15th World Congress on Computational Mechanics
Minisymposium "Multi-level iterative solvers for finite element systems"
Yokohama | Japan | 31 July - 06 August 2022

der Bundeswehr
Universität München

Physics-based block preconditioning for beam/solid interaction

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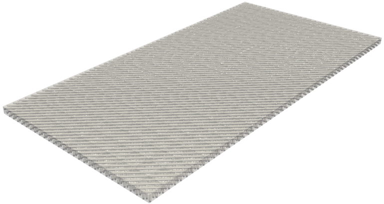
Motivation & Goal

- Beam / Solid interactions occur in a wide variety of scenarios:
  - Engineering (steel-reinforced concrete, composite materials)
  - Biomechanics (collagen fibers in connective tissue)
- Time-to-solution dominated by cost for linear solver
  - Scalability through multilevel methods
  - Algebraic Multigrid (AMG) for its flexibility
  - But: Ill-conditioned matrix due to discretization and penalty regularization prohibit out-of-the-box block smoothing



Goal

Scalable AMG method for beam / solid interaction problems in penalty formulation

Problem description

The coupling of beam-like structures with solid continua is described with the following coupled linearized system for beam/solid interaction:

$$\begin{pmatrix} \mathbf{K}_S + \epsilon \mathbf{M}^T \kappa^{-1} \mathbf{M} & -\epsilon \mathbf{M}^T \kappa^{-1} \mathbf{D} \\ -\epsilon \mathbf{D}^T \kappa^{-1} \mathbf{M} & \mathbf{K}_B + \epsilon \mathbf{D}^T \kappa^{-1} \mathbf{D} \end{pmatrix} \begin{pmatrix} \Delta \mathbf{d}_S \\ \Delta \mathbf{d}_B \end{pmatrix} = - \begin{pmatrix} \mathbf{r}_S \\ \mathbf{r}_B \end{pmatrix}$$

Legend

$(.)_S$	solid contribution
$(.)_B$	beam contribution
$\mathbf{d}$	displacement DOFs
$\mathbf{r}$	residual
ϵ	penalty parameter
κ	scaling factor

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- **Solid DOFs**
- **Beam DOFs**
- **Coupling constraints**

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$(.)_S$	solid contribution
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Challenges:

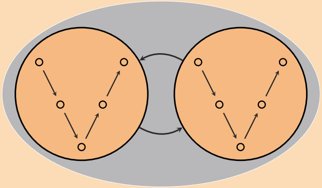
- Highly non-diagonal and ill-conditioned block matrix due to penalty regularization
- Block matrix may be nonsymmetric due to beam formulation

One-level block preconditioning

Approximations in Schur complement preconditioners

1. Approximation $\widehat{A}$ to form Schur complement $\tilde{S}$
   - $\Rightarrow$ Governed by the type of block method
   - $\Rightarrow$ e.g. $A \approx \widehat{A} := diag(A)$
2. Approximate block inverses within Schur complement preconditioner by standard AMG
   - $\Rightarrow$ Approximation quality can be controlled through the AMG settings

BlockMethod(AMG)[1]



- Coupling constraints are considered on fine level only
- Block method can be:
  - $\Rightarrow$ Block LU
  - $\Rightarrow$ Uzawa
  - $\Rightarrow$ SIMPLE

[1] Wiesner, T. A.; Mayr, M.; Popp, A.; Gee, M. W. and Wall, W. A. (2021): "Algebraic multigrid methods for saddle point systems arising from mortar contact formulations", *Numerical Methods in Engineering*, 122, 15:3749-3779

Sparse approximate invese

Due to the penalty regularization using just a diagonal approximation of the inverse inside the Schur complement calculation is not sufficient:

- Sparse approximate inverse methods (SPAI[2]) can produce better approximation
- Use matrix graph of A to calculate inverse $\widehat{A}$ on this sparsity pattern
- Based on Frobenius norm minimization:

$$\min_{\widehat{A} \in S} ||A\widehat{A} - I||_F$$

with S being the set of all sparse matrices with some known structure

Parallel computation

Decomposition into several least squares problems makes it inherently parallel:

$$||A\widehat{A} - I||_F^2 = \sum_{k=1}^{n} ||(A\widehat{A} - I)e_k||_2^2,$$

for each row k solve

$$\min_{\widehat{a}_k} ||A\widehat{a}_k - e_k||_2$$

with QR-decomposition

[2] Grothe, M. J. and Huckle, T. (1997): "Parallel preconditioning with sparse approximate inverses", *Journal Of Scientific Computing, 18, 3:838-853*

A priori pattern selection

Using just the pattern of A as input might not result in a satisfactory result, the matrix pattern needs to be enriched for a good sparse inverse approximation:

