7.22) is provided by Lemma 5.2.2

$$\|C_h(Sv) - S_h v\|_{H^s(\Omega)} \leq ch^{1/2-s}\|v\|_{L^2(\Gamma)}.$$

This concludes the proof. □

For the FE discretization of the optimal state we have the following result. This result is motivated by Lemma 7.2.5, where we apply the boundedness of the discrete solution operator P_h between the $H^\varepsilon(\Omega)$-norm and the $W^{1,\infty}(\Omega)$-seminorm.

Lemma 7.2.3. *Let $\bar{y} = S\bar{u}$ be the optimal state and $\bar{y}_h = S_h\bar{u}_h$ the discrete optimal state associated with the optimal control $\bar{u}$ given by (7.8) and the discrete optimal control $\bar{u}_h$ defined by (7.17), respectively. Furthermore, let the optimal control has a finite number of kink points. Then the discretization error estimate*

$$\|S\bar{u} - S_h\bar{u}\|_{H^\varepsilon(\Omega)} \leq ch^{\min(2,\lambda)-2\varepsilon}$$

holds, where $0 < \varepsilon < \min(1, \lambda - 1)$.

Proof. Since we consider only the case $\lambda > 1$, the optimal state is at least an $H^1(\Omega)$-function, and the desired result can be deduced from the proof of [4, Lemma 4.2], which is based on the considerations from [19]. We have

$$\|S\bar{u} - S_h\bar{u}\|_{L^2(\Omega)} \le ch^{\min(2,\lambda)-\varepsilon},$$

and

$$\|S\bar{u} - S_h\bar{u}\|_{H^1(\Omega)} \le ch^{\min(1,\lambda-1)-\varepsilon}.$$

The desired estimate follows from a standard interpolation argument between the spaces $L^2(\Omega)$ and $H^1(\Omega)$. □

In comparison to the Neumann boundary control problem, the derivation of pointwise discretization error estimates for the optimal control depends on both an $L^2(\Gamma)$-error estimate for the optimal control and an $L^2(\Omega)$-error estimate for the optimal state. These estimates can be deduced from [4, Theorem 5.1].

Theorem 7.2.4. *Let $\bar{y} = S\bar{u}$ be the optimal state and $\bar{y}_h = S_h\bar{u}_h$ the discrete optimal state associated with the optimal control $\bar{u}$ given by (7.8) and the discrete optimal control $\bar{u}_h$ defined by (7.17). Moreover, let the optimal control has a finite number of kink points. Then the discretization error estimates*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^{\min(1,\lambda-1/2-\varepsilon)}|\ln h|^r$$

hold, where $0 < \varepsilon < \lambda - 1/2$ and

$$r = \begin{cases} 1 & \text{if } \lambda \in (3/2, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{7.23}$$

Remember, in case of pointwise error estimates for Neumann optimal control problems the estimate for the control depends on the estimate of the same quality for the adjoint state. Here, the pointwise error estimate for the control depends on the pointwise estimate for the gradient of the error of the adjoint state.

Lemma 7.2.5. *Let $\bar{p} = P(S\bar{u} - y_d)$ be the adjoint state and $\bar{p}_h = P_h(S_h\bar{u}_h - y_d)$ be the discrete adjoint state associated with the optimal control $\bar{u}$ given by (7.8) and the discrete optimal control $\bar{u}_h$ defined by (7.17). Furthermore, let the optimal control has a finite number of kink points. Then the dicretization error estimate*

$$\|\nabla(\bar{p} - \bar{p}_h)\|_{L^\infty(\Omega)} \le ch^{\min(1,\lambda-1-\varepsilon)}$$

holds, provided that $0 < \varepsilon < \lambda - 1$.

Proof. We start this proof as usual, by introducing auxiliary functions and using the triangle inequality

$$\begin{aligned} \|\nabla(\bar{p} - \bar{p}_h)\|_{L^\infty(\Omega)} &= |P(S\bar{u} - y_d) - P_h(S_h\bar{u}_h - y_d)|_{W^{1,\infty}(\Omega)} \\ &\le |(P - P_h)(S\bar{u} - y_d)|_{W^{1,\infty}(\Omega)} + |P_h(S\bar{u} - S_h\bar{u})|_{W^{1,\infty}(\Omega)} \\ &\quad + |P_hS_h(\bar{u} - \bar{u}_h)|_{W^{1,\infty}(\Omega)}. \end{aligned} \tag{7.24}$$

The estimate for the first term in (7.24) follows from the best approximation property given in Theorem 5.1.3, interpolation error estimates from Lemma 3.2.2, and the regularity result from Lemma 7.1.6. We have

$$|(P - P_h)(S\bar{u} - y_d)|_{W^{1,\infty}(\Omega)} \leq ch^{\min(1,\lambda-1-\varepsilon)}. \tag{7.25}$$

