# Relational Semantics for Processes

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**Abstract**

In this paper we give a relation-algebraic model of processes. All standard operations (including parallel composition/interleaving) of the Calculus of Communicating Systems CCS are interpreted by purely relational terms without any inductive methods. We also introduce the notion of a relational bisimulation which leads to a canonical representative of a bisimulation-class of processes.

## 1 Introduction

The standard model for the Calculus of Communicating Systems CCS is the synchronization tree model [4, 8], i.e., operational trees modulo bisimulation. One of the drawbacks of this approach is that there is no canonical representative of a bisimulation class and that the definition of interleaving requires inductive methods.

Our relational approach introduces a category of transition graphs with graph homomorphisms. On this category a notion of bisimulation is established, and it is proven that a bisimulation class, seen as a subcategory, has a terminal object which serves as a canonical representative of this class. We will interprete the standard operations of CCS: prefixing, relabelling, hiding, sum, interleaving by purely relational methods as functors on our category. Every process term $P[X]$ has therefore an associated functor $F(X)$. The semantics of a recursive process definition $X = P[X]$ is defined to be the terminal object of the bisimulation class of the final $F$-coalgebra.

The paper is structured as follows: Based on some fundamentals on heterogeneous relation algebras introduced in the second section we define a category $\mathcal{G}$ of labelled graphs and graph homomorphisms over a given relation algebra in Section 3.

A main contribution of this paper is the introduction of the notion of a relational bisimulation on $\mathcal{G}$ in Section 4. It is proven by purely relational means that every equivalence class of bisimilar graphs has a terminal representative.

In Section 5 we define the standard operations of process calculi as suitable functors on this category. Thereby, the interleaving functor $| : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ corresponding to the parallel composition of two processes is defined without any inductive methods (e.g. using the expansion law [4]). Furthermore, we show that the transition rules associated with each operation are satisfied.

We assume that the reader is familiar with basic notions of the theory of heterogeneous relation algebras and allegories (cf. [2, 5, 7]). We use the notation from [5].

# 2 Heterogeneous Relation Algebras

In this section we recall some fundamentals on heterogeneous relation algebras.

**Definition 2.1** *A (heterogeneous abstract) relation algebra is a locally small category $\mathcal{R}$ consisting of a class $\mathrm{Obj}_{\mathcal{R}}$ of objects and a set $\mathcal{R}[A, B]$ of morphisms for all $A, B \in \mathrm{Obj}_{\mathcal{R}}$ (we also use the notation $R : A \leftrightarrow B$ to indicate that $R \in \mathcal{R}[A, B]$). The morphisms are usually called relations. Composition is denoted by ";" and identities are denoted by $\mathbb{I}_A \in \mathcal{R}[A, A]$. In addition, there is a totally defined unary operation $\breve{\ }_{AB} : \mathcal{R}[A, B] \longrightarrow \mathcal{R}[B, A]$ between the sets of morphisms, called conversion. The operations satisfy the following rules:*

1. *Every set $\mathcal{R}[A, B]$ carries the structure of a complete atomic boolean algebra with operations $\sqcup_{AB}, \sqcap_{AB}, \overline{\phantom{x}}_{AB}$, zero element $\perp\!\!\!\perp_{AB}$, universal element $\top\!\!\!\top_{AB}$, and inclusion ordering $\sqsubseteq_{AB}$.*

2. *The Schröder equivalences*

$$Q; R \sqsubseteq_{AC} S \iff Q\breve{\ }; \overline{S} \sqsubseteq_{BC} \overline{R} \iff \overline{S}; R\breve{\ } \sqsubseteq_{AB} \overline{Q}$$

   *hold for relations $Q : A \leftrightarrow B, R : B \leftrightarrow C$ and $S : A \leftrightarrow C$.*

3. *The Tarski rule*

$$R \neq \perp\!\!\!\perp_{AB} \Longrightarrow \top\!\!\!\top_{CA}; R; \top\!\!\!\top_{BD} = \top\!\!\!\top_{CD}$$

   *holds for all $R \in \mathcal{R}[A, B]$ and $C, D \in \mathrm{Obj}_{\mathcal{R}}$.*

*All the indices of elements and operations are usually omitted for brevity and can easily be reinvented.*

One might ask for the greatest solution of $Q; X \sqsubseteq R$. Using the Schröder equivalences one gets $X = \overline{Q\breve{\ }; \overline{R}}$. This operation is called the right residual. By duality one defines the left residual

$$Q\backslash R := \overline{Q^{\smile};\overline{R}}, \qquad S/T := \overline{\overline{T};S^{\smile}}.$$

A symmetric version of the residuals is the symmetric quotient

$$\mathrm{syQ}(Q,R) := Q\backslash R \sqcap Q^{\smile}/R^{\smile}.$$

By definition this relation is the greatest solution of the inclusions $Q;X \sqsubseteq R$ and $X;R^{\smile} \sqsubseteq Q^{\smile}$.

As usual we define the concept of mappings.

**Definition 2.2** *A relation $R \in \mathcal{R}[A,B]$ is called*

1. *univalent (or partial function) iff $R^{\smile};R \sqsubseteq \mathbb{I}_B$,*
2. *total iff $\mathbb{I}_A \sqsubseteq R;R^{\smile}$,*
3. *injective iff $R^{\smile}$ is univalent,*
4. *surjective iff $R^{\smile}$ is total,*
5. *a mapping iff it is univalent and total.*

*We also use the notation $f : A \longrightarrow B$ to indicate that $f$ is a mapping in $\mathcal{R}[A,B]$.*

If $Q : A \leftrightarrow B$ is univalent the equation $(R \sqcap S;Q^{\smile});Q = R;Q \sqcap S$ is valid for suitable $R$ and $S$, and if $Q$ is total we have $Q;\mathbb{T}_{BC} = \mathbb{T}_{AC}$. A proof of these properties can be found in [7].

Another important class of relations are equivalence relations.

**Definition 2.3** *A relation $R \in \mathcal{R}[A,A]$ is called*

1. *reflexive iff $\mathbb{I}_A \sqsubseteq A$,*
2. *symmetric iff $R^{\smile} \sqsubseteq R$,*
3. *transitive iff $R;R \sqsubseteq R$,*
4. *idempotent iff $R;R = R$,*
5. *an equivalence relation iff it is reflexive, transitive and symmetric.*

The reflexive and transitive closure $R^*$ of a relation $R$ is defined as the least relation containing $R$ which is both reflexive and transitive. It may be computed by $R^* = \bigsqcup\limits_{i\in\mathbb{N}} R^i$ where $R^i := R;\ldots;R$ ($i$ times).

We now introduce the notion of unit objects which are the abstract version of singleton sets.

**Definition 2.4** *An object* $\mathrm{I}$ *is called a unit iff* $\mathbb{I}_{\mathrm{I}}$ *is the greatest morphism in* $\mathcal{R}[\mathrm{I},\mathrm{I}]$ *and for every object* $A$ *there is at least one total morphism in* $\mathcal{R}[A,\mathrm{I}]$*.* $\mathcal{R}$ *is called unitary iff it has a unit.*

The unit I may also be characterized as a terminal object in the subcategory of mappings, and is, hence, unique up to isomorphism. Following the categorical notion of elements, we define a point as follows.

**Definition 2.5** *A mapping* $x : \mathrm{I} \longrightarrow A$ *is called a point.*

One might be interested in the set of all points included in an arbitrary relation. The so-called point axiom guarantees that this set is not empty. It may be formulated as follows:

**Point Axiom:** *For every relation* $Q \neq \perp\!\!\!\perp$ *there are two points* $x, y$ *such that* $x\breve{\ }; y \sqsubseteq Q$*.*

Notice, that the point axiom implies representability [6].

The relational description of pairing is the relational product [5, 7]. This construction corresponds to the categorical product in the subcategory of mappings.

**Definition 2.6** *An object* $A \times B$ *together with two relations* $\pi \in \mathcal{R}[A \times B, A]$ *and* $\rho \in \mathcal{R}[A \times B, B]$ *is called a relational product of* $A$ *and* $B$ *iff*

$$\pi\breve{\ }; \pi = \mathbb{I}_A, \qquad \rho\breve{\ }; \rho = \mathbb{I}_B,$$
$$\pi\breve{\ }; \rho = \top\!\!\!\top_{AB}, \qquad \pi; \pi\breve{\ } \sqcap \rho; \rho\breve{\ } = \mathbb{I}_{A \times B}.$$

$\mathcal{R}$ *has relational products iff for every pair of objects the relational product exists.*

The relational product of two objects is unique (up to isomorphism) [10]. We use the following notations

$$<P, Q> := P; \pi\breve{\ } \sqcap Q; \rho\breve{\ }, \qquad R \times S := <\pi; R, \rho; S>,$$

whenever the projections exist. It is easy to see that

$$<P, Q>; <R, S>\breve{\ } \sqsubseteq P; R\breve{\ } \sqcap Q; S\breve{\ }.$$

The validity of the converse inclusion is called the sharpness problem of relational products. A set of sufficient conditions for sharpness can be found in [1]. Notice, that sharpness implies the following equalities

$$<P, Q>; (R \times S) = <P; R, Q; S>,$$
$$(P \times Q); (R \times S) = (P; R \times Q; S).$$

However, if $P$ is total we have

$$<P, Q>; \rho = P; \pi\breve{\ }; \rho \sqcap Q = P; \top\!\!\!\top \sqcap Q = Q$$

and analogously $<P, Q>; \pi = P$ if $Q$ is total.

The relational description of disjoint unions is the relational sum [5, 10]. This construction corresponds to the categorical product[1]. Here we want to generalize this concept to not necessarily finite sets of objects.

**Definition 2.7** *Let $\{A_i \mid i \in I\}$ be a set of objects indexed by a set $I$. An object $\sum\limits_{i\in I} A_i$ together with relations $\iota_j \in \mathcal{R}[A_j, \sum\limits_{i\in I} A_i]$ for all $j \in I$ is called a relational sum of $\{A_i \mid i \in I\}$ iff for all $i, j \in I$ with $i \neq j$ the following holds*

$$\iota_i; \iota_i^{\smile} = \mathbb{I}_{A_i}, \qquad \iota_i; \iota_j^{\smile} = \perp\!\!\!\perp_{A_i A_j}, \qquad \bigsqcup_{i\in I} \iota_i^{\smile}; \iota_i = \mathbb{I}_{\sum\limits_{i\in I} A_i}.$$

*$\mathcal{R}$ has relational sums iff for every set of objects the relational sum does exist.*

For a set of two objects $\{A, B\}$ this definition corresponds to usual the definition of the relational sum. We use the following notations

$$\bigvee_{i\in I} P_i := \bigsqcup_{i\in I} \iota_i^{\smile}; P_i, \qquad \sum_{i\in I} R_i := \bigvee_{i\in I} R_i; \iota_i,$$

whenever the injections exist. In the binary case we also write $\iota, \kappa$ instead of $\iota_1, \iota_2$, $[P_1, P_2]$ instead of $\bigvee\limits_{i\in\{1,2\}} P_i$ and $R_1 + R_2$ instead of $\sum\limits_{i\in\{1,2\}} R_i$. It is easy to verify that

$$\iota_j; \bigvee_{i\in I} P_i = P_j, \qquad \iota_j; \textstyle\sum_{i\in I} R_i = R_j; \iota_j,$$

$$(\bigvee_{i\in I} P_i); Q = \bigvee_{i\in I} P_i; Q, \qquad (\textstyle\sum_{i\in I} R_i); (\bigvee_{i\in I} P_i) = \bigvee_{i\in I} R_i; P_i.$$

As known, categorical products and hence relational sums are unique up to isomorphism. Furthermore, every relation algebra may be embedded into one with relational sums (cf. [2, 9]).

