p-ADICALLY CONVERGENT LOCI IN VARIETIES ARISING FROM PERIODIC CONTINUED FRACTIONS

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Abstract. Inspired by several alternative definitions of continued fraction expansions for elements in $\mathbb{Q}_p$, we study p-adically convergent periodic continued fractions with partial quotients in $\mathbb{Z}[1/p]$. To this end, following a previous work by Brock, Elkies, and Jordan, we consider certain algebraic varieties whose points represent formal periodic continued fractions with period and preperiod of fixed lengths, satisfying a given quadratic equation. We then focus on the *p-adically convergent loci* of these varieties, characterizing the zero and one-dimensional cases.

1. Introduction

Given an element $\alpha \in \mathbb{R}$, the classical *(Archimedean) continued fraction expansion* of α is the (possibly infinite) sequence $[c_1, c_2, \dots]$ computed by means of the following algorithm:

$$\begin{cases} \alpha_0 = \alpha \\ c_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \frac{1}{\alpha_n - c_n} \quad \text{if } \alpha_n - c_n \neq 0, \end{cases} \tag{1}$$

stopping if $\alpha_n = c_n$. It is well known that α is rational if and only if its continued fraction expansion is finite, and, if α is not rational, then the sequence of *n-th convergents*

$$c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}$$

converges to α. In the latter case, eventually periodic sequences (PCFs) are of particular interest: their study dates back to Lagrange [Lag70], who proved that the continued fraction expansion of α is eventually periodic if and only if α is a quadratic irrational.

In [BEJ21] the authors investigate periodic continued fractions using the language and techniques of algebraic geometry. More specifically, given a multi-set $\{\beta, \beta^*\} \subseteq \overline{\mathbb{Q}}$, and two positive integers N and k, the authors define a variety $V_{N,k}(\beta, \beta^*) \subseteq \mathbb{A}^{N+k}$, called *PCF variety*, whose points, seen as continued fractions of pre-period N and period k, *formally* satisfy the quadratic equation defined by $\{\beta, \beta^*\}$, namely $(x-\beta)(x-\beta^*)$. They particularly focus on PCF varieties of dimension ≤ 1 having their points in some ring of S-integers $\mathcal{O}_S$ in a number field K. We stress that these sequences need not arise as continued fraction expansions nor be convergent. For any ring $A \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}$, the set of A-rational points has a notable subset: the *convergent locus*, whose points correspond to continued fractions that converge in $\mathbb{C}$. In [BEJ21, §4] the authors also give an algorithm to detect the convergence of a given continued fraction.

We lay out a p-adic analogue of [BEJ21], where p is an odd prime. Various constructions of continued fractions can be carried out in the p-adic setting: the classical ones

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date back to Mahler [Mah40], Ruban [Rub70], Browkin [Bro78] and Schneider [Sch70]. More recently, periodic p-adic continued fractions have been investigated by [Bed90; Oot17; CVZ19; CMT23; MRS23].

In this work, we consider the p-adic convergence of *any* eventually periodic sequence $[c_1, c_2, \dots]$ of elements in $\mathbb{Z}[1/p]$. We remark that our approach differs from previous ones since we do not consider a fixed algorithm to generate a single expansion of any given number, but we consider every eventually periodic sequence of partial quotients; therefore, our results apply to the aforementioned expansions by Ruban and Browkin, regardless of the floor function or the 'continued fraction expansion algorithm'. To this aim, we keep the same definition of PCF varieties as in [BEJ21] and study their *p-adically convergent locus*, i.e. the set of points corresponding to continued fractions which converge w.r.t. the p-adic – rather than the Archimedean – absolute value. As in [BEJ21], we focus on PCF varieties of dimension ≤ 1 and study their p-adically convergent loci. Finally, we consider some 1-dimensional subvarieties of $V_{1,3}(\beta, \beta^*)$, that is a PCF variety of dimension 2.

The structure of this paper is the following. In Section 2 we recall the main properties of PCF varieties. Section 3 provides a criterion (Proposition 3.1) for the p-adic convergence of a periodic continued fraction. This is extensively used in Sections 4 and 5 to study the p-adically convergent loci of $V_{N,k}(\beta, \beta^*)$ for small values of N and k. Namely, we give complete characterizations of the p-adically convergent loci of $V_{N,k}(\beta, \beta^*)$ for $(N, k) \in \{(0, 1), (1, 1), (2, 1), (0, 2)\}$. For $(N, k) \in \{(1, 2), (0, 3), (1, 3)\}$ we focus instead on special cases in which modular arithmetic and the theory of Pell equations can be leveraged to prove finiteness and/or non-emptiness of p-adically convergent loci.

Finally, we remark that our notation slightly differs from that in [BEJ21]: rather than labelling the PCF varieties by the multiset $\mathcal{B} = \{\beta, \beta^*\}$ consisting of the two roots of some polynomial $F = x^2 - Ax + B \in \mathbb{Q}[x]$, we will instead use F itself and write $V(F)_{N,k}$.

2. Preliminaries

Let K be a field and $c_1, c_2, \dots$ be a sequence of non-zero elements in K. To start with, we define a *continued fraction* $C = [c_1, c_2, \dots]$ as the formal expression

$$c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots}}. \tag{2}$$

Just as a formal series can converge to some limit, we now want to see how C can "represent" some value α lying in (an extension of) K. Whenever this is the case, with a slight abuse of terminology we will simply write $C = \alpha$.

Two situations can prevent (2) from representing an actual element of K: the appearance of zero denominators, and the fact that the sequence $c_1, c_2, \dots$, and thus C, might be infinite. As for the first case, we will simply focus on the continued fractions where it does not occur. For the second question, we need to introduce a suitable notion of convergence. For $n \geq 1$, we consider the *n-th convergent*

$$[c_1, c_2, \dots, c_n] = c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}},$$

(which can be seen as a finite continued fraction in its own right). Recalling the classic sequences

$$\begin{array}{llll} A_0 = 1, & A_1 = c_1, & A_n = A_{n-1}c_n + A_{n-2} & \text{and} \\ B_0 = 0, & B_1 = 1, & B_n = B_{n-1}c_n + B_{n-2} & \text{for } n \geq 2, \end{array}$$

one can easily see that $[c_1, c_2, \dots, c_n] = A_n/B_n$, and we will always assume that $B_n \neq 0$ for every $n \geq 1$.

If we assume that K is endowed with an absolute value, we can consider the sequence $\{[c_1, \dots, c_n]\}_{n=1}^{\infty}$: if it converges, we say that C *converges* (and, as before, we identify C with its value).

Finally, for a continued fraction $C = [c_1, c_2, \dots]$ we define the matrices

$$D_n = D_n(C) = \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_n = M_n(C) = \begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} \quad \text{for } n \geq 1. \tag{3}$$

If $C = [c_1, \dots, c_n]$ is finite, we simply write M for M_n.
The following standard properties of D_n and M_n are easily derived from the definitions.

Proposition 2.1. *Let $\{c_n\}_{n\in\mathbb{N}}$ be any sequence of elements in a ring, and M_n, D_n be defined as in (3). Then, for each $n \geq 1$,*

- $M_n = D_1 \cdots D_n$,
- $M_n = M([c_1, \dots, c_j])M([c_{j+1}, \dots, c_n])$ *for each* $j < n$,
- ${M_n}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot M([-c_1, \dots, -c_n]) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
- $\det(M_n) = (-1)^n$.

2.1. **PCF varieties.** A *periodic continued fraction* (or *PCF* for short) *of type* (N, k) is a continued fraction of the form

$$P = [\underbrace{b_1, \dots, b_N}_{\text{preperiod}}, \underbrace{a_1, ..., a_k}_{\text{period}}, a_1, a_2, \dots],$$

and is usually denoted by $[b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$. When there is no preperiod, i.e. $N = 0$, we say that P is *purely periodic* (or *PPCF* for short).