- Static approch by using recursive powers of graph of matrix $A \rightarrow$ recursion depth defined as level l
- Combining rows of graph $J(A)$ such that[3]:

$$J(A^l_{k,:}) = J(A^{l-1}_{k,:})J(A^{l-1})$$

- Pre- and post filtering of input graph and sparse inverse approximation with threshold value τ

SPAI with static pattern selection

1. Tresholding of $J(A)$
2. Determine graph of powers of A: $J(A^l)$
3. Calculate sparse inverse approximation $\widehat{A}$
4. Post filtering of $\widehat{A}$

[3] Chow E. (2001): "Parallel implementation and practical use of sparse approximate inverse preconditioners with a priori sparsity patterns", *The International Journal Of High Performance Computing Applications, 15:56-74*

Sparse approximate inverse smoother

Not all block inverses need to be approximated with a full multigrid cycle:

- Due to Schur complement calculation, good approximation of one matrix block is already available
- Using this information for smoothing results in the following SPAI smoother[4]:

$$x^{k+1} = x^k - \widehat{A}(Ax^k - b)$$

with $\widehat{A}$ being a sparse approximate inverse

Solve "Schur complement" equation with a conventional AMG method:

- Standard relaxation methods don't converge due to non-diagonal dominance
- Polynomial smoothers like the Chebychev iteration provide decent results

[4] Bröker O. and Grote, M. J. (2002): "Sparse approximate inverse smoothers for geometric and algebraic multigrid", *Applied Numerical Mathematics, 41:61-80*

Solid block with random fiber placement

Settings

Discretization

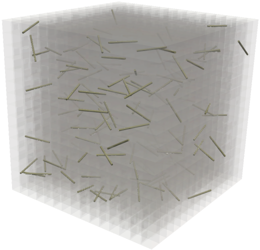
# Solid DOFs:	12288
# Beam DOFs:	1860
# procs:	6

Solver

Newton convergence:	10^{-6} (rel)
GMRES convergence:	10^{-8} (rel)

Material Parameters

Solid:	$E_S = 100 \frac{N}{m^2}, \nu_S = 0.3$ hyperelastic Saint Venant-Kirchhoff model
Beam:	$E_B = 1000 \frac{N}{m^2}, \nu_B = 0.0$ torsion-free Kirchhoff-Love model
Penalty:	$p = 1000 \frac{N}{m}$



- Minimal working example to investigate the influence of the sparse inverse approximation to the block smoothing schemes
- Sub-solves are done with a direct method

Comparison of different sparsity patterns

Block LU

τ	$1e^{-1}$	$1e^{-2}$	$1e^{-3}$	$1e^{-4}$	$1e^{-5}$	$1e^{-6}$
$J(A^0)$	184	184	184	184	184	184
$J(A^1)$	171	74	23	21	21	21
$J(A^2)$	193	102	14	9	9	8
$J(A^3)$	174	15	15	6	3	2

- Shows the averaged iterations of the linear solver in the first newton step
- Sparsity pattern level on the left side and the threshold value for pre- and post filtration on the top
  - $\Rightarrow$ $J(A^0)$ is equivalent to the matrix diagonal $J(diag(A))$
  - $\Rightarrow$ $J(A^1)$ resembles the sparsity pattern $J(A)$
  - $\Rightarrow$ The patterns $J(A^2)$ and $J(A^3)$ are the respective combinations on level 2 and 3

Comparison of different sparsity patterns

SIMPLE ($s = 3, \omega = 0.7$)

τ	$1e^{-1}$	$1e^{-2}$	$1e^{-3}$	$1e^{-4}$	$1e^{-5}$	$1e^{-6}$
$J(A^0)$	-	-	-	-	-	-
$J(A^1)$	-	-	-	369	68	48
$J(A^2)$	-	-	320	11	11	11
$J(A^3)$	-	-	-	7	5	4

value "-" means no convergence

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Comparison of different sparsity patterns

Uzawa ($s = 3, \omega = 0.7$)

τ	$1e^{-1}$	$1e^{-2}$	$1e^{-3}$	$1e^{-4}$	$1e^{-5}$	$1e^{-6}$
$J(A^0)$	-	-	-	-	-	-
$J(A^1)$	-	-	-	287	47	38
$J(A^2)$	-	-	268	11	11	11
$J(A^3)$	-	-	-	7	5	4

value "-" means no convergence

- Shows the averaged iterations of the linear solver in the first newton step
- Sparsity pattern level on the left side and the threshold value for pre- and post filtration on the top

$\Rightarrow$ $J(A^0)$ is equivalent to the matrix diagonal $J(diag(A))$

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Steel-Reinforced Concrete Beam