As in set theory, relational products distribute over arbitrary sums. The induced isomorphism is defined by

$$\mathrm{distr} := <\bigvee_{i\in I} \pi_i, \sum_{i\in I} \rho_i> : \sum_{i\in I} A \times B_i \to A \times \sum_{i\in I} B_i.$$

We have the following property of distr

$$\mathrm{distr}; (R \times \sum_{i\in I} S_i) = (\sum_{i\in I} R \times S_i); \mathrm{distr}.$$

Given a symmetric idempotent (also known as a partial equivalence relation) one might consider the object of (existing) equivalence classes and the corresponding partial function mapping each element to its equivalence class.

[1] By conversion, a relation algebra is self-dual. Therefore a product is also a coproduct and hence a biproduct.

**Definition 2.8** *A relation $S : A \leftrightarrow A$ is called a split iff there is an object $B$ and a relation $R : B \leftrightarrow A$ such that*

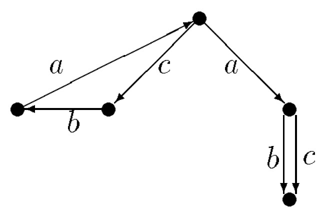
$$R^{\smile}; R = S, \qquad R; R^{\smile} = \mathbb{I}_B.$$

It can be shown that the object $B$ in the definition above is also unique up to isomorphism (cf. [2, 9]).

# 3 Labelled Graphs

Throughout this paper let $\mathcal{R}$ be a unitary heterogeneous relation algebra with a fixed object $L$ from $\mathcal{R}$ such that the relational product $L \times A$ for every object $A$ exists[2]. Furthermore, we suppose that every symmetric idempotent is a split. As shown in [2, 9], every relation algebra can be embedded into another one such that the latter property holds.

In contrast to the synchronization tree approach, we model processes by labelled graphs, also called transition graphs. For example, the recusive defined process $P = c.b.a.P + a.(b.0 \mid c.0)$ may be modeled by the following graph.




We consider a labelled graph on a set of nodes $Z$ as a relation from $Z$ to $L \times Z$. To obtain a convenient category we will consider suitable transition preserving (relational) homomorphisms [5].

**Definition 3.1** *The category $\mathcal{G}$ is defined as follows:*

1. *An object of $\mathcal{G}$ is a pair $(G, w)$ consisting of a relation $G : Z \leftrightarrow L \times Z$ and a point $w : I \longrightarrow Z$. $G = (G, w)$ is called a $L$-graph with root $w$ over the state space $Z$.*

2. *A morphism $f : G_1 \longrightarrow G_2$ is a mapping $f : Z_1 \longrightarrow Z_2$ in $\mathcal{R}$ such that*

$$G_1; (\mathbb{I}_L \times f) \sqsubseteq f; G_2 \quad \textit{and} \quad w_1; f = w_2.$$

*$f$ is called a homomorphism from $G_1$ to $G_2$.*

[2]This requirement gives us sharpness, but do not imply representability (cf. [1, 5]).

An easy verification shows that $\mathcal{G}$ is indeed a category.

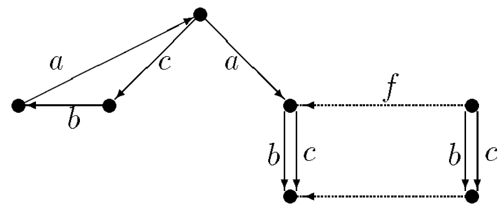
In the category $\mathcal{G}$ a subobject describes a subgraph starting at the same root. To model a transition $P \xrightarrow{a} P'$ we are interested in subgraphs such that the new root is successor of the original root.

**Definition 3.2** *Let $a : I \longrightarrow L$ be a point. An injective morphism $f$ from $G_1 : Z_1 \leftrightarrow L \times Z_1$ to $G_2 : Z_2 \leftrightarrow L \times Z_2$ with*

$$f; G_2 = G_1; (\mathbb{I}_L \times f) \quad \textit{and} \quad w_1; f \sqsubseteq w_2; G_2; (a\breve{\ }; a \times \mathbb{I}_{Z_2}); \rho$$

*is called a transition (in resp. with $a$) $f : G_1 \overset{a}{\rightarrowtail} G_2$. We write $G_1 \overset{a}{\rightarrowtail} G_2$ if such a morphism exists.*

Notice, that a transition morphism is not a morphism of the category $\mathcal{G}$ (but of $\mathcal{R}$). Furthermore, the direction of arrows is reversed. Intuitively, $G_1 \overset{a}{\rightarrowtail} G_2$ indicates that $G_1$ is an $a$-derivative of $G_2$. For example, the process $P$ defined above may reduce (by an $a$-action) to $b.0 \mid c.0$. This situation is modeled by the following $a$-transition$f$.

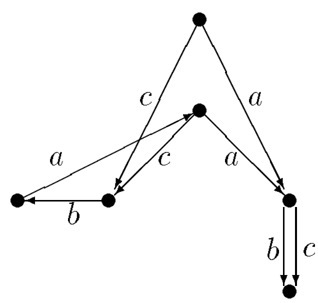



Within a graph there may be an edge targetting at the root. For several purposes we need to seperate the root from the rest of the graph in the sense that there are no edges of this kind.

**Definition 3.3** *Suppose $\mathcal{R}$ has relational sums. Then the extension of a graph $G = (G, w)$ $\mathrm{ext}(G) = (\mathrm{ext}(G), \mathrm{ext}(w))$ is defined by*

1. $\mathrm{ext}(G) := [w; G, G]; (\mathbb{I}_L \times \kappa) : \mathrm{I} + Z \leftrightarrow L \times (\mathrm{I} + Z)$,
2. $\mathrm{ext}(w) := \iota : \mathrm{I} \leftrightarrow \mathrm{I} + Z$.

In our example we gain the following graph.



In the next section we will show that $G$ and $\mathrm{ext}(G)$ are bisimular. For the moment we have to be satisfied with the following lemma.

**Lemma 3.4** *Let* $g : G_1 \overset{a}{\rightarrowtail} G_2$ *be a transition morphism. Then the mapping* $g;\kappa : Z_1 \to \mathrm{I} + Z_2$ *is a transition morphism from* $G_1$ *to* $\mathrm{ext}(G_2)$ *with* $g;\kappa;\mathrm{ext}(w_2)^{\smile} = \perp\!\!\!\perp$.

**Proof:** Notice, that the composition of two injective mappings is a injective mapping again. The calculations

$$\begin{aligned} g;\kappa;\mathrm{ext}(G_2) &= g;\kappa;[w_2;G_2,G_2];(\mathbb{I}_L \times \kappa) \\ &= g;G_2;(\mathbb{I}_L \times \kappa) \\ &= G_1;(\mathbb{I}_L \times g);(\mathbb{I}_L \times \kappa) \\ &= G_1;(\mathbb{I}_L \times g;\kappa) \end{aligned}$$

and

$$\begin{aligned} w_1;g;\kappa &\sqsubseteq w_2;G_2;(a^{\smile};a \times \mathbb{I}_{Z_2});\rho;\kappa \\ &= w_2;G_2;(a^{\smile};a \times \mathbb{I}_{Z_2});(\mathbb{I}_L \times \kappa);\rho \\ &= w_2;G_2;(a^{\smile};a \times \kappa);\rho \\ &= w_2;G_2;(\mathbb{I}_L \times \kappa);(a^{\smile};a \times \mathbb{I}_{Z_2});\rho \\ &= \iota;[w_2;G_2,G_2];(\mathbb{I}_L \times \kappa);(a^{\smile};a \times \mathbb{I}_{Z_2});\rho \\ &= \mathrm{ext}(w_2);\mathrm{ext}(G_2);(a^{\smile};a \times \mathbb{I}_{Z_2});\rho \end{aligned}$$

show that $g;\kappa$ is indeed a transition morphism. The required property follows from

$$g;\kappa;\mathrm{ext}(w_2)^{\smile} = g;\kappa;\iota^{\smile} = g;\perp\!\!\!\perp = \perp\!\!\!\perp.$$ □

# 4 Relational Bisimulation

An important class of equivalence relations on processes are strong bisimulations. We are now going to establish a corresponding notion on $\mathcal{G}$. First we modify the definition of a covering [5] of two graphs. We want to allow a covering to identify two subgraphs which are identically labelled. This reflects the fact that the corresponding processes are bisimular.

**Definition 4.1** *A surjective homomorphism* $f$ *from* $G_1$ *to* $G_2$ *with*

$$f;G_2 \sqsubseteq G_1;(\mathbb{I}_L \times f)$$

*is called a* $L$*-covering.*

We write $f : G_1 \overset{\approx}{\to} G_2$ if $f$ is a $L$-covering from $G_1$ to $G_2$, $G_1 \overset{\approx}{\to} G_2$ if such a morphism exists and $G_1 \approx G_2$ if there is a $G_3$ such that $G_1 \overset{\approx}{\to} G_3$ and $G_2 \overset{\approx}{\to} G_3$.

The next lemma is used several times throughout the paper. For this reason we do not explicitly mention it in any case.

**Lemma 4.2** *Let be* $f : G_1 \overset{\approx}{\to} G_2$*. Then we have*

1. $f^\smile ; G_1 \sqsubseteq G_2; (\mathbb{I}_L \times f^\smile)$,
2. *If* $f$ *is injective then* $f^\smile : G_2 \overset{\approx}{\to} G_1$.