As shown in [BEJ21], each type (N, k) and quadratic polynomial $F(x) \in \mathbb{Q}[x]$ give rise to an algebraic variety whose points represent *formal* PCFs of type (N, k), *disregarding the matter of convergence.*

Let $P = [b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$ be a periodic continued fraction. We define the 2×2 matrix E:

$$\begin{pmatrix} E_{11}(P) & E_{12}(P) \\ E_{21}(P) & E_{22}(P) \end{pmatrix} = M([b_1, \dots, b_N])M([a_1, \dots, a_k])M([b_1, \dots, b_N])^{-1}. \tag{4}$$

Proposition 2.2 (Proposition 2.9 [BEJ21])**.** *If P converges to α, then α is a root of the quadratic polynomial*

$$\mathrm{Quad}(P) = E_{21}(P)x^2 + (E_{22} - E_{11})(P)x - E_{12}(P). \tag{5}$$

We gather in Table 1 the polynomials $\mathrm{Quad}(P)$ for the types that will be considered in the remainder of this article.

Thus we associated a quadratic polynomial to each PCF. Now, following [BEJ21, §3], we want to do the opposite: given a quadratic polynomial $F(x) = Ux^2 + Vx + W \in \mathbb{Q}[x]$ and a type (N, k), we consider the set of $(N + k)$-tuples $(b_1, \dots, b_N, a_1, \dots, a_k)$ corresponding to periodic continued fractions $P = [b_1, \dots, b_N, \overline{a_1, \dots, a_k}]$ such that the

P	$\mathrm{Quad}(P)$
$[\overline{a_1}]$	$x^2 - a_1x - 1$
$[b_1, \overline{a_1}]$	$x^2 + (a_1 - 2b_1)x + b_1^2 - a_1b_1 - 1$
$[b_1, b_2, \overline{a_1}]$	$(b_2a_1 - b_2^2 + 1)x^2 + (-2a_1b_1b_2 + 2b_1b_2^2 - a_1 - 2b_1 + 2b_2)x +$ $a_1b_1^2b_2 - b_1^2b_2^2 + a_1b_1 + b_1^2 - 2b_2b_1 - 1$
$[\overline{a_1, a_2}]$	$a_2x^2 - a_1a_2x - a_1$
$[b_1, \overline{a_1, a_2}]$	$a_1x^2 + (a_2a_1 - 2b_1a_1)x - a_1a_2b_1 + a_1b_1^2 - a_2$
$[\overline{a_1, a_2, a_3}]$	$(a_2a_3 + 1)x^2 + (-a_1a_2a_3 - a_1 + a_2 - a_3)x - a_2a_1 - 1$

TABLE 1. Some polynomials $\mathrm{Quad}(P)$.

polynomial $\mathrm{Quad}(P)$ from (5) is a scalar multiple of $F(x)$. They all belong to the algebraic variety $V(F)_{N,k}$ in $\mathbb{A}^{N+k}$ defined by the equations

$$
\begin{cases} U(E_{22} - E_{11})(P) = VE_{21}(P), \\ -UE_{12}(P) = WE_{21}(P), \\ -VE_{12}(P) = W(E_{22} - E_{11})(P), \end{cases} \tag{6}
$$

which has generically dimension $N + k - 2$ according to [BEJ21, §3.1].

Having attached points of $\mathbb{A}^{N+k}$ to (formal) continued fractions, we now study their p-adic convergence. Inspired by the most studied continued fraction algorithms in the p-adic setting, we will consider partial quotients in $\mathbb{Z}[1/p]$. Therefore we denote by $V(F)^{\mathrm{con}}_{N,k}$ the subset of $V(F)_{N,k}(\mathbb{Z}[1/p])$ consisting of the points with non-zero coordinates and corresponding to p-adically convergent PCFs.

Remark 2.3. If F has no root in $\mathbb{Q}_p$, then $V(F)^{\mathrm{con}}_{N,k}$ is trivially empty.

3. Convergence criterion for PCF varieties

Since it is clear that the convergence of a PCF depends only on its periodic part, we can temporarily restrict our attention to a PPCF $P = [\overline{a_1, \dots, a_k}]$. In this case, the matrix E defined in (4) is simply M_k and, consequently, the polynomial $\mathrm{Quad}(P)$ from (5) is $B_kx^2 + (B_{k-1} - A_k)x - A_{k-1}$.

Proposition 3.1. *$P = [\overline{a_1, \dots, a_k}]$ is p-adically convergent if and only if $|A_k + B_{k-1}|_p > 1$.*

Proof. Assume P convergent and $|A_k + B_{k-1}|_p \le 1$. The characteristic polynomial of the matrix

$$M_k = \begin{pmatrix} A_k & A_{k-1} \\ B_k & B_{k-1} \end{pmatrix}$$

has two roots μ, ν such that $\mu\nu = (-1)^k$ and $|\mu + \nu|_p \le 1$, therefore $|\mu|_p = |\nu|_p = 1$. We then distinguish two cases.

(a) If $\mu = \nu$, then $\mu^2 = \pm 1$ and M_k is conjugate to a matrix of the form

$$\mu \cdot \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix},$$

and since we are assuming convergence, $\delta \neq 0$. Indeed, if $\delta = 0$, then M_k is a scalar matrix and, in particular, B_k is 0 – contradicting the convergence of P. Therefore, M_k is conjugate to $\mu \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\mathcal{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be a matrix such that

$$M_k = \mathcal{A}^{-1} \mu \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{A};$$

then, for any $\ell > 0$,

$$M_k^\ell = \mu^\ell \mathcal{A}^{-1} \cdot \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \mathcal{A}$$
$$= \frac{\mu^\ell}{\det(\mathcal{A})} \begin{pmatrix} \det(\mathcal{A}) + zw\ell & w^2\ell \\ -z^2\ell & \det(\mathcal{A}) - zw\ell \end{pmatrix}.$$

Now, for $\ell = p^h$, the quotient of the first column, i.e. the kp^hth-convergent of P, tends to ∞, while the quotient of the second column, i.e. the preceding convergent, tends to 0, which contradicts the assumption of convergence.

(b) if $\mu \neq \nu$, then M_k is diagonalizable. Notice that the second component of each eigenvector cannot be zero, otherwise $B_{\ell k} = 0$ for every $\ell > 0$ and P would not converge; therefore we can write two linearly independent eigenvectors as

$$\begin{pmatrix} \beta \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \beta^* \\ 1 \end{pmatrix}.$$

Then,

$$M_k^\ell = \begin{pmatrix} \beta & \beta^* \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mu^\ell & 0 \\ 0 & \nu^\ell \end{pmatrix} \begin{pmatrix} \beta & \beta^* \\ 1 & 1 \end{pmatrix}^{-1}$$
$$= \frac{1}{\beta - \beta^*} \begin{pmatrix} \beta\mu^\ell - \beta^*\nu^\ell & -\beta\beta^*(\mu^\ell - \nu^\ell) \\ \mu^\ell - \nu^\ell & \nu^\ell\beta - \mu^\ell\beta^* \end{pmatrix}.$$

The ratio of the first column is $\frac{\beta(\mu\nu^{-1})^\ell - \beta^*}{(\mu\nu^{-1})^\ell - 1}$. By assumption this ratio p-adically converges for $\ell \to \infty$. Then, since $\beta \neq \beta^*$ also $(\mu\nu^{-1})^\ell$ p-adically converges, but this implies $|\mu\nu^{-1}|_p < 1$, a contradiction.