In order to estimate the second term in (7.24), we use Lemma 7.2.2 and Lemma 7.2.3

$$|P_h(S\bar{u} - S_h\bar{u})|_{W^{1,\infty}(\Omega)} \leq c\|S\bar{u} - S_h\bar{u}\|_{H^\varepsilon(\Omega)} \leq h^{\min(2,\lambda)-2\varepsilon}. \tag{7.26}$$

The estimate for the third term follows from a double application of Lemma 7.2.2 with $s = \varepsilon$ and the estimate for the control from Theorem 7.2.4

$$\begin{aligned} |P_h S_h(\bar{u} - \bar{u}_h)|_{W^{1,\infty}(\Omega)} &\leq c\|S_h(\bar{u} - \bar{u}_h)\|_{H^\varepsilon(\Omega)} \\ &\leq c\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \\ &\leq ch^{\min(1,\lambda-1/2-\varepsilon)}|\ln h|^r. \end{aligned} \tag{7.27}$$

The desired assertion follows from estimates (7.25), (7.26) and (7.27). Note that, due to the definition of the parameter r, the logarithmic term is always hidden in $h^{-\varepsilon}$. □

Now, we are able to formulate the main result of this chapter.

Theorem 7.2.6. *Let the optimal control $\bar{u}$ be given by (7.8) and the discrete optimal control $\bar{u}_h$ be defined by (7.17). Moreover, let the optimal control has a finite number of kink points. Then the dicretization error estimate*

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq ch^{\min(1,\lambda-1-\varepsilon)}$$

holds, provided that $0 < \varepsilon < \lambda - 1$.

Before we proceed with auxiliary results for the proof of the theorem above, we observe that the projection formula does not hold for the discrete optimal control, i.e.,

$$\bar{u}_h \neq \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h\right)$$

in general if the control constraints are present. The idea of the proof for the desired estimate is as follows. We introduce the auxiliary function

$$\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h\right),$$

and use the triangle inequality to get

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq \|\bar{u} - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h\right)\|_{L^\infty(\Gamma)} + \|\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h\right) - \bar{u}_h\|_{L^\infty(\Gamma)}. \tag{7.28}$$

We show this estimate at the nodal points in Lemma 7.2.11 and afterwards, exploiting the fact that the discrete optimal control is a piecewise linear function, we show the desired result.

In the next lemma we estimate the first term on the right hand side of the (7.28).

Lemma 7.2.7. *Let $\bar{p} = P(S\bar{u} - y_d)$ be the adjoint state and $\bar{p}_h = P_h(S_h\bar{u}_h - y_d)$ be the discrete adjoint state associated with the optimal control $\bar{u}$ given by (7.8) and the discrete optimal control $\bar{u}_h$ defined by (7.17). Furthermore, let the optimal control has a finite number of kink points. Then the dicretization error estimate*

$$\|\bar{u} - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h\right)\|_{L^\infty(\Gamma)} \leq ch^{\min(1,\lambda-1-\varepsilon)}$$

holds, provided that $0 < \varepsilon < \lambda - 1$.

Proof. We start the proof with the estimate

$$\|\bar{u} - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h\right)\|_{L^\infty(\Gamma)} \leq c\|\partial_n \bar{p} - \partial_n^h \bar{p}_h\|_{L^\infty(\Gamma)}, \tag{7.29}$$

where we used projection formula (7.8) and the Lipschitz continuity of the corresponding projection operator. We proceed estimating (7.29) by introducing the intermediate function $I_h^\partial(\partial_n \bar{p})$ and using the triangle inequality

$$\|\partial_n \bar{p} - \partial_n^h \bar{p}_h\|_{L^\infty(\Gamma)} \leq \|\partial_n \bar{p} - I_h^\partial(\partial_n \bar{p})\|_{L^\infty(\Gamma)} + \|I_h^\partial(\partial_n \bar{p}) - \partial_n^h \bar{p}_h\|_{L^\infty(\Gamma)}, \tag{7.30}$$

where the interpolation operator I_h^∂ is given by (3.7). The estimate for the first term on the right hand side of (7.30) follows the interpolation error estimates from Lemma 3.2.2 and the regularity result from Lemma 7.1.6

$$\|\partial_n \bar{p} - I_h^\partial(\partial_n \bar{p})\|_{L^\infty(\Gamma)} \leq ch^{\min(1,\lambda-1-\varepsilon)}. \tag{7.31}$$

To estimate the second term in (7.30) we denote by

$$e_h := I_h^\partial(\partial_n \bar{p}) - \partial_n^h \bar{p}_h$$

and by $E_* \in \mathcal{E}_h$ the element where e_h admits its maximum. Using Theorem 3.2.1, we get

$$\|e_h\|_{L^\infty(\Gamma)} \leq c|E_*|^{-1}\|e_h\|_{L^1(E_*)} = c\left(e_h, \delta^h\right)_{L^2(E_*)},$$

where

$$\delta^h := \begin{cases} |E_*|^{-1}\operatorname{sgn}(e_h) & \text{on } E_*, \\ 0 & \text{elsewhere} \end{cases} \tag{7.32}$$

is the so-called regularized Dirac function.