**Proof:**

1. The assertion follows from

$$\begin{aligned} f^\smile ; G_1 &\sqsubseteq f^\smile ; G_1; (\mathbb{I}_L \times f; f^\smile) \\ &= f^\smile ; G_1; (\mathbb{I}_L \times f); (\mathbb{I}_L \times f^\smile) \\ &= f^\smile ; f; G_2; (\mathbb{I}_L \times f^\smile) \\ &= G_2; (\mathbb{I}_L \times f^\smile). \end{aligned}$$

2. If $f$ is injective the $\sqsubseteq$ in the proof of 1. is an equality and we have $w_2; f^\smile = w_1; f; f^\smile = w_1$. □

As mentioned in the last section there is a $L$-covering $f$ from $\mathrm{ext}(G)$ to $G$. This morphism identifies the new with the old root of the graph. In our example the states at the top of the graph are mapped to the original root.

**Lemma 4.3** $\mathrm{ext}(G) \overset{\approx}{\to} G$.

**Proof:** Consider the surjective mapping $[w, \mathbb{I}_Z] : \mathrm{I} + Z \longrightarrow Z$. Then we have

$$\begin{aligned} [w, \mathbb{I}_Z]; G &= [w; G, G] \\ &= [w; G, G](\mathbb{I}_L \times \mathbb{I}_Z) \\ &= [w; G, G](\mathbb{I}_L \times \kappa; [w, \mathbb{I}_Z]) \\ &= [w; G, G](\mathbb{I}_L \times \kappa); (\mathbb{I}_L \times [w, \mathbb{I}_Z]) \\ &= \mathrm{ext}(G); (\mathbb{I}_L \times [w, \mathbb{I}_Z]) \end{aligned}$$

and

$$\mathrm{ext}(w); [w, \mathbb{I}_Z] = \iota; [w, \mathbb{I}_Z] = w.$$ □

The identification of subgraphs can be seen as a reduction process. In the next lemma we show that this process is confluent. The required graph $G_4$ is just the graph which is obtained from $G_1$ and the equivalence relation induced by the $L$-coverings $f : G_1 \overset{\approx}{\to} G_2$ and $g : G_1 \overset{\approx}{\to} G_3$.

**Lemma 4.4** *If* $G_1 \overset{\approx}{\to} G_2$ *and* $G_1 \overset{\approx}{\to} G_3$ *then there is a* $G_4$ *such that* $G_2 \overset{\approx}{\to} G_4$ *and* $G_3 \overset{\approx}{\to} G_4$.

**Proof:** Suppose $f : G_1 \overset{\approx}{\to} G_2$ and $g : G_1 \overset{\approx}{\to} G_3$. Then the relation $A := (f; f^{\smile} \sqcup g; g^{\smile})^*$ is an equivalence relation on $Z_1$. Futhermore, suppose $R$ splits $A$. Then $R^{\smile}$ is a surjective mapping. Define

$$\begin{aligned} G_4 &:= R; G_1; (\mathbb{I} \times R^{\smile}), \\ w_4 &:= w_1; R^{\smile}, \\ h &:= f^{\smile}; R^{\smile}, \\ k &:= g^{\smile}; R^{\smile}. \end{aligned}$$

Then $w_4$ is a point because we have

$$\begin{aligned} w_4^{\smile}; w_4 &= R; w_1^{\smile}; w_1; R^{\smile} \\ &\sqsubseteq R; R^{\smile} \\ &= \mathbb{I}, \\ w_4; w_4^{\smile} &= w_1; R^{\smile}; R; w_1^{\smile} \\ &= w_1; A; w_1^{\smile} \\ &\sqsupseteq w_1; w_1^{\smile} \\ &\sqsupseteq \mathbb{I}. \end{aligned}$$

Now, we want to show that

$$(*) \qquad R = R; f; f^{\smile}.$$

The first inclusion $R \sqsubseteq R; f; f^{\smile}$ is given by the totality of $f$ and the other one by

$$R; f; f^{\smile} = R; R^{\smile}; R; f; f^{\smile} = R; A; f; f^{\smile} \sqsubseteq R; A; A = R; A = R; R^{\smile}; R = R.$$

Using $(*)$ we conclude

$$\begin{aligned} h^{\smile}; h &= R; f; f^{\smile}; R^{\smile} \\ &= R; R^{\smile} \\ &= \mathbb{I}, \\ h; h^{\smile} &= f^{\smile}; R^{\smile}; R; f \\ &= f^{\smile}; A; f \\ &\sqsupseteq f^{\smile}; f \\ &= \mathbb{I} \end{aligned}$$

that $h$ is a surjective mapping. The property $w_1; f = w_2$ gives us $w_1 \sqsubseteq w_1; f; f^{\smile} = w_2; f^{\smile}$ and finaly

$$\begin{aligned} w_2; h &= w_2; f^{\smile}; R^{\smile} \\ &\sqsupseteq w_1; R^{\smile} \\ &= w_4. \end{aligned}$$

The fact that both $w_4$ and $w_2;h$ are mappings shows that $w_2;h = w_4$. Again using $(*)$ we have

$$\begin{aligned}
f;f^{\smile};G_1;(\mathbb{I}\times R^{\smile}) &= f;f^{\smile};G_1;(\mathbb{I}\times f;f^{\smile};R^{\smile})\\
&= f;f^{\smile};G_1;(\mathbb{I}\times f);(\mathbb{I}_L\times f^{\smile};R^{\smile})\\
&= f;f^{\smile};f;G_2;(\mathbb{I}_L\times f^{\smile};R^{\smile})\\
&= f;G_2;(\mathbb{I}_L\times f^{\smile};R^{\smile})\\
&= G_1;(\mathbb{I}_L\times f);(\mathbb{I}_L\times f^{\smile};R^{\smile})\\
&= G_1;(\mathbb{I}_L\times f;f^{\smile};R^{\smile})\\
&= G_1;(\mathbb{I}_L\times R^{\smile})
\end{aligned}$$

and analogously $g;g^{\smile};G_1;(\mathbb{I}\times R^{\smile}) = G_1;(\mathbb{I}_L\times R^{\smile})$. This implies

$$\begin{aligned}
A;G_1;(\mathbb{I}\times R^{\smile}) &= (f;f^{\smile}\sqcup g;g^{\smile})^*;G_1;(\mathbb{I}\times R^{\smile})\\
&= \bigsqcup_{i\in\mathbb{N}}(f;f^{\smile}\sqcup g;g^{\smile})^i;G_1;(\mathbb{I}\times R^{\smile})\\
&= G_1;(\mathbb{I}\times R^{\smile}).
\end{aligned}$$

Finaly, the following computation shows that $h$ is a $L$-covering

$$\begin{aligned}
h;G_4 &= f^{\smile};R^{\smile};R;G_1;(\mathbb{I}\times R^{\smile})\\
&= f^{\smile};A;G_1;(\mathbb{I}\times R^{\smile})\\
&= f^{\smile};G_1;(\mathbb{I}\times R^{\smile})\\
&= f^{\smile};G_1;(\mathbb{I}\times f;f^{\smile};R^{\smile})\\
&= f^{\smile};G_1;(\mathbb{I}\times f);(\mathbb{I}_L\times f^{\smile};R^{\smile})\\
&= f^{\smile};f;G_2;(\mathbb{I}_L\times h)\\
&= G_2;(\mathbb{I}_L\times h).
\end{aligned}$$

The required properties of $k$ are shown analogously. □

In the last lemma we have shown that the identification process of subgraphs is confluent. Furthermore, as we will show this process is terminating. In the language of categories this property is expressed by the existence of suitable terminal objects.

Given a graph $G$ we denote with $\mathcal{G}_G$ the subcategory of $\mathcal{G}$ which objects are all graphs $G' \approx G$ and morphisms are $L$-coverings.

**Theorem 4.5** *The category $\mathcal{G}_G$ has a terminal object.*

**Proof:** Let $G : Z \leftrightarrow L\times Z$ be a graph. Consider the operation[3]

$$\tau(R) := (G^{\smile}\backslash(\mathbb{I}_L\times R);G^{\smile})\sqcap(G;(\mathbb{I}_L\times R)/G).$$

[3]The definition of this operation and its greatest fixpoint $A$ was motivated by a simular definition in [3].

on $\mathcal{R}[Z,Z]$. Obviously, $\tau$ is monotonic (wrt. $\sqsubseteq$) and hence has a greatest fixpoint $A$. Notice that $A$ is by definition the greatest relation $X$ in $\mathcal{R}[Z,Z]$ with $X \sqsubseteq \tau(X)$. This and the universal properties of the residuals show that the inclusions $G^{\smile};X \sqsubseteq (\mathbb{I}_L \times X);G^{\smile}$ and $X;G \sqsubseteq G;(\mathbb{I}_L \times X)$ are sufficient for $X \sqsubseteq A$. Furthermore, if $X$ is symmetric these inclusions are equivalent and only one has to be mentioned. Conversely, we have $A \sqsubseteq A$ such that $A;G \sqsubseteq G;(\mathbb{I}_L \times A)$ holds.

First, we want to show that $A$ is an equivalence relation.

$A$ **is reflexive:** Since $\mathbb{I}_Z$ is symmetric and $\mathbb{I}_Z;G \sqsubseteq G$ holds we have $\mathbb{I}_Z \sqsubseteq A$.

$A$ **is transitive:** We have

$$\begin{aligned} & G^{\smile};(G^{\smile}\backslash(\mathbb{I}_L \times A);G^{\smile});(G^{\smile}\backslash(\mathbb{I}_L \times A);G^{\smile}) \\ \sqsubseteq\ & (\mathbb{I}_L \times A);G^{\smile};(G^{\smile}\backslash(\mathbb{I}_L \times A);G^{\smile}) \\ \sqsubseteq\ & (\mathbb{I}_L \times A);(\mathbb{I}_L \times A);G^{\smile} \\ =\ & (\mathbb{I}_L \times A;A);G^{\smile}. \end{aligned}$$

This gives us

$$\begin{aligned} A;A &= \tau(A);\tau(A) \\ &\sqsubseteq (G^{\smile}\backslash(\mathbb{I}_L \times A);G^{\smile});(G^{\smile}\backslash(\mathbb{I}_L \times A);G^{\smile}) \\ &\sqsubseteq G^{\smile}\backslash(\mathbb{I}_L \times A;A);G^{\smile}. \end{aligned}$$

Analogously we compute $A;A \sqsubseteq G;(\mathbb{I}_L \times A;A)/G$. Together we have $A;A \sqsubseteq \tau(A;A)$ and hence $A;A \sqsubseteq A$.

$A$ **is symmetric:** For an abitrary relation $R$ we have

$$\begin{aligned} \tau(R)^{\smile} &= (G^{\smile}\backslash(\mathbb{I}_L \times R);G^{\smile})^{\smile} \sqcap (G;(\mathbb{I}_L \times R)/G)^{\smile} \\ &= (G;(\mathbb{I}_L \times R^{\smile})/G) \sqcap (G^{\smile}\backslash(\mathbb{I}_L \times R^{\smile});G^{\smile}) \\ &= \tau(R^{\smile}). \end{aligned}$$

This implies $A^{\smile} = \tau(A)^{\smile} = \tau(A^{\smile})$ and hence $A^{\smile} \sqsubseteq A$.