Conversely, assume that $|A_k + B_{k-1}|_p > 1$. We want to show that P is p-adically convergent. The eigenvalues of M_k, say μ and ν, must satisfy $|\mu + \nu|_p > 1$ and $\mu\nu = (-1)^k$. Therefore they are distinct and $|\mu|_p > 1$, $|\nu|_p < 1$. In (b) we have already proven the equality

$$M_k^\ell = \frac{1}{\beta - \beta^*} \begin{pmatrix} \beta\mu^\ell - \beta^*\nu^\ell & -\beta\beta^*(\mu^\ell - \nu^\ell) \\ \mu^\ell - \nu^\ell & \nu^\ell\beta - \mu^\ell\beta^* \end{pmatrix}$$

for each $\ell > 0$. In this case, recalling that $\nu^\ell \to 0$, the ratios of the first and second columns converges to β, i.e.

$$\frac{\beta\mu^\ell - \beta^*\nu^\ell}{\mu^\ell - \nu^\ell} \xrightarrow{p} \beta \quad \text{and} \quad \frac{-\beta\beta^*(\mu^\ell - \nu^\ell)}{\nu^\ell\beta - \mu^\ell\beta^*} \xrightarrow{p} \beta.$$

Therefore, we find that

$$\frac{A_{\ell k}}{B_{\ell k}} \xrightarrow{p} \beta \qquad \text{and} \qquad \frac{A_{\ell k-1}}{B_{\ell k-1}} \xrightarrow{p} \beta.$$

The same considerations hold for the transposed matrix, so that we also find

$$\frac{A_{\ell k}}{A_{\ell k-1}} \xrightarrow{p} \eta \qquad \text{and} \qquad \frac{B_{\ell k}}{B_{\ell k-1}} \xrightarrow{p} \eta.$$

Moreover, for any $r \in \{1, \dots, \ell - 1\}$,

$$\begin{aligned} M_{\ell k + r} &= M_{\ell k} M_r \\ &= M_k^\ell M_r \\ &= \begin{pmatrix} A_{\ell k} & A_{\ell k-1} \\ B_{\ell k} & B_{\ell k-1} \end{pmatrix} \begin{pmatrix} A_r & A_{r-1} \\ B_r & B_{r-1} \end{pmatrix} \\ &= \begin{pmatrix} A_{\ell k} A_r + A_{\ell k-1} B_r & A_{\ell k} A_{r-1} + A_{\ell k-1} B_{r-1} \\ B_{\ell k} A_r + B_{\ell k-1} B_r & B_{\ell k} A_{r-1} + B_{\ell k-1} B_{r-1} \end{pmatrix}. \end{aligned}$$

The ratio of the first column is

$$\frac{A_{\ell k+r}}{B_{\ell k+r}} = \frac{A_{\ell k-1}}{B_{\ell k-1}} \left(\frac{\frac{A_{\ell k}}{A_{\ell k-1}} A_r + B_r}{\frac{B_{\ell k}}{B_{\ell k-1}} A_r + B_r} \right) \to \beta \text{ for } \ell \to \infty,$$

because expression in parentheses tends to 1 for $\ell \to \infty$. □

4. Families of PCF of type (N, k) when $N + k \leq 3$

Let $\mathcal{B} = \{\beta, \beta^*\}$ be the multi-set of roots of some quadratic polynomial $F(x) \in \mathbb{Q}[x]$. Suppose that F is monic, i.e.

$$F(x) = x^2 - Ax + B \qquad \text{where } A = \beta + \beta^* \text{ and } B = \beta\beta^*.$$

In this section, we consider the same types as in [BEJ21, §5-8], i.e. the types such that $V(F)_{N,k}$ is either zero or one dimensional, and study them in the p-adic setting. In particular, we will characterize the cases in which $V(F)^{\text{con}}_{N,k}$ is non-empty and β, β^* are rationals or square roots of rationals.

4.1. **Type** $(0, 1)$**.** Using Table 1 and (6), the equations of $V(F)_{0,1}$ are

$$\begin{cases} a_1 & = A \\ B & = -1. \end{cases} \tag{7}$$

Suppose that $[\overline{a_1}]$ converges to β; then, $\beta = a_1 + 1/\beta$ is a root of $x^2 - a_1 x - 1$. Therefore, it follows from Proposition 3.1 that $V(F)^{\text{con}}_{0,1}$ is non-empty if and only if $B = -1$ and $|A|_p > 1$. In this case $V(F)^{\text{con}}_{0,1} = \{(A)\}$ and the continued fraction $[\overline{A}]$ converges to the only root, among β and $-1/\beta$, with negative p-adic valuation.

Proposition 4.1. *Suppose that F is reducible over $\mathbb{Q}$. Then, $V(F)^{\text{con}}_{0,1}$ is non-empty if and only if its multi-set of roots is $\mathcal{B} = \{\pm 1/p^k, \mp p^k\}$ for some $k > 0$ and $V(F)^{\text{con}}_{0,1} = \{(\pm(1 - p^{2k})/p^k)\}$.*

Proof. $V(F)^{\text{con}}_{0,1}$ contains a rational point (a_1) if and only if $F(x) = x^2 - a_1 x - 1$ and its discriminant $a_1^2 + 4$ is a square, that is

$$\frac{a^2}{p^{2k}} + 4 = \frac{h^2}{p^{2\ell}} \tag{8}$$

for $a \in \mathbb{Z}$ and $h, k, \ell \in \mathbb{Z}_{>0}$ such that $a_1 = a/p^k$ and a, h are coprime with p. Observing that $\ell = k$, we can rewrite (8) as

$$(h + a)(h - a) = 4p^{2k}.$$

Since p is odd and coprime with a and h, it can divide only one number between $(h + a)$ and $(h - a)$. Moreover, both factors must be even because their sum and their product are. Therefore, up to changing the sign of a, we have $h + a = 2p^{2k}$ and $h - a = 2$ and the thesis follows by substituting the second equation in the first. □

Notice that if β is an irrational square root, i.e. $\beta^2 \in \mathbb{Q}\setminus\mathbb{Q}^2$, it cannot satisfy $F(\beta)=0$ without contradicting $|a_1|_p > 1$.

4.2. **Type** $(1,1)$**.** Using Table 1 and (6), the equations of $V(F)_{1,1}$ are

$$\begin{cases} a_1 - 2b_1 + A &= 0 \\ b_1^2 - a_1 b_1 - B - 1 &= 0. \end{cases} \tag{9}$$

The convergence of a PCF of type $(1,1)$ depends only on its purely periodic part and can be therefore studied in the light of type $(0,1)$.

Proposition 4.2. *$V(F)_{1,1}^{\text{con}}$ is not empty if and only if $A, B \in \mathbb{Z}[1/p]$ and the quantity A^2-4B-4 is a square in $\mathbb{Z}[1/p]$ with $|A^2-4B-4|_p > 1$. In this case, it contains precisely the two points $(b_1, 2b_1 - A), (A-b_1, A-2b_1)$ where b_1 is a root of $F(x)+1$. One of the two associated continued fractions converges to β, and the other to β^*.*

Proof. Suppose first that A^2-4B-4 is a square, say a_1^2 with $a_1 \in \mathbb{Z}[1/p]$, such that $|A^2-4B-4|_p > 1$. Then $|a_1|_p > 1$, which in turn implies that $[\overline{a_1}]$ converges. Moreover, since A^2-4B-4 is the discriminant of $F(x)+1$, this polynomial has roots b_1 and $b_1 - a_1$, where

$$b_1 = \frac{A+a_1}{2}.$$

We have thus proven that (a_1, b_1) satisfies (9) and $[\overline{a_1}]$ converges, so that $(b_1, a_1) \in V_{1,1}^{\text{con}}(F)$.
Conversely, suppose that $(b_1, a_1) \in V_{1,1}^{\text{con}}(F)$. Since b_1 and $b_1 - a_1$ are roots of $F(x)+1$, whose discriminant is $A^2-4B-4 = a_1^2$, we immediately deduce that A^2-4B-4 must be a square in $\mathbb{Z}[1/p][x]$. Finally, the convergence of the $[\overline{a_1}]$ implies $|A^2-4B-4|_p > 1$. In this case, $V_{1,1}^{\text{con}}(F)$ also contains the point $(A-b_1, -a_1)$. To prove this, observe that the purely periodic continued fraction $[\overline{a_1}]$ converges to some α which is – as we saw for type $(0,1)$ – the p-adically dominant root of the polynomial $x^2 - a_1 x - 1$. It follows that $-\alpha$ is the limit of the continued fraction $[\overline{-a_1}]$. Therefore, if β is the limit of the continued fraction $[b_1, \overline{a_1}]$, then $[A-b_1, \overline{-a_1}]$ converges to $A - b_1 - \frac{1}{\alpha} = A - \beta = \beta^*$, so that $(A-b_1, -a_1) \in V_{1,1}^{\text{con}}(F)$. $\square$

When $A=0$, the above proposition provides information about the square roots of rationals which can be expressed as p-adically convergent continued fractions of type $(1,1)$.