Next, we introduce the $L^2(\Gamma)$-projection operator

$$Q_h \colon L^2(\Gamma) \to U_h$$

defined by

$$(v - Q_h v, v_h)_{L^2(\Gamma)} = 0 \quad \forall v_h \in U_h. \tag{7.33}$$

For the operator Q_h the stability estimate

$$\|Q_h v\|_{L^p(\Gamma)} \le c\|v\|_{L^p(\Gamma)}, \quad p \in [1,\infty] \tag{7.34}$$

holds [34], see also [88, Lemma 3.5] or [89, Theorem 3.2.3].

Using the definition of the $L^2(\Gamma)$-projection Q_h given above and introducing the auxiliary function $\partial_n \bar{p}$, we proceed with

$$\begin{aligned}\left(e_h, \delta^h\right)_{L^2(E_*)} = \left(e_h, Q_h(\delta^h)\right)_{L^2(\Gamma)} &= \left(I_h^\partial(\partial_n \bar{p}) - \partial_n \bar{p}, Q_h(\delta^h)\right)_{L^2(\Gamma)} \\ &+ \left(\partial_n \bar{p} - \partial_n^h \bar{p}_h, Q_h(\delta^h)\right)_{L^2(\Gamma)}.\end{aligned} \tag{7.35}$$

The first integral in (7.35) can be estimated using the Hölder inequality, stability of the $L^2(\Gamma)$-projection from (7.34), estimate (7.31) and the definition of the function δ^h

$$\begin{aligned}\left(I_h^\partial(\partial_n \bar{p}) - \partial_n \bar{p}_h, Q_h(\delta^h)\right)_{L^2(\Gamma)} &\le c\|I_h^\partial(\partial_n \bar{p}) - \partial_n \bar{p}\|_{L^\infty(\Gamma)}\|Q_h(\delta^h)\|_{L^1(\Gamma)} \\ &\le ch^{\min(1,\lambda-1-\varepsilon)}\|\delta^h\|_{L^1(\Gamma)} \\ &\le ch^{\min(1,\lambda-1-\varepsilon)}.\end{aligned}$$

It remains to estimate the second integral in (7.35). For this reason we introduce the extension operator $\tilde{S}_h$, which extends by zero a function from U_h to a function from V_h. Now, testing both (7.9) and (7.18) with $\tilde{S}_h Q_h(\delta^h) \in V_h$, we get

$$\left(\partial_n \bar{p}, Q_h(\delta^h)\right)_{L^2(\Gamma)} = -\left(\bar{y} - y_d, \tilde{S}_h Q_h(\delta^h)\right)_{L^2(\Omega)} + \left(\nabla \bar{p}, \nabla \tilde{S}_h Q_h(\delta^h)\right)_{L^2(\Omega)} \tag{7.36}$$

and

$$\left(\partial_n^h \bar{p}_h, Q_h(\delta^h)\right)_{L^2(\Gamma)} = -\left(\bar{y}_h - y_d, \tilde{S}_h Q_h(\delta^h)\right)_{L^2(\Omega)} + \left(\nabla \bar{p}_h, \nabla \tilde{S}_h Q_h(\delta^h)\right)_{L^2(\Omega)}. \tag{7.37}$$

Subtracting (7.37) from (7.36) we arrive at

$$\begin{aligned}\left(\partial_n \bar{p} - \partial_n^h \bar{p}_h, Q_h(\delta^h)\right)_{L^2(\Gamma)} &= -\left(\bar{y} - \bar{y}_h, \tilde{S}_h Q_h(\delta^h)\right)_{L^2(\Omega)} + \left(\nabla(\bar{p} - \bar{p}_h), \tilde{S}_h Q_h(\delta^h)\right)_{L^2(\Omega)} \\ &\le c\Big(\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}\|\tilde{S}_h Q_h(\delta^h)\|_{L^2(\Omega)} \\ &+ \|\nabla(\bar{p} - \bar{p}_h)\|_{L^\infty(\Omega)}\|\nabla \tilde{S}_h Q_h(\delta^h)\|_{L^1(\Omega)}\Big),\end{aligned} \tag{7.38}$$

where in the last step we used the Hölder inequality. Let us first estimate the terms involving the extension operator $\tilde{S}_h$. We start with the norm $\|\tilde{S}_h Q_h(\delta^h)\|_{L^2(\Omega)}$, and observe that it does not vanish only on the boundary strip consisting of elements touching the boundary. Having this observation in mind, on each element from the boundary strip we can apply the transformation to a reference element from Theorem 3.2.1 and the norm equivalence in finite dimensional spaces, and summing over all elements, we get

$$\|\tilde{S}_h Q_h(\delta^h)\|_{L^2(\Omega)} \le c\|Q_h(\delta^h)\|_{L^1(\Gamma)}.$$