Suppose $R$ splits $A$. Then we define

$$\begin{aligned} G_{\mathrm{t}} &:= R;G;(\mathbb{I}_L \times R^{\smile}), \\ w_{\mathrm{t}} &:= w;R^{\smile}. \end{aligned}$$

By definition $R^{\smile}$ is a surjective mapping. The computations

$$\begin{aligned} R^{\smile};G_{\mathrm{t}} &= R^{\smile};R;G;(\mathbb{I}_L \times R^{\smile}) \\ &= A;G;(\mathbb{I}_L \times R^{\smile}) \\ &= G;(\mathbb{I}_L \times A);(\mathbb{I}_L \times R^{\smile}) \\ &= G;(\mathbb{I}_L \times A;R^{\smile}) \\ &= G;(\mathbb{I}_L \times R^{\smile};R;R^{\smile}) \\ &= G;(\mathbb{I}_L \times R^{\smile}) \end{aligned}$$

and $w; R^{\smile} = w_{\mathrm{t}}$ show that $R^{\smile}$ is a $L$-covering from $G$ to $G_{\mathrm{t}}$.

We need some technical properties of $G_{\mathrm{t}}$ for proving that this graph is a terminal object in $\mathcal{G}_G$. First, consider the operation $\tau_{\mathrm{t}}$ simular to $\tau$ on $\mathcal{R}[Z_{\mathrm{t}}, Z_{\mathrm{t}}]$ defined by

$$\tau_{\mathrm{t}}(R) := (G_{\mathrm{t}}^{\smile}\backslash(\mathbb{I}_L \times R); G_{\mathrm{t}}^{\smile}) \sqcap (G_{\mathrm{t}}; (\mathbb{I}_L \times R)/G_{\mathrm{t}})$$

and its greatest fixpoint $A_{\mathrm{t}}$. As above $A_{\mathrm{t}}$ is an equivalence relation. We want to show that $A_{\mathrm{t}} = \mathbb{I}_{Z_{\mathrm{t}}}$ holds. Notice, that we have $A_{\mathrm{t}}; G_{\mathrm{t}} \sqsubseteq G_{\mathrm{t}}; (\mathbb{I}_L \times A_{\mathrm{t}})$. The computation

$$\begin{aligned}
R^{\smile}; A_{\mathrm{t}}; R; G &= R^{\smile}; A_{\mathrm{t}}; R; R^{\smile}; R; G \\
&= R^{\smile}; A_{\mathrm{t}}; R; A; G \\
&\sqsubseteq R^{\smile}; A_{\mathrm{t}}; R; G; (\mathbb{I}_L \times A) \\
&= R^{\smile}; A_{\mathrm{t}}; R; G; (\mathbb{I}_L \times R^{\smile}); (\mathbb{I}_L \times R) \\
&= R^{\smile}; A_{\mathrm{t}}; G_{\mathrm{t}}; (\mathbb{I}_L \times R) \\
&\sqsubseteq R^{\smile}; G_{\mathrm{t}}; (\mathbb{I}_L \times A_{\mathrm{t}}); (\mathbb{I}_L \times R) \\
&= R^{\smile}; R; G; (\mathbb{I}_L \times R^{\smile}); (\mathbb{I}_L \times A_{\mathrm{t}}; R) \\
&= A; G; (\mathbb{I}_L \times R^{\smile}; A_{\mathrm{t}}; R) \\
&\sqsubseteq G; (\mathbb{I}_L \times A); (\mathbb{I}_L \times R^{\smile}; A_{\mathrm{t}}; R) \\
&= G; (\mathbb{I}_L \times A; R^{\smile}; A_{\mathrm{t}}; R) \\
&= G; (\mathbb{I}_L \times R^{\smile}; R; R^{\smile}; A_{\mathrm{t}}; R) \\
&= G; (\mathbb{I}_L \times R^{\smile}; A_{\mathrm{t}}; R)
\end{aligned}$$

and the symmetry of $R^{\smile}; A_{\mathrm{t}}; R$ gives us $R^{\smile}; A_{\mathrm{t}}; R \sqsubseteq A$. We follow

$$A_{\mathrm{t}} = R; R^{\smile}; A_{\mathrm{t}}; R; R^{\smile} \sqsubseteq R; A; R^{\smile} = R; R^{\smile}; R; R^{\smile} = \mathbb{I}_{Z_{\mathrm{t}}}$$

and hence $A_{\mathrm{t}} = \mathbb{I}_{Z_{\mathrm{t}}}$.

Suppose $g : G \overset{\approx}{\rightarrow} G_{\mathrm{t}}$. Using Lemma 4.2 we have

$$\begin{aligned}
R; g; G_{\mathrm{t}} &= R; G; (\mathbb{I}_L \times g) \\
&\sqsubseteq G_{\mathrm{t}}; (\mathbb{I}_L \times R); (\mathbb{I}_L \times g) \\
&= G_{\mathrm{t}}; (\mathbb{I}_L \times R; g), \\
G_{\mathrm{t}}^{\smile}; R; g &= (g^{\smile}; R^{\smile}; G_{\mathrm{t}})^{\smile} \\
&= (g^{\smile}; G; (\mathbb{I}_L \times R^{\smile}))^{\smile} \\
&\sqsubseteq (G_{\mathrm{t}}; (\mathbb{I}_L \times g^{\smile}); (\mathbb{I}_L \times R^{\smile}))^{\smile} \\
&= (\mathbb{I}_L \times R; g); G_{\mathrm{t}}^{\smile}.
\end{aligned}$$

This implies $R; g \sqsubseteq A_{\mathrm{t}} = \mathbb{I}_{Z_{\mathrm{t}}}$ and hence $g \sqsubseteq A; g = R^{\smile}; R; g = R^{\smile}$. Since $g$ and $R^{\smile}$ are mappings they are equal. This shows that $R^{\smile}$ is the unique $L$-covering from $G$ to $G_{\mathrm{t}}$.

Suppose $G \approx G'$. Then there is a graph $G''$ and $L$-coverings $h, k$ with $h : G \overset{\approx}{\rightarrow} G''$ and $k : G' \overset{\approx}{\rightarrow} G''$. First, we want to show that $R; h; h^{\smile} = R$. The computation

$$h; h^{\smile}; G \sqsubseteq h; h^{\smile}; G; (\mathbb{I}_L \times h; h^{\smile})$$

$$\begin{aligned}
&= h;h^{\smile};G;(\mathbb{I}_L \times h);(\mathbb{I}_L \times h^{\smile}) \\
&= h;h^{\smile};h;G';(\mathbb{I}_L \times h^{\smile}) \\
&= h;G';(\mathbb{I}_L \times h^{\smile}) \\
&= G;(\mathbb{I}_L \times h);(\mathbb{I}_L \times h^{\smile}) \\
&= G;(\mathbb{I}_L \times h;h^{\smile})
\end{aligned}$$

shows that $h;h^{\smile} \sqsubseteq A$. This implies our assertition by

$$R;h;h^{\smile} \sqsubseteq R;A = R;R^{\smile};R = R \sqsubseteq R;h;h^{\smile}.$$

Now, we are ready to prove that $k;h^{\smile};R^{\smile} : G' \overset{\approx}{\rightarrow} G_{\mathrm{t}}$ as follows

$$\begin{aligned}
(k;h^{\smile};R^{\smile})^{\smile};k;h^{\smile};R^{\smile} &= R;h;k^{\smile};k;h^{\smile};R^{\smile} \\
&= R;h;h^{\smile};R^{\smile} \\
&= R;R^{\smile} \\
&= \mathbb{I}_Z, \\
k;h^{\smile};R^{\smile};(k;h^{\smile};R^{\smile})^{\smile} &= k;h^{\smile};R^{\smile};R;h;k^{\smile} \\
&\sqsupseteq k;h^{\smile};h;k^{\smile} \\
&= k;k^{\smile} \\
&= \mathbb{I}_{Z'}, \\
k;h^{\smile};R^{\smile};G_{\mathrm{t}} &= k;h^{\smile};G;(\mathbb{I}_L \times R^{\smile}) \\
&= k;h^{\smile};G;(\mathbb{I}_L \times h;h^{\smile};R^{\smile}) \\
&= k;h^{\smile};G;(\mathbb{I}_L \times h);(\mathbb{I}_L \times h^{\smile};R^{\smile}) \\
&= k;h^{\smile};h;G'';(\mathbb{I}_L \times h^{\smile};R^{\smile}) \\
&= k;G'';(\mathbb{I}_L \times h^{\smile};R^{\smile}) \\
&= G';(\mathbb{I}_L \times k;h^{\smile};R^{\smile}), \\
w';k;h^{\smile};R^{\smile} &= w'';h^{\smile};R^{\smile} \\
&\sqsupseteq w;R^{\smile} \\
&= w_{\mathrm{t}}.
\end{aligned}$$

Suppose $l : G'' \overset{\approx}{\rightarrow} G_{\mathrm{t}}$. First, we want to show that $k;k^{\smile};l = l$. The symmetry of $l^{\smile};k;k^{\smile};l$ and

$$\begin{aligned}
l^{\smile};k;k^{\smile};l;G_{\mathrm{t}} &= l^{\smile};k;k^{\smile};G';(\mathbb{I}_L \times l) \\
&\sqsubseteq l^{\smile};k;G'';(\mathbb{I}_L \times k^{\smile};l) \\
&= l^{\smile};G';(\mathbb{I}_L \times k;k^{\smile};l) \\
&\sqsubseteq G_{\mathrm{t}};(\mathbb{I}_L \times l^{\smile};k;k^{\smile};l)
\end{aligned}$$

implies $l^{\smile};k;k^{\smile};l \sqsubseteq A_{\mathrm{t}} = \mathbb{I}_{Z_{\mathrm{t}}}$. This gives us

$$k;k^{\smile};l \sqsubseteq l;l^{\smile};k;k^{\smile};l \sqsubseteq l \sqsubseteq k;k^{\smile};l.$$

By using this property a computation simular to the one above shows that $h; k^{\smile}; l$ is a $L$-covering from $G$ to $G_t$. We conclude $h; k^{\smile}; l = R^{\smile}$ and

$$k; h^{\smile}; R^{\smile} = k; h^{\smile}; h; k^{\smile}; l = k; k^{\smile}; l = l.$$

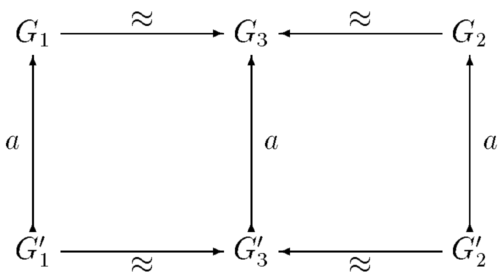
This implies that $G_t$ is indeed a terminal object in $\mathcal{G}_G$. □

In contrast to the tree approach this terminal object can be taken as a canonical representation of its equivalence class.