Corollary 4.3. *Suppose $A=0$; then, $V(F)_{1,1}^{\text{con}}$ is non-empty if and only if there exists $b_1 \in \mathbb{Z}[1/p]$ with $|b_1|_p > 1$ such that $-B = b_1^2+1$. In this case $V(F)_{1,1}^{\text{con}} = \{\pm(b_1, 2b_1)\}$.*

4.3. **Type** $(2,1)$**.** The equations of $V(F)_{2,1}$ are

$$\begin{cases} -Aa_1b_2 + Ab_2^2 + 2a_1b_1b_2 - 2b_1b_2^2 - A + a_1 + 2b_1 - 2b_2 &= 0 \\ a_1b_1^2b_2 - b_1^2b_2^2 - Ba_1b_2 + Bb_2^2 + a_1b_1 + b_1^2 - 2b_1b_2 - B - 1 &= 0. \end{cases} \tag{10}$$

As for type $(1,1)$, the convergence of a PCF of type $(2,1)$ depends only on its purely periodic part and can be therefore studied in the light of type $(0,1)$.

Proposition 4.4. *$V(F)_{2,1}^{\text{con}}$ is not empty if and only if (10) has a solution and*

$$|Ab_2^2 - 2b_1b_2^2 - A + 2b_1 - 2b_2|_p > |Ab_2 - 2b_1b_2 - 1|_p.$$

Proof. The first equation of (10) can be rewritten as

$$a_1(-Ab_2 + 2b_1b_2 + 1) + Ab_2^2 - 2b_1b_2^2 - A + 2b_1 - 2b_2 = 0.$$

If $-Ab_2 + 2b_1b_2 + 1$ was zero for some solution of the above equation, then also $Ab_2^2 - 2b_1b_2^2 - A + 2b_1 - 2b_2$ would be zero and, as a consequence, $b_1^2 = -1$, contradicting the fact that we are only taking solutions in $\mathbb{Z}[1/p]$. Therefore we may rewrite the first equation of (10) as

$$a_1 = \frac{Ab_2^2 - 2b_1b_2^2 - A + 2b_1 - 2b_2}{Ab_2 - 2b_1b_2 - 1}.$$

The statement follows from the fact that the continued fraction $[b_1, b_2, \overline{a_1}]$ converges if and only if $|a_1|_p > 1$. □

4.4. **Type** $(0,2)$**.** The equations of $V(F)_{0,2}$ are

$$\begin{cases} -A + a_1 & = 0 \\ -Ba_2 - a_1 & = 0. \end{cases} \tag{11}$$

$V(F)^{\mathrm{con}}_{0,2}$ is non-empty if and only if

$$\begin{cases} A \in \mathbb{Z}[1/p], & \\ |A|_q \leq |B|_q & \text{for all } q \neq p \\ |A|_p^2 > |B|_p & \text{by Proposition 3.1.} \end{cases} \tag{12}$$

In this case,

$$V(F)^{\mathrm{con}}_{0,2} = \{(A, -A/B)\}. \tag{13}$$

(in particular, if $B = 0$ then $V(F)^{\mathrm{con}}_{0,2}$ is empty since (11) implies $A = 0$, which does not satisfy (12)).

When $B = -1$, the continued fraction has in fact type $(0,1)$. We notice that, by (12), the continued fraction $[\overline{A, -A/B}]$ converges to a root of F if and only if the continued fraction $[\overline{-A/B, A}]$ converges to a root of the polynomial $G(x) = Bx^2 + Ax + 1$; therefore $V(F)^{\mathrm{con}}_{0,2}$ is non-empty if and only if $V(G)^{\mathrm{con}}_{0,2} = \{(-A/B, A)\}$.

When F is reducible over $\mathbb{Q}$, we have the following.

Proposition 4.5. *Suppose that F is reducible over $\mathbb{Q}$ and $V(F)^{\mathrm{con}}_{0,2}$ is non-empty. Then, $V(F)^{\mathrm{con}}_{0,2} = \{(a_1, a_2)\}$ where the product a_1a_2 satisfies*

$$a_1a_2 = \pm\frac{(p^k - 1)^2}{4^{\epsilon}p^k},$$

with $k \in \mathbb{Z}_{>0}$ and $\epsilon \in \{0, 1\}$.

Proof. F splits over $\mathbb{Q}$ if and only if its discriminant is a square. Recalling that in the case $(0,2)$ we have $A = a_1$ and $B = -a_1/a_2$, the discriminant of F becomes

$$\Delta = a_1^2 + 4\frac{a_1}{a_2};$$

by multiplying by a_2^2 we have that Δ is a square if and only if there exist $R, S \in \mathbb{Z}$ coprime with p and $k, \ell > 0$ with $a_1a_2 = S/p^k$ such that

$$\frac{S}{p^k}\left(\frac{S}{p^k} + 4\right) = \frac{R^2}{p^{2\ell}}. \tag{14}$$

Now, it is easy to see that k and ℓ must be equal, so that (14) becomes

$$S(S + 4p^k) = R^2. \tag{15}$$

Since S is coprime with p, the only possible common divisors between the two factors above are, up to a sign, 2 and 4; we consider the three cases separately.

(i) If S and $S+4p^k$ are coprime, then S is odd and we can write

$$S = T^2 \qquad \text{and} \qquad S+4p^k = U^2$$

for some coprime odd integers T and U not divided by p. These two equations yield

$$4p^k = (U+T)(U-T),$$

and the only possibility, up to the signs of U and T, is $U+T = 2p^k$ and $U-T = 2$. Therefore $S = (p^k-1)^2$.

(ii) If the greatest common divisor between S and $S+4p^k$ is 2, then $S = 2S'$ for some odd S' and we can write

$$S' = T^2 \qquad \text{and} \qquad S'+2p^k = U^2$$

for some coprime odd integers T and U not divided by p. However, this leads to a contradiction since $(U+T)(U-T) = 2p^k$ implies that only one between $(U+T)$ and $(U-T)$ is even, which is clearly false.

(iii) If the greatest common divisor between S and $S+4p^k$ is 4, then $S = 4S'$ for some odd S', then we can write

$$S' = T^2 \qquad \text{and} \qquad S'+p^k = U^2$$

for some coprime odd integers T and U not divided by p. These two equations yield

$$p^k = (U+T)(U-T),$$

and the only possibility, up to the signs of U and T, is $U+T = p^k$ and $U-T = 1$. Therefore $S = ((p^k-1)/2)^2$. □

Finally, notice that, if β is a square root of a rational number, i.e. $\beta^2 \in \mathbb{Q}$, it cannot satisfy $F(\beta) = 0$ without contradicting the third condition of (12); this implies that the square roots of rational numbers have never type $(0,2)$.

4.5. **Type** $(1,2)$**.** The equations of $V(F)_{1,2}$ are

$$\begin{cases} -Aa_1 - a_1a_2 + 2a_1b_1 & = 0, \\ -a_1a_2b_1 + a_1b_1^2 - Ba_1 - a_2 & = 0. \end{cases} \tag{16}$$

Let $(b_1, a_1, a_2) \in V(F)_{1,2}^{\mathrm{con}}$ and let β be the root of F to which the continued fraction converges; since $a_2 \neq 0$, conditions (16) imply that b_1 is not a root of F. Then, $1/(\beta - b_1)$ is a root of

$$x^2 - \frac{A-2b_1}{B-Ab_1+b_1^2}x + \frac{1}{B-Ab_1+b_1^2}.$$

On the other hand, as a purely periodic continued fraction of type $(0,2)$, $1/(\beta - b_1)$ is also a root of

$$a_2x^2 - a_2a_1x - a_1.$$

Therefore, from (11) and (12) we deduce the following.