Using the stability of the $L^2(\Gamma)$-projection and the definition of the function δ^h, we obtain

$$\|\tilde{S}_h Q_h(\delta^h)\|_{L^2(\Omega)} \le c\|Q_h(\delta^h)\|_{L^1(\Gamma)} \le c\|\delta^h\|_{L^1(\Gamma)} \le c. \tag{7.39}$$

The estimate for the norm $\|\nabla \tilde{S}_h Q_h(\delta^h)\|_{L^1(\Omega)}$ can be shown in a similar way to (7.39). We have

$$\|\nabla \tilde{S}_h Q_h(\delta^h)\|_{L^1(\Omega)} \le c\|Q_h(\delta^h)\|_{L^1(\Gamma)} \le c\|\delta^h\|_{L^1(\Gamma)} \le c. \tag{7.40}$$

Finally, Theorem 7.2.4, Lemma 7.2.5 as well as estimates (7.39) and (7.40) yield

$$\left(\partial_n \bar{p} - \partial_n^h \bar{p}_h, Q_h(\delta^h)\right)_{L^2(\Gamma)} \le h^{\min(1,\lambda-1-\varepsilon)},$$

which concludes the proof. □

In order to be able to estimate the second term on the right hand side of (7.28) at the nodal points, we need the following lemma, which exploits the result from the previous one.

Lemma 7.2.8. *Let the assumptions of Lemma 7.2.7 be satisfied. Then for all nodal points $x_i \in \Gamma$ the estimate*

$$\frac{1}{\nu}|\partial_n^h \bar{p}_h(x_i) - \partial_n^h \bar{p}_h(x_{i-1})| \le ch^{\min(1,\lambda-1-\varepsilon)}$$

holds, where x_{i-1} denotes one of the neighboring nodes of x_i, and $0 < \varepsilon < \lambda - 1$.

Proof. Introducing auxiliary functions and applying the Hölder inequality, we get

$$\begin{aligned} |\partial_n^h \bar{p}_h(x_i) - \partial_n^h \bar{p}_h(x_{i-1})| \le{}& |\partial_n^h \bar{p}_h(x_i) - \partial_n \bar{p}(x_i)| + |\partial_n \bar{p}(x_i) - \partial_n \bar{p}(x_{i-1})| \\ &+ |\partial_n \bar{p}(x_{i-1}) - \partial_n^h \bar{p}_h(x_{i-1})|, \end{aligned} \tag{7.41}$$

where the estimate for the first and the third terms on the right hand side of (7.41) follows from (7.30)

$$|\partial_n^h \bar{p}_h(x_i) - \partial_n \bar{p}(x_i)| + |\partial_n \bar{p}(x_{i-1}) - \partial_n^h \bar{p}_h(x_{i-1})| \le ch^{\min(1,\lambda-1-\varepsilon)},$$

and the estimate for the second term on the right hand side of (7.41) follows from the Lipschitz/Hölder continuity of $\partial_n \bar{p}$ from Lemma 7.1.6. Namely, we have

$$|\partial_n \bar{p}(x_i) - \partial_n \bar{p}(x_{i-1})| \le c|x_i - x_{i-1}|^{\min(1,\lambda-1-\varepsilon)} \le ch^{\min(1,\lambda-1-\varepsilon)}.$$

□

Lemma 7.2.9. *Let the assumptions of Lemma 7.2.7 be fulfilled. Then the estimate*

$$\max_{x_i \in \Gamma} \left| \bar{u}_h(x_i) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_i)\right)\right| \le ch^{\min(1,\lambda-1-\varepsilon)}$$

is valid, provided that $0 < \varepsilon < \lambda - 1$.

Proof. First, we assume that

$$M := \max_{x_i \in \Gamma} \left| \bar{u}_h(x_i) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_i)\right)\right| = \left| \bar{u}_h(x_k) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_k)\right)\right| > 0. \tag{7.42}$$

Note that if $M = 0$, the desired estimate is trivially fulfilled. Next, we want to show that

$$\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_k)\right) = \frac{1}{\nu}\partial_n^h \bar{p}_h(x_k). \tag{7.43}$$

Without loss of generality we assume that

$$\left| \bar{u}_h(x_k) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_k)\right)\right| = \bar{u}_h(x_k) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_k)\right) > 0.$$

The inequality above and the fact that $\bar{u}_h \le u_b$ imply (7.43).