The following theorem shows that $\mathcal{G}$ provides convenient models for processes.

**Theorem 4.6** *If the point-axiom is valid, then the relation $\approx$ is a strong bisimulation with respect to $\overset{a}{\rightarrowtail}$.*

Given $G_1 \approx G_2$ and $G'_1 \overset{a}{\rightarrowtail} G_1$ the idea of the proof is reflected by the following diagram:




The existence of $G'_2$ and $G'_3$ is guaranteed by the following lemma.

**Lemma 4.7** *1. If $G'_1 \overset{a}{\rightarrowtail} G_1$ and $G_1 \overset{\approx}{\to} G_3$ then there is a $G'_3$ such that $G'_1 \overset{\approx}{\to} G'_3$ and $G'_3 \overset{a}{\rightarrowtail} G_3$;*

*2. If $G_2 \overset{\approx}{\to} G_3, G'_3 \overset{a}{\rightarrowtail} G_3$ and the point-axiom is valid then there is a $G'_2$ such that $G'_2 \overset{a}{\rightarrowtail} G_2$ and $G'_2 \overset{\approx}{\to} G'_3$.*

**Proof:**

1. Suppose $f$ is a $a$-transition from $G'_1$ to $G_1$ and $g$ is a $L$-covering from $G_1$ to $G_3$. Then the relation $A := g^{\smile}; f^{\smile}; f; g$ is partial identity on $Z_3$. Suppose that $h$ splits $A$. Then $h$ is a injective mapping and we define $G'_3 := h; G_3; (\mathbb{I}_L \times h^{\smile})$ and $w'_3 := w'_1; f; g; h^{\smile}$. From

$$\begin{aligned} w'_3; w'^{\smile}_3 &= w_1; f; g; h^{\smile}; h; g^{\smile}; f^{\smile}; w_1^{\smile} \\ &= w_1; f; g; A; g^{\smile}; f^{\smile}; w_1^{\smile} \\ &= w_1; f; g; g^{\smile}; f^{\smile}; f; g; g^{\smile}; f^{\smile}; w_1^{\smile} \\ &\sqsupseteq w_1; f; f^{\smile}; f; f^{\smile}; w_1^{\smile} \\ &= w_1; w_1^{\smile} \end{aligned}$$

$$
\begin{array}{rcl}
 & \sqsupseteq & \mathbb{I}_I, \\
w_3'^{\smile}; w_3' & = & h; g^{\smile}; f^{\smile}; w_1^{\smile}; w_1; f; g; h^{\smile} \\
 & \sqsubseteq & h; h^{\smile} \\
 & = & \mathbb{I}
\end{array}
$$

we conclude that $w_3'$ is a point and hence $G_3'$ a $L$-graph. Next, we want to show that $h; G_3 = h; G_3; (\mathbb{I}_L \times A)$ holds. First we have

$$f; g; A = f; g; g^{\smile}; f^{\smile}; f; g \sqsupseteq f; f^{\smile}; f; g = f; g,$$

and $f; g; A \sqsubseteq f; g$. Furthermore, we get

$$
\begin{array}{rcl}
h^{\smile}; h; G_3 & = & A; G_3 \\
 & = & g^{\smile}; f^{\smile}; f; g; G_3 \\
 & = & g^{\smile}; f^{\smile}; f; G_1; (\mathbb{I}_L \times g) \\
 & = & g^{\smile}; f^{\smile}; G_1'; (\mathbb{I}_L \times f; g)
\end{array}
$$

which gives us $h; G_3 = h; h^{\smile}; h; G_3 = h; g^{\smile}; f^{\smile}; G_1'; (\mathbb{I}_L \times f; g)$. Together we aim

$$
\begin{array}{rcl}
h; G_3; (\mathbb{I}_L \times A) & = & h; g^{\smile}; f^{\smile}; G_1'; (\mathbb{I}_L \times f; g; A) \\
 & = & h; g^{\smile}; f^{\smile}; G_1'; (\mathbb{I}_L \times f; g) \\
 & = & h; G_3.
\end{array}
$$

Now, the computation

$$
\begin{array}{rcl}
G_3'; (\mathbb{I}_L \times h) & = & h; G_3; (\mathbb{I}_L \times h^{\smile}); (\mathbb{I}_L \times h) \\
 & = & h; G_3; (\mathbb{I}_L \times h^{\smile}; h) \\
 & = & h; G_3; (\mathbb{I}_L \times A) \\
 & = & h; G_3, \\
w_3'; h & = & w_1'; f; g; h^{\smile}; h \\
 & = & w_1'; f; g; A \\
 & \sqsubseteq & w_1'; f; g \\
 & \sqsubseteq & w_1; G_1; (a^{\smile}; a \times \mathbb{I}); \rho; g \\
 & = & w_1; G_1; (a^{\smile}; a \times \mathbb{I}); (\mathbb{I} \times g); \rho \\
 & = & w_1; G_1; (a^{\smile}; a \times g); \rho \\
 & = & w_1; G_1; (\mathbb{I} \times g); (a^{\smile}; a \times \mathbb{I}); \rho \\
 & = & w_1; g; G_3; (a^{\smile}; a \times \mathbb{I}); \rho \\
 & = & w_3; G_3; (a^{\smile}; a \times \mathbb{I}); \rho
\end{array}
$$

shows that $h$ is a $a$-transition.

To see that $G_1' \overset{\approx}{\to} G_3'$ define $k := f; g; h^{\smile}$. By the definition of $w_3'$ the equation $w_1'; k = w_1'; f; g; h^{\smile} = w_3'$ holds. The computations

$$k^{\smile}; k \quad = \quad h; g^{\smile}; f^{\smile}; f; g; h^{\smile}$$

$$
\begin{aligned}
&= h; A; h^{\smile} \\
&= h; h^{\smile}; h; h^{\smile} \\
&= \mathbb{I}, \\
k; k^{\smile} &= f; g; h^{\smile}; h; g^{\smile}; f^{\smile} \\
&= f; g; g^{\smile}; f^{\smile}; f; g; g^{\smile}; f^{\smile} \\
&\sqsupseteq f; f^{\smile}; f; f^{\smile} \\
&= \mathbb{I}, \\
k; G_3' &= f; g; h^{\smile}; h; G_3; (\mathbb{I}_L \times h^{\smile}) \\
&= f; g; A; G_3; (\mathbb{I}_L \times h^{\smile}) \\
&= f; g; G_3; (\mathbb{I}_L \times h^{\smile}) \\
&= f; G_1; (\mathbb{I}_L \times g; h^{\smile}) \\
&= G_1'; (\mathbb{I}_L \times f; g; h^{\smile}) \\
&= G_1'; (\mathbb{I}_L \times k)
\end{aligned}
$$

gives us the assertation.

2. Suppose $f$ is a $L$-covering from $G_2$ to $G_3$, and $g$ is a $a$-transition from $G_3'$ to $G_3$. Then the relation $A := f; g^{\smile}; g; f^{\smile} \sqcap \mathbb{I}_{Z_2}$ is a partial identity on $Z_2$. Suppose $h$ splits $A$ and define

   $G_2' := h; G_2; (\mathbb{I}_L \times h^{\smile})$,
   $w_2'$ as a point contained in $w_3'; f; g^{\smile}$,
   $k := h; f; g^{\smile}$.

   Here the point axiom is used to obtain the root $w_2'$. In general, the relation $w_3'; f; g^{\smile}$ is not univalent. Simular to point 1. it is shown that $h$ is a $a$-transition from $G_2'$ to $G_2$ and $k$ is a $L$-covering from $G_2'$ to $G_3'$. □

# 5 $\mathcal{G}$ as a Model of Processes

Now, we want to define functors corresponding to the standard operations of CCS. Furthermore, we show that every transition rule associated with each operation is fulfilled.

**Definition 5.1** *Suppose $\mathcal{R}$ has relational sums, and let $a : \mathrm{I} \longrightarrow L$ be a point. The prefixing functor $P_a : \mathcal{G} \longrightarrow \mathcal{G}$ is defined by*

1. $P_a(G) := [<a, w>, G]; (\mathbb{I}_L \times \kappa) : \mathrm{I} + Z \leftrightarrow L \times (\mathrm{I} + Z)$,
2. $P_a(w) := \iota : \mathrm{I} \longrightarrow \mathrm{I} + Z$,
3. $P_a(f) := \mathbb{I}_{\mathrm{I}} + f$,

*for* $G : Z \leftrightarrow L \times Z, w : \mathrm{I} \longrightarrow Z$ *and homomorphism* $f$.

The transition rule associated with the prefixing is just the simple axiom

$$\frac{\text{---}}{a.P \stackrel{a}{\longrightarrow} P}.$$

$P_a(G)$ is, by definition, the graph containing a new root and exactly one transition from this new root to old one via the label $a$.

**Lemma 5.2** *$P_a$ is a functor such that $G \stackrel{a}{\rightarrowtail} P_a(G)$.*

**Proof:** Let $f$ be a homomorphism from $G_1 : Z_1 \leftrightarrow L \times Z_1$ to $G_2 : Z_2 \leftrightarrow L \times Z_2$, and $(\mathrm{I} + Z_1, \iota_1, \kappa_1)$ respectively $(\mathrm{I} + Z_2, \iota_2, \kappa_2)$ be relational sums. Then we have

$$P_a(w_1); P_a(f) = \iota_1; (\mathbb{I}_\mathrm{I} + f) = \iota_2 = P_a(w_2),$$

and from

$$\begin{aligned}
P_a(G_1); (\mathbb{I}_L \times P_a(f)) &= [< a, w_1 >, G_1]; (\mathbb{I}_L \times \kappa_1); (\mathbb{I}_L \times P_a(f)) \\
&= [< a, w_1 >, G_1]; (\mathbb{I}_L \times \kappa_1; (\mathbb{I}_\mathrm{I} + f)) \\
&= [< a, w_1 >, G_1]; (\mathbb{I}_L \times f; \kappa_2) \\
&= [< a, w_1 >, G_1]; (\mathbb{I}_L \times f); (\mathbb{I}_L \times \kappa_2) \\
&= [< a, w_1 >; (\mathbb{I}_L \times f), G_1; (\mathbb{I}_L \times f)]; (\mathbb{I}_L \times \kappa_2) \\
&= [< a, w_1; f >, G_1; (\mathbb{I}_L \times f)]; (\mathbb{I}_L \times \kappa_2) \\
&= [< a, w_2 >, G_1; (\mathbb{I}_L \times f)]; (\mathbb{I}_L \times \kappa_2) \\
&\sqsubseteq [< a, w_2 >, f; G_2]; (\mathbb{I}_L \times \kappa_2) \\
&= (\mathbb{I}_\mathrm{I} + f); [< a, w_2 >, G_2]; (\mathbb{I}_L \times \kappa_2) \\
&= P_a(f); P_a(G_2)
\end{aligned}$$

we conclude that $P_a(f)$ is a homomorphism between $P_a(G_1)$ and $P_a(G_2)$. The other properties of a functor are easely verified and therefore omitted.