Theorem 4.6. *Assume $b_1 \in \mathbb{Z}[1/p]$ and $F(b_1) \neq 0$. Then, there exists a point $(b_1, a_1, a_2) \in V(F)_{1,2}^{\mathrm{con}}$ if and only if the following conditions hold:*

$$\begin{cases} \frac{A-2b_1}{B-Ab_1+b_1^2} \in \mathbb{Z}[1/p], \\ A \in \mathbb{Z}[1/p], \\ |A-2b_1|_p^2 > |B-Ab_1+b_1^2|_p. \end{cases} \tag{17}$$

In this case

$$a_1 = \frac{A-2b_1}{B-Ab_1+b_1^2}, \qquad a_2 = -A+2b_1.$$

The third condition in (17) implies that $|b_1|_p \leq \max(|A|_p, |B|_p^{1/2})$, since, otherwise, both the left-hand side and right-hand side of the inequality would be equal to $|b_1|_p^2$. We remark that, by Proposition 4.5, conditions (17) can be satisfied even when the polynomial $F(x) = x^2 - Ax + B$ is reducible.
Furthermore, if conditions (17) are satisfied for b_1, then they are also satisfied for $A - b_1$, so that

$$\left(b_1, \frac{A-2b_1}{B-Ab_1+b_1^2}, -A+2b_1\right), \left(A-b_1, \frac{-A+2b_1}{B-Ab_1+b_1^2}, A-2b_1\right) \in V(F)_{1,2}^{\mathrm{con}}.$$

It is easy to see that the two continued fractions

$$\left[b_1, \overline{\frac{A-2b_1}{B-Ab_1+b_1^2}, -A+2b_1}\right] \tag{18}$$

$$\left[A-b_1, \overline{\frac{-A+2b_1}{B-Ab_1+b_1^2}, A-2b_1}\right]$$

converge respectively to the two different roots β, β^* of F.

Remark 4.7. The variety $V(F)_{1,2}$ always has a component given by the line $a_1 = a_2 = 0$. If $A^2 - 4B \neq 0$, the variety has exactly one additional component, which is a curve of genus 0 and 3 points at infinity, therefore it has at most finitely many $\mathbb{Z}[1/p]$-points by Siegel's theorem. If $A^2 = 4B$, there are exactly two additional components: one is the line $a_2 = 0, b_1 = A/2$ and the other is the rational curve with two points at infinity given by $a_1 a_2 = -4, b_1 = \frac{A+a_2}{2}$.

Examples 4.8.

a) Assume that $F(x) = x^2 + B$ with $B = -b_1^2 \pm 2^i p^k$ where $b_1 \in \mathbb{Z}[1/p]$, $i \in \{0,1\}$ and $k > 2v_p(b_1)$. Then, conditions (17) are satisfied and, by (18),

$$\left(b_1, -\frac{2b_1}{\pm 2^i p^k}, 2b_1\right) \in V(F)_{1,2}^{\mathrm{con}}.$$

b) Assume again that $F(x) = x^2 - B$ with $B \in \mathbb{Z}$ such that $p \nmid B$. Then we see that $V(F)_{1,2}^{\mathrm{con}} \neq \emptyset$ if and only if there exist $b \in \mathbb{Z}$ such that $|b|_p = 1$ and $c \in \mathbb{Z}$ a divisor of $\gcd(B, b)$ such that $B + b^2 = 2^i c p^k$ with $k \geq 1$ and $i \in \{0,1\}$. In this case,

$$\left(b, \frac{2b}{2^i c p^k}, 2b\right) \in V(F)_{1,2}^{\mathrm{con}}.$$

Notice that this implies

$$\left(\frac{b}{c}, \frac{2b}{2^i p^k}, 2\frac{b}{c}\right) \in V(G)_{1,2}^{\mathrm{con}},$$

where $G(x) = x^2 - \frac{B}{c^2}$.

c) As an example of polynomial $F(x) = x^2 - Ax + B$ having roots in $\mathbb{Q}_p$ and such that $V(F)_{1,2}^{\mathrm{con}}$ is empty, we notice that $V(F)_{1,2}^{\mathrm{con}} = \emptyset$ with $F(x) = x^2 + 5$ for every $p \equiv 1 \pmod{35}$. We need to show that $b^2 = 2^i c p^r + 5$ has no solutions with $b \in \mathbb{Z}$, $i \in \{0,1\}$, $k \geq 1$ and $c \in \{1,5\}$. If $c = 5$, this gives $\frac{b^2}{5} = 2^i p^k + 1$, that is $0 \equiv 2^i + 1 \equiv \pm 2 \pmod 5$, a contradiction. If $c = 1$ and $i = 0$, this gives a contradiction modulo 7. If $c = 1$ and $i = 1$, this gives a contradiction modulo 5. (This example can be generalized to any polynomial $F(x) = x^2 + m$ with m

prime such that none of $m, m+1, m+2$ is a square and p in some arithmetic progression).

d) It is possible to find a polynomial F such that $|V(F)^{\mathrm{con}}_{1,2}| > 2$. Indeed, we have that $[7, \overline{\frac{2}{27}, 2}]$, $[5, \overline{-\frac{2}{27}, -2}]$ and $[11, \overline{\frac{10}{3}, 10}]$, $[1, \overline{-\frac{10}{3}, -10}]$ all converge to 3-adic solutions of $F(x) = x^2 - 12x + 8$, i.e. $6 \pm 2\sqrt{7}$.

e) For a polynomial $F(x) = x^2 - B$, through the identity

$$2p^{r+s} - \left(\frac{2p^r + p^s - 1}{2}\right)^2 = 2p^r - \left(\frac{2p^r - p^s + 1}{2}\right)^2$$

and point (a) one can find examples of B such that $|V(F)^{\mathrm{con}}_{1,2}| > 2$. For example, $[10, \overline{-\frac{10}{27}, 20}]$, $[8, \overline{-\frac{8}{9}, 16}]$ and $[-10, \overline{\frac{10}{27}, -20}]$, $[-8, \overline{\frac{8}{9}, -16}]$ all converge 3-adically to a square root of 46.

4.6. **Type** $(0,3)$**.** The equations of $V(F)_{0,3}$ are

$$\begin{cases} Aa_2a_3 - a_1a_2a_3 + A - a_1 + a_2 - a_3 & = 0, \\ Ba_2a_3 + a_1a_2 + B + 1 & = 0. \end{cases} \tag{19}$$

By Proposition 3.1 we have that $[\overline{a_1, a_2, a_3}]$ converges p-adically if and only if $v_p(a_1a_2a_3 + a_1 + a_2 + a_3) < 0$. Notice that this condition holds up to permuting a_1, a_2 and a_3. It is still an open question whether the p-adic convergence can be expressed only in terms of A and B – as we did, for example, for type $(0,2)$. Here we will focus on a particular case, namely $A = 0$ and $B = -d$ for some square-free integer d.

4.6.1. *Pure integer radicals.* Equation (19), with $A = 0$ gives the system of equations

$$\begin{cases} a_2 - a_1 - a_3 - a_1a_2a_3 = 0, \\ -(a_2a_3 + 1)B = a_1a_2 + 1. \end{cases} \tag{20}$$

Eliminating a_2 from (20), we get

$$B = -\frac{a_1^2 + 1}{a_3^2 + 1}, \tag{21}$$

from which it is immediate to see that B must be $-d$ for some positive integer d. Then, we only have to search for positive integers d such that $\sqrt{d}$ has an expansion of type $(0,3)$.

Proposition 4.9. *Let d be a positive integer. All converging p-adic expansions of $\sqrt{d}$ of type $(0,3)$ are of the form*

$$\left[\overline{a_1, \frac{v}{p^s}, a_3}\right],$$

where $a_1, a_3, v, s \in \mathbb{Z}$, $s \geq 1$, $p \nmid a_1a_3v$, $v \mid d-1$ and

$$a_1^2 - da_3^2 = d - 1, \tag{22}$$

$$a_1 - da_3 = \frac{d-1}{v}p^s. \tag{23}$$

Furthermore, every such quadruple a_1, a_3, v, s gives rise to a converging p-adic expansion of $\sqrt{d}$ of type $(0,3)$ for some $d \in \mathbb{Z}$.

Proof. We notice that (22) is obtained from (21) by setting $B = -d$, which in turn implies

$$(a_1 + a_3)(a_1 - da_3) = (d-1)(1 - a_1a_3). \tag{24}$$

From the first equation of (20) and (24) we get

$$a_2 = \frac{a_1 + a_3}{1 - a_1 a_3} = \frac{d-1}{a_1 - d a_3}. \tag{25}$$

The quantity $a_1a_2a_3 + a_1 + a_2 + a_3$ is equal to $2a_2$, which is also equal to $2\frac{a_1+a_3}{1-a_1a_3}$ so that $[\overline{a_1, a_2, a_3}]$ converges p-adically if and only if $v_p(a_2) < 0$. If one between a_1 or a_3 has negative p-adic valuation, so has the other and they are equal (we recall that $p \neq 2$); but then we see that $v_p(\frac{a_1+a_3}{1-a_1a_3}) > 0$ and the fraction does not converge p-adically. Therefore $v_p(a_1), v_p(a_3) \geq 0$.