Settings

Discretization

# Solid DOFs:	5376
# Beam DOFs:	1686
# procs:	6

Solver

Newton convergence:	10^{-6} (rel)
BiCGSTAB convergence:	10^{-8} (rel)

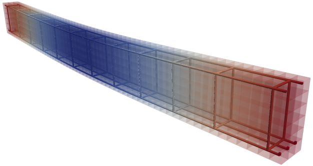
Material Parameters

Solid: $E_S = 30\frac{N}{m^2}, \nu_S = 0.3$
hyperelastic Saint Venant-Kirchhoff model

Beam: $E_B = 210\frac{N}{m^2}, \nu_B = 0.0$
torsion-free Kirchhoff-Love model

Penalty: $p = 1000\frac{N}{m}$

Four-point bending test under static loading[5]



[5] Braml, T.; Wimmer, J. and Varabei, Y. (2022): "Erfordernisse an die Datenaufnahme und -verarbeitung zur Erzeugung von intelligenten Digitalen Zwillingen", *Innsbrucker Bautage 2022 (eds Berger, J.) (Studia, 2022), 31-49*

Fiber-Reinforced Composite Plate[6]

Settings

Discretization

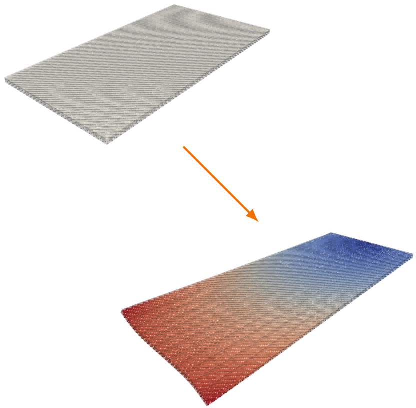
# Solid DOFs:	1950
# Beam DOFs:	10992
# procs:	6

Solver

Newton convergence:	10^{-6} (rel)
BiCGSTAB convergence:	10^{-8} (rel)

Material Parameters

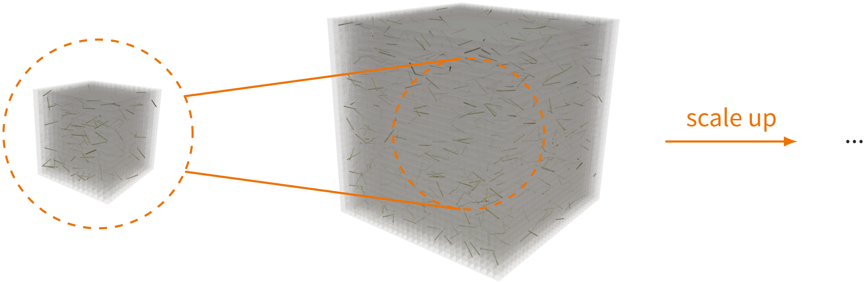
Solid:	$E_S = 10\frac{N}{m^2}, \nu_S = 0.3$ hyperelastic Saint Venant-Kirchhoff model
Beam:	$E_B = 1000\frac{N}{m^2}, \nu_B = 0.0$ torsion-free Kirchhoff-Love model
Penalty:	$p = 1000\frac{N}{m}$



Deformation of the plate due to tensile load

[6] Steinbrecher, I.; Mayr, M.; Grill, M. J.; Kremheller, J.; Meier, C. and Popp, A. (2020): "A mortar-type finite element approach for embedding 1D beams into 3D solid volumes", *Computational Mechanics, 66:1377-1398*

Weak scaling study: Cube filled with randomly placed and oriented fibers.



1 x 1 x 1 domain

- $\sim$ 25.000 Dofs
- 1 Processor

2 x 2 x 2 subdomains

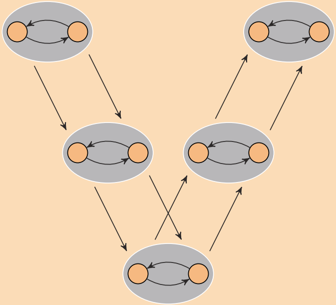
- $\sim$ 200.000 Dofs
- 8 Processors

n x n x n subdomains

- $\sim$? Dofs
- n^3 Processors

Outlook

AMG(BlockMethod)



- Consider coupling constraints on all levels
- Assembly of the beam DOFs nullspace specific to beam formulation

- For now only considered torsion-free Kirchhoff–Love beam elements:
  - ⇒ Sufficient for a broad range of applications
  - ⇒ Restriction to straight center line in reference configuration

  Extend to other beam formulations in the near future.

Thank you!

Collaborators:

- Matthias Mayr
- Alexander Popp
- Ivo Steinbrecher

References:
Open-source implementation will be available in TrilinosMueLu: https://trilinos.github.io/muelu.html

Contact:

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