Obviously, $\kappa : Z \longrightarrow \mathrm{I} + Z$ is injective. Furthermore, we have

$$\kappa; P_a(G) = \kappa; [< a, w >, G]; (\mathbb{I}_L \times \kappa) = G; (\mathbb{I}_L \times \kappa)$$

and

$$\begin{aligned}
P_a(w); P_a(G); (a^\smile; a \times \mathbb{I}); \rho &= \iota; [< a, w >, G]; (\mathbb{I} \times \kappa); (a^\smile; a \times \mathbb{I}); \rho \\
&= < a, w >; (\mathbb{I} \times \kappa); (a^\smile; a \times \mathbb{I}); \rho \\
&= < a, w >; (a^\smile; a \times \kappa); \rho \\
&= < a; a^\smile; a, w; \kappa >; \rho \\
&= < a, w; \kappa >; \rho \\
&= w; \kappa.
\end{aligned}$$

Hence, $\kappa$ is the required transition morphism. □

Relabelling and hiding are simular operations, i.e., hiding is a relabelling with a partial identity. This implies that both operations can be described by a common class of functors.

**Definition 5.3** *Let $l : L \leftrightarrow L$ be univalent. Then the functor $F_l : \mathcal{G} \longrightarrow \mathcal{G}$ is defined by*

1. $F_l(G) := G; (l \times \mathbb{I}_Z)$,
2. $F_l(w) := w$,
3. $F_l(f) := f$,

*for $G : Z \leftrightarrow L \times Z, w : \mathrm{I} \longrightarrow Z$ and homomorphism $f$.*

Given a relabelling function $f$ and a set $\mathcal{L}$ of labels, the corresponding transition rules are

$$\frac{P \xrightarrow{a} P'}{P[f] \xrightarrow{f(a)} P'[f]} \qquad\qquad \frac{P \xrightarrow{a} P'}{P\backslash\mathcal{L} \xrightarrow{a} P'\backslash\mathcal{L}}\ a \notin \mathcal{L}.$$

The second rule indicates that hiding has to be interpreted by $F_{\bar{l}\sqcap\mathbb{I}}$ where $l$ is the partial identity induced by $\mathcal{L}$.

**Lemma 5.4** *$F_l$ is a functor such that*

1. *if $l$ is total then $G_1 \overset{a}{\rightarrowtail} G_2$ implies $F_l(G_1) \overset{a;l}{\rightarrowtail} F_l(G_2)$,*
2. *if $l \sqsubseteq \mathbb{I}_L$ then $G_1 \overset{a}{\rightarrowtail} G_2$ and $a^{\smile}; a \sqcap l = \perp\!\!\!\perp$ implies $F_{\bar{l}\sqcap\mathbb{I}}(G_1) \overset{a}{\rightarrowtail} F_{\bar{l}\sqcap\mathbb{I}}(G_2)$.*

**Proof:** Let $f$ be a homomorphism from $G_1 : Z_1 \leftrightarrow L \times Z_1$ to $G_2 : Z_2 \leftrightarrow L \times Z_2$. Then we have

$$F_l(w_1); f = w_1; f = w_2 = F_l(w_2)$$

and from

$$\begin{aligned} F_l(G_1); (\mathbb{I}_L \times F_l(f)) &= F_l(G_1); (\mathbb{I}_L \times f) \\ &= G_1; (l \times \mathbb{I}_{Z_1}); (\mathbb{I}_L \times f) \\ &= G_1; (l \times f) \\ &= G_1; (\mathbb{I}_L \times f); (l \times \mathbb{I}_{Z_2}) \\ &\sqsubseteq f; G_2; (l \times \mathbb{I}_{Z_2}) \\ &= F_l(f); F_l(G_2) \end{aligned}$$

we conclude that $F_l(f)$ is a homomorphism between $F_l(G_1)$ and $F_l(G_2)$. The other properties of a functor are trivial.

1. Suppose $l$ is total and $g : G_1 \overset{a}{\rightarrowtail} G_2$. Then we have

$$\begin{aligned} g; F_l(G_2) &= g; G_2; (l \times \mathbb{I}_{Z_2}) \\ &= G_1; (\mathbb{I}_L \times g); (l \times \mathbb{I}_{Z_2}) \\ &= G_1; (l \times g) \\ &= G_1; (l \times \mathbb{I}_{Z_1}); (\mathbb{I}_L \times g) \\ &= F_l(G_1); g. \end{aligned}$$

From the computation

$$\begin{aligned} F_l(w_1); g &= w_1; g \\ &\sqsubseteq w_2; G_2; (a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &\sqsubseteq w_2; G_2; (l \times \mathbb{I}_{Z_2}); (l^\smile \times \mathbb{I}_{Z_2}); (a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &= F_l(w_2); F_l(G_2); (l^\smile; a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &= F_l(w_2); F_l(G_2); (\pi; (a; l)^\smile; a; \pi^\smile \sqcap \rho; \rho^\smile); \rho \\ &= F_l(w_2); F_l(G_2); (\pi; (a; l)^\smile; a; \mathbb{T} \sqcap \rho) \\ &= F_l(w_2); F_l(G_2); (\pi; (a; l)^\smile; a; l; \mathbb{T} \sqcap \rho) \\ &= F_l(w_2); F_l(G_2); (\pi; (a; l)^\smile; a; l; \pi^\smile \sqcap \rho; \rho^\smile); \rho \\ &= F_l(w_2); F_l(G_2); ((a; l)^\smile; a; l \times \mathbb{I}_{Z_2}); \rho \end{aligned}$$

we conclude $g : F_l(G_1) \overset{a;l}{\rightarrowtail} F_l(G_2)$.

2. Suppose $l \sqsubseteq \mathbb{I}_L, g : G_1 \overset{a}{\rightarrowtail} G_2$ and $a^\smile; a \sqcap l = \perp\!\!\!\perp$. Analogously to 1., we have $g; F_{\bar{l} \sqcap \mathbb{I}}(G_2) = F_{\bar{l} \sqcap \mathbb{I}}(G_1); g$. Using

$$\begin{aligned} a^\smile; a \sqcap l = \perp\!\!\!\perp &\Leftrightarrow a^\smile; a \sqcap l \sqsubseteq \perp\!\!\!\perp \\ &\Leftrightarrow a^\smile; a \sqsubseteq \perp\!\!\!\perp \sqcup \bar{l} = \bar{l} \\ &\Rightarrow a^\smile; a \sqsubseteq \bar{l} \sqcap \mathbb{I}_L. \end{aligned}$$

and the inclusion[4] $a \sqsubseteq a; a^\smile; a$ we conclude from

$$\begin{aligned} F_{\bar{l} \sqcap \mathbb{I}}(w_1); g &= w_1; g \\ &\sqsubseteq w_2; G_2; (a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &\sqsubseteq w_2; G_2; (a^\smile; a; a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &= w_2; G_2; (a^\smile; a \times \mathbb{I}_{Z_2}); (a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &\sqsubseteq w_2; G_2; ((\bar{l} \sqcap \mathbb{I}_L) \times \mathbb{I}_{Z_2}); (a^\smile; a \times \mathbb{I}_{Z_2}); \rho \\ &= F_{\bar{l} \sqcap \mathbb{I}}(w_2); F_{\bar{l} \sqcap \mathbb{I}}(G_2); (a^\smile; a \times \mathbb{I}_{Z_2}); \rho, \end{aligned}$$

that $g : F_{\bar{l} \sqcap \mathbb{I}}(G_1) \overset{a}{\rightarrowtail} F_{\bar{l} \sqcap \mathbb{I}}(G_2)$. □

[4]This formula is valid for all relations. A proof can be found [2, 5, 7, 9]

The sum operation of CCS is defined for arbitary sets $I$ of processes. In our framework a set of objects (resp. morphisms) of $\mathcal{G}$ is represented by a function from a set $I$ to the objects (resp. morphisms) of $\mathcal{G}$. The obvious definition of composition and the identities turns this structure into a category $\mathcal{G}^I$. We call this category the $I$'s power of $\mathcal{G}$. For simplicity we use a tuple-like notation $(\ldots, G_i, \ldots)$.

**Definition 5.5** *Suppose $\mathcal{R}$ has relational sums, and let $I$ be a set. The functor $S_I : \mathcal{G}^I \longrightarrow \mathcal{G}$ is defined by*

1. $S_I(\ldots, G_i, \ldots) := R; (\sum_{i \in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile})$,
2. $S_I(\ldots, w_i, \ldots) := (R; \bigvee_{i \in I} \mathrm{ext}(w_i)^{\smile})^{\smile}$,
3. $S_I(\ldots, f_i, \ldots) := R; (\sum_{i \in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}$,

*where $R : C \leftrightarrow \sum_{i \in I} \mathrm{I} + Z_i$ resp. $R' : C' \leftrightarrow \sum_{i \in I} \mathrm{I} + Z'_i$ splits the equivalence relation*

$$\begin{aligned} S &:= (\bigsqcup_{i,j \in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \mathrm{ext}(w_j); \iota_j) \sqcup \mathbb{I}_{\sum_{i \in I} \mathrm{I}+Z_i} \\ \textit{resp.} \quad S' &:= (\bigsqcup_{i,j \in I} \iota_i^{\smile}; \mathrm{ext}(w'_i)^{\smile}; \mathrm{ext}(w'_j); \iota_j) \sqcup \mathbb{I}_{\sum_{i \in I} \mathrm{I}+Z'_i}. \end{aligned}$$

The sum of a set of graphs is, roughly, given by the disjoint union of the graphs and identifying all roots. Associated with the sum operation is the following transition rule

$$\frac{P_i \stackrel{a}{\longrightarrow} P'_i}{\sum_{i \in I} P_i \stackrel{a}{\longrightarrow} P'_i}.$$

**Lemma 5.6** *$S_I$ is a functor such that $G'_i \stackrel{a}{\rightarrowtail} G_i$ implies $G'_i \stackrel{a}{\rightarrowtail} S_I(\ldots, G_i, \ldots)$.*