If at least one between $v_p(a_1), v_p(a_3)$ is positive, then $v_p(\frac{a_1+a_3}{1-a_1a_3}) \geq 0$ and again the fraction does not converge p-adically; therefore $v_p(a_1) = v_p(a_3) = 0$.

Finally, since $a_2 \in \mathbb{Z}[1/p]$ we must have that $a_1 - da_3 = wp^t$ for some w dividing $d-1$, prime to p, and $t > v_p(d-1)$. Let v be the prime-to-p part of $\frac{d-1}{w}$, and $s = t - v_p(d-1)$. Then, we get $a_2 = \frac{v}{p^s}$. □

Corollary 4.10. *Suppose that $d \in \mathbb{Z}$ is divided by 4 or by any prime congruent to -1 modulo 4. Then, for every prime p, $\sqrt{d}$ does not have a p-adic expansion of type $(0, 3)$.*

Proof. If $\sqrt{d}$ has a p-adic expansion of type $(0, 3)$, then equation (22) modulo 4 (resp. any prime divisor q of d) implies that $d \not\equiv 0 \pmod 4$ (resp. $q \not\equiv -1 \pmod 4$). □

Proposition 4.11. *Let d be a positive square-free integer, and suppose that $\sqrt{d}$ has a converging p-adic expansion $\left[\overline{a_1, \frac{v}{p^s}, a_3}\right]$ as in Proposition 4.9. Let k be the greatest common divisor of $a_1 + a_3$ and $1 - a_1a_3$. Then*

$$p^s \leq k \frac{(d+1)^{3/2} + 2d}{|d+1|_p^2}. \tag{26}$$

Proof. Let k be the greatest common divisor between $a_1 + a_3$ and $1 - a_1a_3$, so that $1 - a_1a_3 = kp^s$ by (25). This equality can be used to eliminate a_3 from (22), yielding

$$a_1^4 - (d-1)a_1^2 - d(kp^s - 1)^2 = 0.$$

Viewing it as a quadratic equation in a_1^2, we have that the discriminant must be a square, hence there exists $u \in \mathbb{Z}$ such that

$$(d-1)^2 + 4d(kp^s - 1)^2 = u^2. \tag{27}$$

Choose u to be positive (it cannot be 0 since the left-hand side of (27) is always ≥ 1). Moreover,

$$\begin{aligned} 4dkp^s(kp^s - 2) &= 4d(kp^s - 1)^2 - 4d \\ &= u^2 - (d-1)^2 - 4d \\ &= u^2 - (d+1)^2 \\ &= (u + d + 1)(u - d - 1). \end{aligned} \tag{28}$$

From this equality we obtain an effective upper bound for s in terms of d. The idea is that the two factors in (28) are almost coprime, so that the factor p^s must go almost entirely in one of them, say for now in the first one. Then we can write

$$\begin{cases} u + d + 1 = 2\frac{ag}{p^x} p^s, \\ u - d - 1 = 2\frac{dkp^x}{ag}(kp^s - 2), \end{cases} \tag{29}$$

where a is a positive divisor of kd, x is an integer between 0 and $v_p(\gcd(u, d+1))$ and g is a positive divisor of $kp^s - 2$. These two factors should have roughly the same size.

Taking the difference we get

$$d+1 = \frac{a^2g^2 - dk^2p^{2x}}{agp^x}p^s + \frac{2dkp^x}{ag}.$$

The coefficient in front of p^s cannot be 0 because d is not a square, so we obtain

$$p^s = \left(d+1-\frac{2dkp^x}{ag}\right)\frac{agp^x}{a^2g^2-dk^2p^{2x}} \le \left|d+1-\frac{2dkp^x}{ag}\right| agp^x \le (d+1)agp^x + 2dkp^{2x}.$$

We are left with the task of bounding g independently of s. To do so, take the quotient of the two equations (29)

$$\begin{aligned}\frac{u-d-1}{u+d+1} &= dk\left(\frac{p^x}{ag}\right)^2\left(k-\frac{2}{p^s}\right)\\ \left(\frac{ag}{p^x}\right)^2 &= dk\left(k-\frac{2}{p^s}\right)\frac{u+d+1}{u-d-1} < dk^2\left(1+\frac{2(d+1)}{u-d-1}\right).\end{aligned}$$

The bound in the statement is larger than $\frac{k(d+1)^2}{\sqrt{d}}+1$, therefore we can now assume freely that $p^s \ge \frac{k(d+1)^2}{\sqrt{d}}+1$, so that, from (27), $u > 2\sqrt{d}(kp^s-1) \ge 2k^2(d+1)^2$. Therefore we get

$$\begin{aligned}\left(\frac{ag}{p^x}\right)^2 &< dk^2\left(1+\frac{2(d+1)}{u-d-1}\right) < dk^2\left(1+\frac{2(d+1)}{2k^2(d+1)^2-d-1}\right)\\ &\le dk^2\left(1+\frac{2}{2k^2(d+1)-1}\right) < dk^2\left(1+\frac{1}{dk^2}\right) \le dk^2+1.\end{aligned}$$

Thus we obtain

$$\begin{aligned}p^s &\le (d+1)agp^x + 2dkp^{2x} = p^{2x}\left((d+1)\frac{ag}{p^x}+2kd\right) < p^{2x}\left((d+1)\sqrt{dk^2+1}+2kd\right)\\ &\le p^{2x}k\left((d+1)\sqrt{d+1}+2d\right) \le k\frac{(d+1)^{3/2}+2d}{|d+1|_p^2}.\end{aligned}$$

Arguing with the role of the two factors of (29) reversed leads to the same bound. □

Setting $k=1$ in the statement of Proposition 4.11 shows that, for fixed d, there is only a finite number of p-adic expansions of $\sqrt{d}$ of type $(0,3)$ of the form

$$\left[\overline{a_1, \frac{a_1+a_3}{p^s}, a_3}\right].$$

For example, for $k=1$ and $3 \le |d| \le 10$ one can check by hand – for the primes allowed by Theorem 4.10 and (26) – that only $\sqrt{10}$ has p-adic expansions of type $(0,3)$, which are

$$\left[\overline{13, \frac{9}{53}, -4}\right]_{53} \quad \text{and} \quad \left[\overline{-13, -\frac{9}{53}, 4}\right]_{53}.$$

We remark that, for $k \neq 1$, one can find other p-adic expansions of $\sqrt{2}, \sqrt{5}$ and $\sqrt{10}$ of type $(0,3)$:

$$\text{(for } k=-1\text{)} \qquad \sqrt{10} = \left[\overline{7, -\frac{9}{13}, 2}\right]_{13},$$

$$\text{(for } k=2\text{)} \qquad \sqrt{5} = \left[\overline{7, \frac{2}{11}, -3}\right]_{11},$$

$$\text{(for } k=5\text{)} \qquad \sqrt{2} = \left[\overline{17, \frac{1}{41}, -12}\right]_{41},$$

$$\text{(for } k=-25\text{)} \qquad \sqrt{10} = \left[\overline{-57, \frac{3}{41}, -18}\right]_{41},$$

$$\text{(for } k=-37\text{)} \qquad \sqrt{10} = \left[\overline{253, -\frac{9}{547}, 80}\right]_{547},$$

$$\text{(for } k=37\text{)} \qquad \sqrt{10} = \left[\overline{487, \frac{9}{2027}, -154}\right]_{2027},$$

$$\text{(for } k=-949\text{)} \qquad \sqrt{10} = \left[\overline{2163, -\frac{3}{1559}, 684}\right]_{1559}.$$

4.6.2. *Finiteness results.* The Generalized Pell equation, i.e. the equation of the form $x^2 - dY^2 = n$ have been widely studied. The solutions correspond to elements of norm n in $\mathbb{Z}[\sqrt{d}]$. When $n = 1$, solutions form a cyclic group $\mathcal{U}_d$ under the Brahmagupta product. For general n, the set of solutions is equipped with an action of $\mathcal{U}_d$. As in [Nag51, Ch. 6, §58], we call *classes of solutions* the orbits of such action. We shall say that a solution (x, y) of $X^2 - dY^2 = n$ is the *fundamental solution* in its class if y is the minimal non-negative among all solutions in the class. If two solutions in the class have the same minimal non-negative y, then the solution with $x > 0$ is the fundamental solution.