Moreover, without loss of generality one can show similarly to [66, Lemma 3.8] that there holds

$$u_b = \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_{k-1})\right) < \frac{1}{\nu}\partial_n^h \bar{p}_h(x_{k-1}) \tag{7.44}$$

for some neighboring node x_{k-1} of x_k. From estimate (7.44) and Lemma 7.2.8 we conclude

$$0 < u_b - \frac{1}{\nu}\partial_n^h \bar{p}_h(x_k) \le c h^{\min(1,\lambda-1-\varepsilon)} \tag{7.45}$$

Note that for h small enough the inequality

$$u_b - c h^{\min(1,\lambda-1-\varepsilon)} > u_a \tag{7.46}$$

always holds. Finally, from the inequality $\bar{u}_h(x_k) \le u_b$, equality (7.43) and inequality (7.45) we get

$$\begin{aligned} 0 < \bar{u}_h(x_k) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_k)\right) &\le u_b - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h \bar{p}_h(x_k)\right) \\ &= u_b - \frac{1}{\nu}\partial_n^h \bar{p}_h(x_k) \\ &\le c h^{\min(1,\lambda-1-\varepsilon)}. \end{aligned}$$

Note that one can argue analogously for the bound u_a. □

Remark 7.2.10. *In the previous proof we stated that (7.44) can be deduced from [66, Lemma 3.8]. This is possible due to the following argument. In [66] a pointwise error estimate for the piecewise linear discretization of the control variable in a distributed control problem is considered. Hence the optimal control in that problem and in our case have a similar structure with the only difference, namely, that the optimal control in case of Dirichlet control problems is defined only on the boundary, whereas the optimal control in case of distributed control problems is defined in the underlying domain. However, for the result exploited in the previous lemma it does not make any big difference.*

In the next lemma we show the desired result at the nodal points.

Lemma 7.2.11. *Let the assumptions of Lemma 7.2.7 hold. Then the estimate*

$$\max_{x_i\in\Gamma}|\bar u(x_i)-\bar u_h(x_i)| \le ch^{\min(1,\lambda-1-\varepsilon)}$$

holds, provided that $0<\varepsilon<\lambda-1$.

Proof. We start this proof like we did in (7.28), but only at the nodal points. We introduce the intermediate value $\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h\bar p_h(x_i)\right)$ and applying the triangle inequality we get

$$\begin{aligned}\max_{x_i\in\Gamma}|\bar u(x_i)-\bar u_h(x_i)| \le{}& \max_{x_i\in\Gamma}\left|\bar u(x_i)-\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h\bar p_h(x_i)\right)\right| \\ &+\max_{x_i\in\Gamma}\left|\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h\bar p_h(x_i)\right)-\bar u_h(x_i)\right|. \end{aligned} \tag{7.47}$$

The estimate for the first term on the right hand side of (7.47) follows from projection formula (7.8), the Lipschitz continuity of the projection operator and Lemma 7.2.7

$$\begin{aligned}\max_{x_i\in\Gamma}\left|\bar u(x_i)-\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h\bar p_h(x_i)\right)\right| &= \max_{x_i\in\Gamma}\left|\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n\bar p(x_i)\right)-\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n^h\bar p_h(x_i)\right)\right| \\ &\le c\|\partial_n\bar p-\partial_n^h\bar p_h\|_{L^\infty(\Gamma)} \\ &\le ch^{\min(1,\lambda-1-\varepsilon)}.\end{aligned}$$

The estimate for the second term on the right hand side of (7.47) is given in Lemma 7.2.9. □

Remark 7.2.12. *Note that, if inequality (7.46) does not hold, the assertion of Lemma 7.2.11 trivially holds true. This means*

$$\max_{x_i\in\Gamma}|\bar u(x_i)-\bar u_h(x_i)| \le u_b-u_a\le ch^{\min(1,\lambda-1-\varepsilon)},$$

where in the last step we used that (7.46) does not hold.

In the previous lemma we showed the pointwise error estimate for the control at the nodal points $x_i\in\Gamma$. To get the estimate of the same quality on the whole boundary Γ, we use fact that the discrete optimal control is linear on each element. The main result of this section can be proven as follows.

Proof of Theorem 7.2.6. Assume that $|\bar u-\bar u_h|(x)$ attains its maximum at some point $x_*\in E_*=(x_*^l,x_*^r)$. Since the discrete optimal control is a piecewise linear function we can argue as follows

$$\begin{aligned}|\bar u(x_*)-\bar u_h(x_*)| &= |\bar u(x_*)-\tilde\lambda\bar u_h(x_*^l)-(1-\tilde\lambda)\bar u_h(x_*^r)| \\ &\le \tilde\lambda\left(|\bar u(x_*)-\bar u(x_*^l)|+|\bar u(x_*^l)-\bar u_h(x_*^l)|\right) \\ &+(1-\tilde\lambda)\left(|\bar u(x_*)-\bar u(x_*^r)|+|\bar u(x_*^r)-\bar u_h(x_*^r)|\right),\end{aligned}\tag{7.48}$$

where $\tilde{\lambda} \in (0,1)$. Using Lemma 7.2.11, we get

$$\tilde{\lambda}|\bar{u}(x_*^l) - \bar{u}_h(x_*^l)| + (1-\tilde{\lambda})|\bar{u}(x_*^r) - \bar{u}_h(x_*^r)| \le ch^{\min(1,\lambda-1-\varepsilon)}. \tag{7.49}$$