**Proof:** Notice, that $R$ is total and surjective, and hence $S_I(\ldots, f_i, \ldots)$ is total. Furthermore, we have[5]

$$\begin{aligned} & S; (\sum_{i \in I} \mathbb{I}_{\mathrm{I}} + f_i) \\ = \; & ((\bigsqcup_{i,j \in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \mathrm{ext}(w_j); \iota_j) \sqcup \mathbb{I}_{\sum_{i \in I} \mathrm{I}+Z_i}); (\sum_{i \in I} \mathbb{I}_{\mathrm{I}} + f_i) \\ = \; & (\bigsqcup_{i,j \in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \mathrm{ext}(w_j); \iota_j); (\sum_{i \in I} \mathbb{I}_{\mathrm{I}} + f_i) \sqcup \sum_{i \in I} \mathbb{I}_{\mathrm{I}} + f_i \end{aligned}$$

[5]Notice, that different occurrences of $\iota$ may point to different injections, i.e., first injections to relational sums with different second component.

$$
\begin{aligned}
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \mathrm{ext}(w_j); (\mathbb{I}_{\mathrm{I}} + f_j); \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \iota; (\mathbb{I}_{\mathrm{I}} + f_j); \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \iota; \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; \mathrm{ext}(w_i)^{\smile}; \mathrm{ext}(w'_j); \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; \iota^{\smile}; \mathrm{ext}(w'_j); \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; (\mathbb{I}_{\mathrm{I}} + f_i); \iota^{\smile}; \mathrm{ext}(w'_j); \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= \bigsqcup_{i,j\in I} \iota_i^{\smile}; (\mathbb{I}_{\mathrm{I}} + f_i); \mathrm{ext}(w'_i)^{\smile}; \mathrm{ext}(w'_j); \iota_j \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); (\bigsqcup_{i,j\in I} \iota_i^{\smile}; \mathrm{ext}(w'_i)^{\smile}; \mathrm{ext}(w'_j); \iota_j) \sqcup \sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i \\
&= (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); S'.
\end{aligned}
$$

We refer to the last property by $(*)$. Using $(*)$ we conclude

$$
\begin{aligned}
&S_I(\ldots, f_i, \ldots)^{\smile}; S_I(\ldots, f_i, \ldots) \\
&\quad = R'; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i)^{\smile}; R^{\smile}; R; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile} \\
&\quad = R'; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i^{\smile}); S; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile} \\
&\quad = R'; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i^{\smile}); (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); S'; R'^{\smile} \\
&\quad \sqsubseteq R'; S'; R'^{\smile} \\
&\quad = R'; R'^{\smile}; R'; R'^{\smile} \\
&\quad = \mathbb{I}_{C'}
\end{aligned}
$$

that $S_I(\ldots, f_i, \ldots)$ is a mapping. Furthermore, we have

$$
\begin{aligned}
&(\sum_{i\in I} \mathrm{ext}(G_i)); (\sum_{i\in I} \mathbb{I}_L \times (\mathbb{I}_{\mathrm{I}} + f_i)) \\
&\quad = \sum_{i\in I} \mathrm{ext}(G_i)); (\mathbb{I}_L \times (\mathbb{I}_{\mathrm{I}} + f_i)) \\
&\quad = \sum_{i\in I} [w_i; G_i, G_i]; (\mathbb{I}_L \times \kappa); (\mathbb{I}_L \times (\mathbb{I}_{\mathrm{I}} + f_i)) \\
&\quad = \sum_{i\in I} [w_i; G_i, G_i]; (\mathbb{I}_L \times \kappa; (\mathbb{I}_{\mathrm{I}} + f_i))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i\in I}[w_i; G_i, G_i]; (\mathbb{I}_L \times f_i; \kappa) \\
&= \sum_{i\in I}[w_i; G_i, G_i]; (\mathbb{I}_L \times f_i); (\mathbb{I}_L \times \kappa) \\
&= \sum_{i\in I}[w_i; G_i; (\mathbb{I}_L \times f_i), G_i; (\mathbb{I}_L \times f_i)]; (\mathbb{I}_L \times \kappa) \\
&\sqsubseteq \sum_{i\in I}[w_i; f_i; G'_i, f_i; G'_i]; (\mathbb{I}_L \times \kappa) \\
&= \sum_{i\in I}[w'_i; G'_i, f_i; G'_i]; (\mathbb{I}_L \times \kappa) \\
&= \sum_{i\in I}(\mathbb{I}_{\mathrm{I}} + f_i); [w'_i; G'_i, G'_i]; (\mathbb{I}_L \times \kappa) \\
&= \sum_{i\in I}(\mathbb{I}_{\mathrm{I}} + f_i); \mathrm{ext}(G'_i) \\
&= (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); (\sum_{i\in I} \mathrm{ext}(G'_i)).
\end{aligned}
$$

Using the last property and $(*)$ we get

$$
\begin{aligned}
&S_I(\ldots, G_i, \ldots); (\mathbb{I}_L \times S_I(\ldots, f_i, \ldots)) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (\mathbb{I}_L \times R; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}; R; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times S; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); S'; R'^{\smile}) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}; R'; R'^{\smile}) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}) \\
&\quad= R; (\sum_{i\in I} \mathrm{ext}(G_i)); (\sum_{i\in I} \mathbb{I}_L \times (\mathbb{I}_{\mathrm{I}} + f_i)); \mathrm{distr}; (\mathbb{I}_L \times R'^{\smile}) \\
&\quad\sqsubseteq R; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); (\sum_{i\in I} \mathrm{ext}(G'_i)); \mathrm{distr}; (\mathbb{I}_L \times R'^{\smile}) \\
&\quad\sqsubseteq R; (\sum_{i\in I} \mathbb{I}_{\mathrm{I}} + f_i); R'^{\smile}; R'; (\sum_{i\in I} \mathrm{ext}(G'_i)); \mathrm{distr}; (\mathbb{I}_L \times R'^{\smile}) \\
&\quad= S_I(\ldots, f_i, \ldots); S_I(\ldots, G'_i, \ldots).
\end{aligned}
$$

The other properties of a functor are trivial. Suppose $h : G'_i \overset{a}{\rightarrowtail} G_i$. Then there is $g : G'_i \overset{a}{\rightarrowtail} \mathrm{ext}(G_i)$ such that $g; \mathrm{ext}(w_i)^\smile = \perp\!\!\!\perp$. The computations

$$\begin{aligned}
(g;\iota_i;R^\smile)^\smile; g;\iota_i;R^\smile &= R;\iota_i^\smile;g^\smile;g;\iota_i;R^\smile \\
&\sqsubseteq R;\iota_i^\smile;\iota_i;R^\smile \\
&\sqsubseteq R;R^\smile \\
&= \mathbb{I}_C, \\
g;\iota_i;R^\smile;(g;\iota_i;R^\smile)^\smile &= g;\iota_i;R^\smile;R;\iota_i^\smile;g^\smile \\
&= g;\iota_i;S;\iota_i^\smile;g^\smile \\
&\sqsupseteq g;\iota_i;\iota_i^\smile;g^\smile \\
&= g;g^\smile \\
&\sqsupseteq \mathbb{I}_{Z'_i}
\end{aligned}$$

show that $g;\iota_i;R^\smile$ is a mapping. Using the fact that $g;\mathrm{ext}(w_i)^\smile = \perp\!\!\!\perp$ we have

$$\begin{aligned}
&g;\iota_i;R^\smile;S_I(\ldots,G_i,\ldots) \\
&= g;\iota_i;R^\smile;R;(\sum_{i\in I}\mathrm{ext}(G_i));\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= g;\iota_i;S;(\sum_{i\in I}\mathrm{ext}(G_i));\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= g;\iota_i;((\bigsqcup_{i,j\in I}\iota_i^\smile;\mathrm{ext}(w_i)^\smile;\mathrm{ext}(w_j);\iota_j)\sqcup\mathbb{I}_{\sum_{i\in I}Z_i});(\sum_{i\in I}\mathrm{ext}(G_i));\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= g;((\bigsqcup_{j\in I}\mathrm{ext}(w_i)^\smile;\mathrm{ext}(w_j);\iota_j)\sqcup\iota_i);(\sum_{i\in I}\mathrm{ext}(G_i));\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= ((\bigsqcup_{j\in I}g;\mathrm{ext}(w_i)^\smile;\mathrm{ext}(w_j);\iota_j)\sqcup g;\iota_i);(\sum_{i\in I}\mathrm{ext}(G_i));\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= g;\iota_i;(\sum_{i\in I}\mathrm{ext}(G_i));\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= g;\mathrm{ext}(G_i);\iota_i;\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= G'_i;(\mathbb{I}_L\times g);\iota_i;\mathrm{distr};(\mathbb{I}_L\times R^\smile) \\
&= G'_i;(\mathbb{I}_L\times g);(\mathbb{I}_L\times\iota_i);(\mathbb{I}_L\times R^\smile) \\
&= G'_i;(\mathbb{I}_L\times g;\iota_i;R^\smile).
\end{aligned}$$

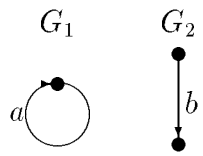
By computing

$$\begin{aligned}
w'_i;g;\iota_i;R^\smile &\sqsubseteq \mathrm{ext}(w_i);\mathrm{ext}(G_i);(a^\smile;a\times\mathbb{I}_{Z_i});\rho;\iota_i;R^\smile \\
&= \mathrm{ext}(w_i);\mathrm{ext}(G_i);(a^\smile;a\times\mathbb{I}_{Z_i});(\mathbb{I}_L\times\iota_i;R^\smile);\rho \\
&= \mathrm{ext}(w_i);\mathrm{ext}(G_i);(a^\smile;a\times\iota_i;R^\smile);\rho \\
&= \mathrm{ext}(w_i);\mathrm{ext}(G_i);(\mathbb{I}_L\times\iota_i);(a^\smile;a\times R^\smile);\rho \\
&= \mathrm{ext}(w_i);\mathrm{ext}(G_i);\iota_i;\mathrm{distr};(a^\smile;a\times R^\smile);\rho \\
&= \mathrm{ext}(w_i);\mathrm{ext}(G_i);\iota_i;\mathrm{distr};(\mathbb{I}_L\times R^\smile);(a^\smile;a\times\mathbb{I}_{\sum_{i\in I}Z_i});\rho
\end{aligned}$$

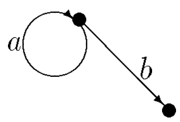
$$
\begin{aligned}
&\sqsubseteq (\bigsqcup_{i\in I} \mathrm{ext}(w_i); \mathrm{ext}(G_i); \iota_i); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho \\
&= (\bigvee_{i\in I} \mathrm{ext}(G_i)^{\smile}; \mathrm{ext}(w_i)^{\smile})^{\smile}; \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho \\
&= ((\sum_{i\in I} \mathrm{ext}(G_i)^{\smile}); (\bigvee_{i\in I} \mathrm{ext}(w_i)^{\smile}))^{\smile}; \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho \\
&= (\bigvee_{i\in I} \mathrm{ext}(w_i)^{\smile})^{\smile}; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho \\
&\sqsubseteq (\bigvee_{i\in I} \mathrm{ext}(w_i)^{\smile})^{\smile}; S; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho \\
&= (R; \bigvee_{i\in I} \mathrm{ext}(w_i)^{\smile})^{\smile}; R; (\sum_{i\in I} \mathrm{ext}(G_i)); \mathrm{distr}; (\mathbb{I}_L \times R^{\smile}); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho \\
&= S_I(\ldots, w_i, \ldots); S_I(\ldots, G_i, \ldots); (a^{\smile}; a \times \mathbb{I}_{\sum_{i\in I} Z_i}); \rho
\end{aligned}
$$

we conclude that $g; \iota_i; R^{\smile}$ is a transition morphism from $G_i'$ to $S(\ldots, G_i, \ldots)$. □

Notice, that using just $G_i$ instead of $\mathrm{ext}(G_i)$ does not give an adequate definition of the sum operation. Consider the processes $P_1 = a.P_1$ and $P_2 = b.0$. They are modeled by the graphs:




Not using the extension of $G_1$ and $G_2$ we would gain the following graph.