If d is not a perfect square, then there are at most finitely many classes of solutions [Nag51, Theorem 109]. Moreover the fundamental solution (x, y) is explicitly bounded. In the case $n \geq 0$, it lies in the rectangle

$$0 < |x| \leq \sqrt{\frac{1}{2}(u_1+1)n},$$
$$0 \leq y \leq \frac{u_2}{\sqrt{2(u_1+1)}} \cdot \sqrt{n},$$

where (u_1, u_2) is the positive generator of $\mathcal{U}_d$.

Starting from a fundamental solution (if any) (U_0, V_0) of $X^2 - dY^2 = d - 1$, we can construct two sequences (U_i, V_i) and (U_{-i}, V_{-i}) obtained respectively via multiplication (Brahmagupta product) by (u_1, u_2) or by its inverse $(u_1, -u_2)$. In other words, if

$$M = \begin{pmatrix} u_1 & du_2 \\ u_2 & u_1 \end{pmatrix},$$

then, for every $i \in \mathbb{Z}$,

$$\begin{pmatrix} U_i \\ V_i \end{pmatrix} = M^i \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}.$$

Notice that the sequences (U_i), (V_i), (U_{-i}), (V_{-i}) satisfy the second order recurrence

$$x_{i+2} = 2u_1 x_{i+1} - x_i,$$

so that the same recurrence relation is satisfied also by the sequences $(U_i - dV_i)$ and $(U_{-i} - dV_{-i})$. We recall the following result concerning the occurrence of powers of integers in linear recursive sequences.

Theorem 4.12 ([Pet82])**.** *Let (x_n) be a sequence of integers satisfying the binary linear recurrence relation*

$$x_n = hx_{n-1} - kx_{n-2},$$

with $h, k \in \mathbb{Z}$ such that $h \neq 0$, $(h, k) = 1$ and $h^2 \neq ik$ for $i \in \{1, 2, 3, 4\}$. Assume that the initial values x_0, x_1 are not both zero and that $h^2 - 4k$ is not a perfect square if the quantity $k(x_1^2 - hx_1x_2 + kx_0^2)$ is zero.

Let S be a finite set of prime numbers and consider the diophantine equation

$$x_n = wz^e, \tag{30}$$

with $n, e \in \mathbb{N}$, $z, w \in \mathbb{Z}$, $e \geq 2$ and w an integer whose prime divisors all lie in S. Then, there are effectively computable constants C_1, C_2, C_3 depending only on h, k, S, x_0, x_1 such that every solution as above of equation (30) satisfies

- $\max\{|w|, |z|, n, e\} < C_1$, *if* $|z| > 1$;
- $\max\{|w|, n\} < C_2$, *if* $|z| = 1$;
- $n < C_3$, *if* $z = 0$.

It is easily seen that the hypotheses of Theorem 4.12 are satisfied for the sequence $(U_n - dV_n)$. We deduce the following result.

Theorem 4.13. *Let d be a square-free positive integer.*

(a) For every p, there are finitely many periodic p-adic continued fractions of type $(0, 3)$ converging to $\sqrt{d}$ (take $S = \{$prime divisors of $d - 1\} \cup \{p\}$ and $z = 1$). Moreover, the maximum exponent s and index n are effectively computable.

(b) For every d, there are finitely many p and periodic p-adic continued fractions of type $(0, 3)$ converging to $\sqrt{d}$ and such that the exponent s of p in the denominator of a_2 is > 1 (take $S = \{$prime divisors of $d - 1\}$).

Although Theorem 4.12 reduces in principle the problem of finding all the solutions of (30) to a finite amount of computations, from a practical point of view the possibility of using brute force is illusory since the computations involved are extremely heavy.

Example 4.14. Consider the case $d = 5$. There are three classes of solutions of $X^2 - 5Y^2 = 4$, represented by the fundamental solutions $(2, 0)$ and $(\pm 3, 1)$. The positive generator of $\mathcal{U}_5$ is $(9, 4)$. Therefore we are looking for prime powers in six sequences $W^i = (W_n^i)$ for $i = 1, \ldots, 6$ satisfying the recurrence formulae $W_{n+2}^i = 18W_{n+1}^i - W_n^i$ with initial conditions, respectively

$$\begin{aligned}
W_0^1 &= 2, & W_1^1 &= -22,\\
W_0^2 &= 2, & W_1^2 &= 58,\\
W_0^3 &= -2, & W_1^3 &= -58,\\
W_0^4 &= -2, & W_1^4 &= 22,\\
W_0^5 &= -8, & W_1^5 &= 8,\\
W_0^6 &= -8, & W_1^6 &= -152.
\end{aligned}$$

Up to sign, it is enough to study the sequences W^1, W^2, and W^5. Therefore we are looking in these sequences for values of the form wp^s, with p prime, w dividing 4, and $s \geq 1$. On the other side, all terms in the sequence W^5 are divisible by 8.

Let us show that there are no solutions with $s > 1$. We are considering solutions of the equations $X^2 - 5Y^2 = 4$ and $X - 5Y = wp^s$ for w a divisor of 4 and p a prime number.

The second equation tells us that $X \equiv 5Y \pmod p$; putting this into the first equation gives $5Y^2 \equiv 1 \pmod p$. Therefore $\left(\frac{5}{p}\right) = 1$ and, by reciprocity, $\left(\frac{p}{5}\right) = 1$, so that $p \equiv \pm 1 \pmod 5$.

The first equation also tells us that $X \equiv \pm 2 \pmod 5$ and therefore from the second equation we obtain that $w = \pm 2$. Eliminating X from the first equation we now find

$$5Y^2 \pm 5p^sY + p^{2s} - 1 = 0.$$

The discriminant of this equation as a quadratic equation in Y is $5(p^{2s}+4)$, therefore we must have that $p^{2s} + 4 = 5u^2$, that is to say that $\frac{p^s+\sqrt{5}u}{2}$ has norm -1 in $\mathbb{Q}(\sqrt{5})$. Let $\phi = \frac{1+\sqrt{5}}{2}$ be the fundamental unit in $\mathbb{Q}(\sqrt{5})$. It can be shown easily by induction that its powers can be written as $\phi^n = \frac{L_n+\sqrt{5}F_n}{2}$, where L_n and F_n are the n-th Lucas and Fibonacci numbers respectively. Therefore we are looking for prime powers in the sequence of Lucas numbers, but it is known from [BMS06] that 1 and 4 are the only Lucas numbers to be perfect powers.

4.6.3. *The case* $d = a^2+1$. In this case the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is $u = a+\sqrt{d}$ and has norm -1. Therefore the fundamental solution of the Pell equation $X^2 - dY^2 = 1$ is $(2a^2+1, 2a)$, and all other positive solutions are obtainable by the recurrence formulae

$$\begin{cases} x_1 &= 2a^2+1, \\ y_1 &= 2a, \\ x_{n+1} &= x_1x_n + dy_1y_n, \\ y_{n+1} &= y_1x_n + x_1y_n. \end{cases}$$

A class of solutions of the Pell equation $X^2 - dY^2 = d-1$ given by

$$\begin{cases} U_n &= ax_n, \\ V_n &= ay_n; \end{cases}$$

then,

$$U_n - dV_n = a(x_n - dy_n).$$

Now, let $f_n(x) \in \mathbb{Z}[x]$ be the sequence of polynomials defined by the recurrence

$$\begin{aligned} f_0(x) &= 1, \\ f_1(x) &= -2x^3 + 2x^2 - 2x + 1, \\ f_{n+2}(x) &= 2(2x^2+1)f_{n+1}(x) - f_n(x), \end{aligned}$$

so that $f_n(a) = x_n - dy_n$.
We have that

$$\frac{d-1}{U_n - dV_n} = \frac{a}{x_n - dy_n} = \frac{a}{f_n(a)},$$

and the greatest common divisor between a and $f_n(a)$ is 1 because $x_n^2 - dy_n^2 = 1$ implies

$$(x_n+y_n)f_n(a) = (x_n+y_n)(x_n-dy_n) = 1 - (d-1)x_ny_n = 1 - a^2x_ny_n.$$

Therefore, for every $n \in \mathbb{N}$ such that $f_n(a)$ is a power of p with exponent ≥ 1, there is a point $\left(ax_n, \frac{a}{f_n(a)}, ay_n\right) \in V(x^2-d)_{0,3}^{\mathrm{con}}$, i.e. a p-adically convergent continued fraction for $\sqrt{d}$.