Next, exploiting the Lipschitz continuity of the projection operator onto the admissible set and the Lipschitz/Hölder continuity of the normal derivative of the optimal adjoint state, we obtain

$$\begin{aligned} |\bar{u}(x_*) - \bar{u}(x_*^l)| &= \left|\Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n\bar{p}(x_*)\right) - \Pi_{[u_a,u_b]}\left(\frac{1}{\nu}\partial_n\bar{p}(x_*^l)\right)\right| \\ &\le c|\partial_n\bar{p}(x_*) - \partial_n\bar{p}(x_*^l)| \\ &\le c|x_* - x_*^l|^{\min(1,\lambda-1-\varepsilon)} \\ &\le ch^{\min(1,\lambda-1-\varepsilon)}. \end{aligned} \tag{7.50}$$

Obviously, there also holds

$$|\bar{u}(x_*) - \bar{u}(x_*^r)| \le ch^{\min(1,\lambda-1-\varepsilon)} \tag{7.51}$$

Inserting estimates (7.49), (7.50) and (7.51) into (7.48) concludes the proof. □

Having the pointwise error estimate for the control, we can show the following pointwise estimate for the state.

Corollary 7.2.13. *Let $\bar{y} = S\bar{u}$ be the optimal state and $\bar{y}_h = S_h\bar{u}_h$ the discrete optimal state associated with the optimal control $\bar{u}$ given by (7.8) and the discrete optimal control $\bar{u}_h$ defined by (7.17). Moreover, let the optimal control has a finite number of kink points. Then the discretization error estimate*

$$\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \le ch^{\min(1,\lambda-1)-\varepsilon}$$

holds, provided that $0 < \varepsilon < \min(1, \lambda - 1)$.

Proof. Using the auxiliary function $S_h\bar{u}$ and applying the triangle inequality, we obtain

$$\|S\bar{u} - S_h\bar{u}_h\|_{L^\infty(\Omega)} \le \|S\bar{u} - S_h\bar{u}\|_{L^\infty(\Omega)} + \|S_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)}. \tag{7.52}$$

The estimate for the first term on the right hand side of (7.52) follows from the best approximation property given in [79, Theorem 2], an interpolation error estimate from Lemma 3.2.2 and the regularity result from Lemma 7.1.8

$$\|S\bar{u} - S_h\bar{u}\|_{L^\infty(\Omega)} \le c|\ln h|\|S\bar{u} - I_h(S\bar{u})\|_{L^\infty(\Omega)} \le ch^{\min(1,\lambda-1)-\varepsilon}.$$

To estimate the second term, we recall that the solution operator S_h coincides with the $L^2(\Gamma)$-projection on the boundary. Hence, the weak discrete maximum principle from [79, Theorem 1], the stability of the $L^2(\Gamma)$-projection and the pointwise error estimate for the control from Theorem 7.2.6 yield the estimate

$$\|S_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)} \le c\|S_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Gamma)} \le \|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \le ch^{\min(1,\lambda-1-\varepsilon)}.$$

□

Remark 7.2.14. *In the proof of the previous result we used that $\bar{y} \in H^{2-\varepsilon}(\Omega)$ for $\lambda > 2$ from Lemma 7.1.8. However, using techniques from [3], it is possible to show that $\bar{y} \in H^2(\Omega)$ for $\lambda > 2$, and consequently recover the convergence order of one in the estimate above for $\lambda > 2$.*

7.3 Numerical example

In this section we confirm the theoretical results proven in the previous section by numerical experiments. We also confirm that in case of control constrained problems we can observe better convergence rates on concave domains.

The optimal control problem we consider in this section reads as

$$\frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_\omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma_\omega)} \to \min!$$

subject to

$$\begin{aligned} -\Delta y &= f \qquad \text{in } \Omega_\omega, \\ y &= u \qquad \text{on } \Gamma_\omega, \end{aligned}$$

$$u \in U_{ad} := \{u \in L^2(\Gamma_\omega) : u_a \le u \le u_b \text{ a.e. on } \Gamma_\omega\}.$$

The optimality system of the optimal control problem above reads as

$$\begin{aligned} -\Delta y &= f && \text{in } \Omega_\omega, \\ y &= u && \text{on } \Gamma_\omega, \\ -\Delta p &= y - y_d && \text{in } \Omega_\omega, \\ p &= 0 && \text{on } \Gamma_\omega, \\ u &= \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\partial_n p\right) && \text{on } \Gamma_\omega. \end{aligned}$$

We consider an example for the unconstrained problem ($-u_a = u_b = +\infty$). This is done due to the following reason. The main goal of our numerical experiments is to observe how the convergence rates depend on the largest interior angle of the underlying domain, see Figure 7.1. In case of the unconstrained problem it is possible to construct the exact solution of the optimality system above, which guarantees more accurate calculation of the experimental order of convergence. Otherwise, if an exact solution is not given, one has to compute a reference solution u_{ref} on a much finer mesh than other solutions u_k computed on, and after calculate the convergence rates using the formula

$$\operatorname{eoc}(L^\infty(\Gamma_\omega)) := \frac{\ln\left(\|u_{ref} - u_{h_{k-1}}\|_{L^\infty(\Gamma_\omega)}/\|u_{ref} - u_{h_k}\|_{L^\infty(\Gamma_\omega)}\right)}{\ln(h_{k-1}/h_k)}, \tag{7.53}$$

where the error $\|u_{ref} - u_{h_k}\|_{L^\infty(\Gamma_\omega)}$ is evaluated at the nodal points of the mesh obtained after k refinements.