This graph seen as a process may produce the stream $ab$, which is not in the behaviour of $P_1 + P_2$.

Last but not least we define the interleaving functor.

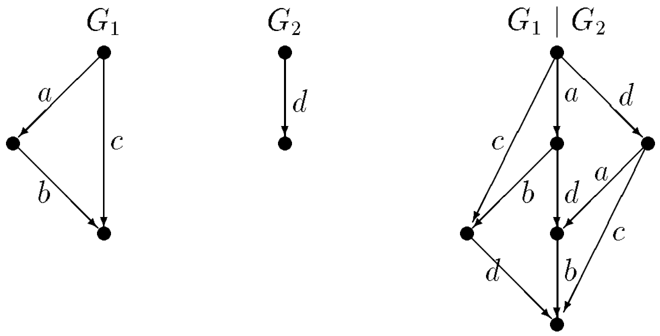
**Definition 5.7** *Suppose $\mathcal{R}$ has relational products. Then the interleaving functor $| : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ is defined by*

1. $G_1 \mid G_2 := (\pi; G_1; (\mathbb{I}_L \times \pi^{\smile}) \sqcap \rho; \rho^{\smile}; \rho^{\smile}) \sqcup (\pi; \pi^{\smile}; \rho^{\smile} \sqcap \rho; G_2; (\mathbb{I}_L \times \rho^{\smile}))$ *a relation in* $\mathcal{R}[Z_1 \times Z_2, L \times (Z_1 \times Z_2)]$,
2. $w_1 \mid w_2 := < w_1, w_2 >$,

*3.* $f_1 \mid f_2 := f_1 \times f_2$,

*for* $G_1 : Z_1 \leftrightarrow L \times Z_1, G_2 : Z_2 \leftrightarrow L \times Z_2$ *and a homomorphism* $f$.

By definition the state space of $G_1 \mid G_2$ is the product $Z_1 \times Z_2$. The next picture shows an example of the interleaving of two graphs.




The transition rule associated with interleaving is

$$\frac{P \xrightarrow{a} P'}{P \mid Q \xrightarrow{a} P' \mid Q} \qquad \frac{Q \xrightarrow{a} Q'}{P \mid Q \xrightarrow{a} P \mid Q'}.$$

The commutativity of relational products implies that our definition of the interleaving lead to a commutative functor. Therefore, it is sufficient to consider the first rule.

**Lemma 5.8** $\mid$ *is a functor such that* $G_1 \overset{a}{\rightarrowtail} G_2$ *implies* $G_1 \mid G_3 \overset{a}{\rightarrowtail} G_2 \mid G_3$.

**Proof:** We only show the second assertion. Suppose $f : G_1 \overset{a}{\rightarrowtail} G_2$. Then we have

$$\begin{aligned}
&(f \times \mathbb{I}_{Z_3}); (G_2 \mid G_3)\\
&= (f \times \mathbb{I}_{Z_3}); ((\pi; G_2; (\mathbb{I}_L \times \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}) \sqcup (\pi; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile})))\\
&= (f \times \mathbb{I}_{Z_3}); (\pi; G_2; (\mathbb{I}_L \times \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}) \sqcup (f \times \mathbb{I}_{Z_3}); (\pi; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}))\\
&= (\pi; f; G_2; (\mathbb{I}_L \times \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}) \sqcup (\pi; f; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}))\\
&= (\pi; G_1; (\mathbb{I}_L \times f); (\mathbb{I}_L \times \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}) \sqcup (\pi; f; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}))\\
&= (\pi; G_1; (\mathbb{I}_L \times f; \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}) \sqcup (\pi; f; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}))\\
&= (\pi; G_1; (\mathbb{I}_L \times \pi^{\smallsmile}; (f \times \mathbb{I}_{Z_3})) \sqcap \rho; \rho^{\smallsmile}; (f \times \mathbb{I}_{Z_3}); \rho^{\smallsmile})\\
&\quad \sqcup (\pi; \pi^{\smallsmile}; (f \times \mathbb{I}_{Z_3}); \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}; (f \times \mathbb{I}_{Z_3})))\\
&= (\pi; G_1; (\mathbb{I}_L \times \pi^{\smallsmile}); (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3})) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}; (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3})))\\
&\quad \sqcup (\pi; \pi^{\smallsmile}; \rho^{\smallsmile}; (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3})) \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}); (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3})))\\
&= ((\pi; G_1; (\mathbb{I}_L \times \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}); (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3}))\\
&\quad \sqcup (\pi; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile})); (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3}))\\
&= ((\pi; G_1; (\mathbb{I}_L \times \pi^{\smallsmile}) \sqcap \rho; \rho^{\smallsmile}; \rho^{\smallsmile}) \sqcup (\pi; \pi^{\smallsmile}; \rho^{\smallsmile} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho^{\smallsmile}))); (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3}))\\
&= (G_1 \mid G_3); (\mathbb{I}_L \times (f \times \mathbb{I}_{Z_3})).
\end{aligned}$$

Since $(a\breve{~}; a \times \mathbb{I}_{Z_2})$ is a partial identity and hence $(a\breve{~}; a \times \mathbb{I}_{Z_2}); \rho$ univalent the computation

$$
\begin{array}{ll}
\multicolumn{2}{l}{(w_1 \mid w_3); (f \times \mathbb{I}_{Z_3})} \\
= & <w_1, w_3>; (f \times \mathbb{I}_{Z_3}) \\
= & <w_1; f, w_3> \\
\sqsubseteq & <w_2; G_2; (a\breve{~}; a \times \mathbb{I}_{Z_2}); \rho, w_3> \\
= & w_2; G_2; (a\breve{~}; a \times \mathbb{I}_{Z_2}); \rho; \pi\breve{~} \sqcap w_3; \rho\breve{~} \\
= & w_2; G_2; (a\breve{~}; a \times \mathbb{I}_{Z_2}); (\mathbb{I}_L \times \pi\breve{~}); \rho \sqcap w_3; \rho\breve{~} \\
= & w_2; G_2; (\mathbb{I}_L \times \pi\breve{~}); (a\breve{~}; a \times \mathbb{I}_{Z_2}); \rho \sqcap w_3; \rho\breve{~} \\
= & (w_2; G_2; (\mathbb{I}_L \times \pi\breve{~}) \sqcap w_3; \rho\breve{~}; \rho\breve{~}; (a\breve{~}; a \times \mathbb{I}_{Z_2})); (a\breve{~}; a \times \mathbb{I}_{Z_2}); \rho \\
\sqsubseteq & (w_2; G_2; (\mathbb{I}_L \times \pi\breve{~}) \sqcap w_3; \rho\breve{~}; \rho\breve{~}); (a\breve{~}; a \times \mathbb{I}_{Z_2}); \rho \\
\sqsubseteq & ((w_2; G_2; (\mathbb{I} \times \pi\breve{~}) \sqcap w_3; \rho\breve{~}; \rho\breve{~}) \sqcup (w_2; \pi\breve{~}; \rho\breve{~} \sqcap w_3; G_3; (\mathbb{I} \times \rho\breve{~}))) \\
 & \quad ; (a\breve{~}; a \times \mathbb{I}_{Z_2 \times Z_3}); \rho \\
= & (<w_2, w_3>; (\pi; G_2; (\mathbb{I} \times \pi\breve{~}) \sqcap \rho; \rho\breve{~}; \rho\breve{~}) \sqcup <w_2, w_3>; (\pi; \pi\breve{~}; \rho\breve{~} \sqcap \rho; G_3; (\mathbb{I} \times \rho\breve{~}))) \\
 & \quad ; (a\breve{~}; a \times \mathbb{I}_{Z_2 \times Z_3}); \rho \\
= & <w_2, w_3>; ((\pi; G_2; (\mathbb{I}_L \times \pi\breve{~}) \sqcap \rho; \rho\breve{~}; \rho\breve{~}) \sqcup (\pi; \pi\breve{~}; \rho\breve{~} \sqcap \rho; G_3; (\mathbb{I}_L \times \rho\breve{~}))) \\
 & \quad ; (a\breve{~}; a \times \mathbb{I}_{Z_2 \times Z_3}); \rho \\
= & (w_2 \mid w_3); (G_2 \mid G_3); (a\breve{~}; a \times \mathbb{I}_{Z_2 \times Z_3}); \rho
\end{array}
$$

shows that $f \mid \mathbb{I}_{Z_3}$ is a $a$-transition. □

We have not introduced a notion of communication. As usual this can be done by splitting $L$ into $L_\mathrm{i} + L_\mathrm{o} + \mathrm{I}$ of input resp. output labels and a distinguished symbol $\tau$.

# 6 Conclusion

We established a categorical model of process calculi based on abstract relation algebra. The main advantage of this approach is the existence of a canonical representation of an equivalance class by its terminal object.

Considering concrete relations, the representative of the interpretation of a process built up by finite summation (but arbitrary recursion) is a finite graph. The equality (up to isomorphism) of finite graphs is decidable and thus implies the decidability of strong bisimulation of such processes. This result corresponds to the theory of finite state machines. But notice, that our graphs are not necessarily finite. As an example consider the graph $S_{\mathbb{N}}(P_0(\emptyset), P_1(\emptyset), P_2(\emptyset)\ldots)$ where $0, 1, 2, \ldots$ are the points induced by the natural numbers.

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