Examples 4.15.

a) For $a = 3, d = 10$, we found solutions for $p = 41, 1559, 3241728359, \ldots$;
b) for $a = 4, d = 17$ we found solutions for $p = 103, 448631, 29602847, \ldots$.

Recall that the *Bunyakowsky Conjecture* [Bou57] asserts that for every nonconstant irreducible polynomial $f(x)$ in $\mathbb{Z}[x]$ such that the values $f(n)$ are coprime for $n = 1, \ldots$, there are infinitely many $n \in \mathbb{Z}$ such that $|f(n)|$ is a prime number.
Then, for instance, since $f_1(x) = -2x^3 + 2x^2 - 2x + 1$ is irreducible, and it has constant term 1, we expect to find infinitely many integers d of the form $a^2 + 1$ and primes p such $|f_1(a)| = p$, so that $V(x^2 - (a^2+1))^{\mathrm{con}}_{0,3}$ contains the point $\left(a(2a^2+1), \frac{a}{p}, 2a^2\right)$. The following conjecture has been tested for n up to 10000.

Conjecture 4.16. *$f_n(x)$ is irreducible in $\mathbb{Z}[x]$ for every $n \in \mathbb{Z}$.*

If Conjecture 4.16 and Bunyakowsky Conjecture are true, then for every $n \in \mathbb{N}$ there are infinitely many integers a such that $f_n(a)$ is a prime p, giving rise to points $\left(ax_n, \frac{a}{p}, ay_n\right) \in V(x^2 - (a^2+1))^{\mathrm{con}}_{0,3}$.
The following conjecture, if true, would show that the condition on the exponent of p in the second part of Theorem 4.13 cannot be dropped.

Conjecture 4.17. *For every integer $a \neq 0$ there are infinitely many $n \in \mathbb{N}$ such that $|f_n(a)|$ is a prime number.*

4.6.4. *The negative Pell equation.* Assume now that the negative Pell equation $X^2 - dY^2 = -1$ admits a solution, and let (s_1, t_1) be the fundamental solution for it. Then the positive solutions of $X^2 - dY^2 = 1$ can be obtained by the recurrence formulae

$$\begin{cases} x_1 &= s_1^2 + dt_1^2, \\ y_1 &= 2s_1t_1, \\ x_{n+1} &= x_1x_n + dy_1y_n, \\ y_{n+1} &= y_1x_n + x_1y_n, \end{cases}$$

and the positive solutions of $X^2 - dY^2 = -1$ can be obtained by the recurrence formulae

$$\begin{cases} s_{n+1} &= x_1s_n + dy_1t_n, \\ t_{n+1} &= y_1s_n + x_1t_n. \end{cases}$$

A class of solutions of the generalized Pell equation $X^2 - dY^2 = d - 1$ is given by (U_n, V_n), where $U_n = s_n + dt_n$, $V_n = s_n + t_n$. We have $\frac{d-1}{U_n - dV_n} = \frac{-1}{s_n}$. Therefore, if $|s_n|$ is a prime number p, we get points $\left(\pm p + dt_n, \mp\frac{1}{p}, \pm p + t_n\right) \in V(x^2 - d)^{\mathrm{con}}_{0,3}$. However, notice that s_1 divides y_1, therefore it also divides s_n for every $n \in \mathbb{N}$. This implies that there is at most one prime in the sequence of s_n's when $s_1 \neq \pm 1$. Finally, $s_1 = \pm 1$ can only occur if $d = 2$: in this case we do find solutions for $p = 7, 41, 239, 9369319, 63018038201, 489133282872437279, \ldots$.

5. Some examples of type (1, 3)

The generic PCF of type (1,3) has the form $[b_1, \overline{a_1, a_2, a_3}.]$
Since the convergence only depends on the purely periodic part, Proposition 3.1 tells us that $[b_1, \overline{a_1, a_2, a_3}]$ converges p-adically if and only if $v_p(a_1a_2a_3 + a_1 + a_2 + a_3) < 0$. We focus on the case $F = x^2 - d$. The equations of $V(F)_{1,3}$ are

$$\begin{cases} a_1a_2a_3 + a_1 + a_3 - a_2 = 2b_1(a_1a_2 + 1), \\ -(a_3a_2 + 1) = (b_1^2 - d)(a_1a_2 + 1), \end{cases}$$

or, equivalently,

$$\begin{cases} a_3 = 2b_1 + \frac{a_2 - a_1}{a_1a_2 + 1}, \\ (b_1^2 - d)(a_1a_2 + 1)^2 + a_2^2 + 2b_1a_2(a_1a_2 + 1) + 1 = 0. \end{cases} \tag{31}$$

If we fix b_1, the second equation of (31) defines a conic in a_2 and $c = a_1a_2 + 1$.

In the remainder of this section, we assume that d has the form $d = S^2 + 1$ for some positive integer S, and we study the integral points of the subvariety of $V(F)_{1,3}$ defined by $b_1 = S$. In this case, the second equation of (31) becomes

$$-a_1^2a_2^2 - 2a_1a_2 + a_2^2 + 2Sa_1a_2^2 + 2Sa_2 = 0. \tag{32}$$

For $a_2 = 0$ we obtain points

$$[S, \overline{a_1, 0, 2S - a_1}],$$

which are convergent if and only if $v_p(S) < 0$.
We now find points with $a_2 \neq 0$. Equation (32) gives

$$a_2 = \frac{2(S - a_1)}{a_1^2 - 2Sa_1 - 1},$$

and the first equation in (31) gives $a_3 = a_1$, so that we get a family of points

$$\left[S, \overline{a_1, \frac{2(S - a_1)}{a_1^2 - 2Sa_1 - 1}, a_1}\right]. \tag{33}$$

The convergence condition in this case is

$$v_p\left(\frac{2a_1^2S + 4a_1 - 2S}{-a_1^2 + 2a_1S + 1}\right) < 0.$$

In particular, if $v_p(S) \geq 0$ and $v_p(a_1) \neq 0$, (33) is not p-adically convergent.

Example 5.1. If $d = 10$, $b_1 = 3, p = 3$

$$\sqrt{10} = \left[3, \overline{1, -\frac{2}{3}, 1}\right],$$

3-adically convergent. The same holds in general for

$$\sqrt{p^2 + 1} = \left[p, \overline{1, -\frac{p-1}{p}, 1}\right].$$

Example 5.2. Setting $a_1 = 2$ we get

$$a_2 = -\frac{2(S - 2)}{4S - 3}.$$

Setting now $S = \frac{p^k+3}{4}$ if p is congruent to 1 modulo 4, or $S = \frac{p^{2k}+3}{4}$ if p is congruent to -1 modulo 4 gives an infinite family of examples of the form

$$\sqrt{\frac{p^{2k} + 6p^k + 25}{16}} = \left[\frac{p^k + 3}{4}, \overline{2, -\frac{p^k - 5}{2p^k}, 2}\right] \qquad \text{if } p \equiv 1 \pmod 4,$$

$$\sqrt{\frac{p^{4k} + 6p^{2k} + 25}{16}} = \left[\frac{p^{2k} + 3}{4}, \overline{2, -\frac{p^{2k} - 5}{2p^{2k}}, 2}\right] \qquad \text{if } p \equiv -1 \pmod 4.$$

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