The optimality system of the optimal control problem read as

$$\begin{aligned} \bar{y} &= -\lambda r^{\lambda-1}(x_1^2-1)(x_2^2-1) + 2r^{\lambda}\sin(\lambda\varphi)\left((x_1^2-1)+(x_2^2-1)\right) && \text{in } \Omega_\omega, \\ \bar{p} &= -r^{\lambda}\sin(\lambda\varphi)(x_1^2-1)(x_2^2-1) && \text{in } \Omega_\omega, \\ \bar{u} &= -\lambda r^{\lambda-1}(x_1^2-1)(x_2^2-1) + 2r^{\lambda}\sin(\lambda\varphi)\left((x_1^2-1)+(x_2^2-1)\right) && \text{on } \Gamma_\omega. \end{aligned}$$

Note that the optimal adjoint state possesses exactly the regularity showed in Lemma 7.1.6. We set $\nu = 1$, and prescribe the input data by the exact solutions

$$\begin{aligned} f &= -\Delta\bar{y} && \text{in } \Omega_\omega, \\ y_d &= \bar{y} - \Delta\bar{p} && \text{in } \Omega_\omega. \end{aligned}$$

The corresponding discrete optimality system can be found e.g. in [58, Section 3]. We solve the system using Matlab, and calculate the experimental order of convergence $\mathrm{eoc}(L^\infty(\Gamma_\omega))$ using formula (6.24). Since we consider only convex domains, we compute numerical solutions on Γ_ω with

$$\omega \in \{\pi/2, 7\pi/12, 2\pi/3, 3\pi/4, 5\pi/6, 11\pi/12\}.$$

Recall that the computational domain Ω_ω is given by (4.61). In Table 7.1 we can see that the calculated convergence rates agree with the theoretical results, which can be also seen in Figure 7.1.

In Table 7.2 one can see the rates of convergence for constrained problems on the non-convex domains $\Gamma_{5\pi/4}$, $\Gamma_{3\pi/2}$ and $\Gamma_{7\pi/4}$. For this computations we used the same input data y_d and f as in the unconstrained one, and also set $\nu = 1$. We solved the discrete optimality system using a modification a primal-dual active set strategy [86], and calculated convergence rates according to (7.53). In this case one can observe that the convergence rates are slightly better than one, which can be explained by the fact that the optimal control is flattened in the neighborhood of concave corners by the projection formula, and hence, is more regular. Theoretical considerations on non-convex domains are postponed to the future work.

$\omega = 90°$			$\omega = 105°$		
mesh size h	e_h	eoc(expect. 1.00)	mesh size h	e_h	eoc(expext. 0.71)
0.031250	0.1301	0.83	0.323523	0.1160	0.85
0.015625	0.0686	0.92	0.161761	0.0609	0.92
0.007812	0.0352	0.96	0.008088	0.0316	0.96
0.003906	0.0178	0.98	0.004044	0.0158	0.98
0.001953	0.0089	0.99	0.002022	0.0079	0.99
0.000976	0.0045	1.00	0.001011	0.0046	0.79

$\omega = 120°$			$\omega = 135°$		
mesh size h	e_h	eoc(expect. 0.50)	mesh size h	e_h	eoc(expect. 0.33)
0.072169	0.1863	0.67	0.088388	0.1667	0.55
0.036084	0.1027	0.86	0.044194	0.0923	0.70
0.018042	0.0537	0.93	0.022090	0.0716	0.85
0.009021	0.0296	0.86	0.011048	0.0563	0.36
0.004510	0.0209	0.50	0.005524	0.0445	0.34
0.002255	0.0148	0.50	0.002762	0.0353	0.34

$\omega = 150°$			$\omega = 165°$		
mesh size h	e_h	eoc(expect. 0.20)	mesh size h	e_h	eoc(expect. 0.09)
0.125000	0.3683	0.30	0.241481	0.3975	0.40
0.062500	0.3124	0.24	0.120740	0.2094	0.21
0.031250	0.2694	0.21	0.060370	0.1293	0.13
0.015625	0.2337	0.20	0.030185	0.1033	0.10
0.007812	0.2031	0.20	0.015093	0.0952	0.09
0.003906	0.1767	0.20	0.007546	0.0925	0.09

Table 7.1: Discretization errors $e_h = u - u_h$ for the unconstrained problem.

	$\omega = 235°$		$\omega = 270°$		$\omega = 315°$	
mesh size h	e_h	eoc(expect. 1.00)	e_h	eoc(1.00)	e_h	eoc(1.00)
0.125000	0.2527	0.50	0.1344	0.37	0.3951	0.02
0.062500	0.2037	0.31	0.0823	0.71	0.2356	0.75
0.031250	0.1505	0.44	0.0506	0.70	0.1681	0.49
0.015625	0.0920	0.71	0.0308	0.72	0.1029	0.71
0.007812	0.0470	0.97	0.0166	0.89	0.0568	0.86
0.003906	0.0224	1.07	0.0071	1.21	0.0263	1.10

Table 7.2: Discretization errors $e_h = u - u_h$ for the constrained problem.

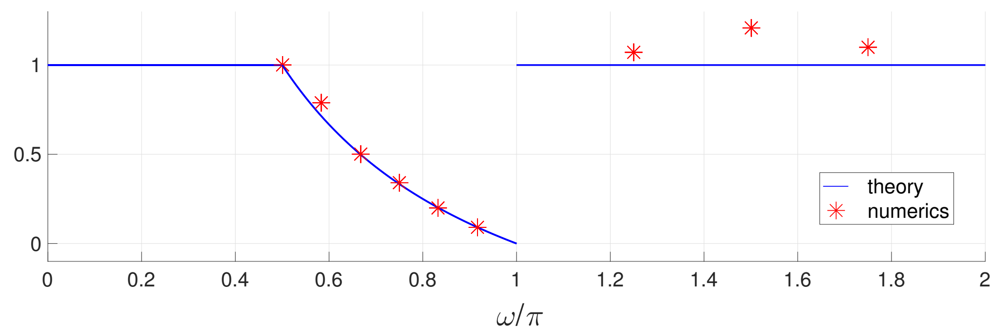

Figure 7.1: Numerically computed convergence rates depending on the largest interior angle.

CHAPTER 8

Conclusion and outlook

In this thesis we derived pointwise error estimates for finite element discretizations of boundary control problems with constant control constraints on polygonal domains. The first main result of this work is the sharp pointwise error estimate on graded meshes and the best possible rates on quasi-uniform meshes for the Neumann boundary value problem using piecewise linear elements. These estimates play a very important role in pointwise estimates for the Neumann boundary control problem, which we discretized using two approaches, namely, the concept of variational discretization and the postprocessing approach. In both cases we obtained the same convergence orders as for the Neumann problem itself. We finished this thesis by showing the best possible rates for pointwise error estimates of the piecewise linear discretization for the Dirichlet control problem on convex domains if control constraints are present. Our theoretical results have been verified with numerical implementations performed in Matlab.

The first possible extension of the present work is a derivation of maximum norm error estimates on graded meshes in three dimensional domains. Note that local error estimate (4.15) holds also in 3D. However, the exponent of the factor d_J in that estimate depends on the dimension of the underlying domain, which might cause additional difficulties in arguments similar to the ones used in Lemma 4.2.15. To the best of our knowledge there is no reference dealing with pointwise estimates for three-dimensional boundary value problems on graded meshes.

A further extension is pointwise error estimates for the constrained Dirichlet boundary control problem on non-convex polygonal domains. In this case one would expect the rate of one if all singular functions in expansion (7.12) are present, see the rates in Figure 7.1. It may, however, happen that one of the leading singularities in that expansion is not the first one, which might reduce the convergence order. In order to show such estimates one has to assume that there is no error in the optimal control in the vicinity of the concave corners, where the leading singularity is the first one, like [4, Assumption 5.2], and use some localization argument.

Another possible extension is obviously $L^\infty(\Gamma)$-norm error estimates for the Dirichlet control problem on graded meshes. In order to achieve this goal one needs fundamental results on graded meshes analogous to Theorem 7.2.4, namely, the $L^2(\Gamma)$- and $L^2(\Omega)$-norm error estimates for the control and state, respectively, which will be available in [5]. Moreover, for the pointwise error estimates one requires the stability of the $L^2(\Gamma)$-projection on graded meshes exploited in Lemma 7.2.7 in the quasi-uniform case. Taking into account that the boundary Γ is a one-dimensional domain, this result can be deduced from [28].

The last extension we would like to mention here is the Stokes problem with the corresponding local and global pointwise estimates on graded meshes. To the best of our knowledge estimates on graded triangulations for the Stokes problem are considered only in [84], where the optimal error estimates in the $L^2(\Omega)^n$-norm with $n = 2, 3$ and the $L^2(\Omega)$-norm are obtained for the velocity field and the pressure, respectively. In [41] the best approximation property for the gradient of the velocity field and the pressure in the maximum norms is shown in polyhedral domains under the assumption that the solution is regular enough. This assumption, however, yields some additional restrictions on the underlying domain. This issue is overcome in [44], where the best approximation property is shown in convex polyhedra.

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