Finite Element Error Analysis for Neumann Boundary Control Problems on Polygonal and Polyhedral Domains

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Abstract

This thesis deals with the numerical solution of optimization problems in function spaces governed by linear elliptic partial differential equations. Many physical processes for instance in thermodynamics, elasticity, fluid mechanics or electrical engineering are modeled by partial differential equations. The aim of optimal control is to regulate occurring parameters or other quantities in such a way that the result of the mathematical model is optimal in a certain sense.

In particular, Neumann boundary control problems are investigated meaning that the flux of the state variable on the boundary of the underlying computational domain can be controlled. Of particular interest are finite element discretizations for these problems on domains having polyhedral shape. Since singularities in a vicinity of corners and edges are expected to be contained in the solution, optimal convergence of the finite element method on quasi-uniform meshes can as a general rule not be guaranteed. For a better description of the occurring singularities weighted Sobolev spaces are used in this thesis, and, exploiting corresponding regularity results finite element error estimates are proved.

Up to now sharp error estimates for the trace of the finite element approximation of the Neumann problem on polyhedral domains were unknown, and the newly developed estimates in this thesis allow an improvement of many convergence results for Neumann boundary control problems. Among others problems with $L^2(\Gamma)$-regularization are considered and improved estimates for the numerical approximation using the full discretization, the postprocessing approach and the variational discretization are derived.

Further, a new energy regularization approach is considered on polygonal domains, where the convergence rate depends in this case solely on the interior angles at the corners of the domain.

The aim of this thesis is always to investigate which convergence rate can be expected on quasi-uniform meshes, and to what extend the best-possible convergence rate can be retained with local mesh refinement, such that methods for the numerical computation of Neumann boundary control problems can be improved significantly.

Zusammenfassung

Diese Dissertation behandelt die numerische Lösung von Optimierungsproblemen in Funktionenräumen, welche durch lineare elliptische partielle Differentialgleichungen beschränkt sind. Viele physikalische Prozesse, zum Beispiel in der Thermodynamik, der Elastizitätstheorie, der Strömungsmechanik oder der Elektrotechnik, werden durch partielle Differentialgleichungen modelliert. Ziel der optimalen Steuerung ist es auftretende Parameter oder andere Einflussgrößen zu kontrollieren, so dass die Lösung des mathematischen Modells in einem bestimmten Sinn optimal wird.

Insbesondere werden Neumann-Randsteuerungsprobleme behandelt, was bedeutet, dass die Änderungsrate der Zustandsgröße auf dem Rand des zugrunde liegenden Gebietes kontrolliert werden kann. Von besonderem Interesse werden Finite-Elemente-Diskretisierungen dieser Probleme auf polyedrisch berandeten Gebieten sein. Da in einer Umgebung von Ecken und Kanten dieser Gebiete Singularitäten in den Lösungen zu erwarten sind, ist eine optimale Konvergenz der Methode der finiten Elemente auf quasi-uniformen Gittern im Allgemeinen nicht sicher gestellt. Um diese Singularitäten genauer zu beschreiben werden in dieser Arbeit gewichtete Sobolevräume verwendet, und unter Ausnutzung entsprechender Regularitätsaussagen Finite-Elemente Fehlerabschätzungen bewiesen.

Bisher unbekannt waren scharfe Abschätzungen für die Spur der Finite-Elemente-Approximation des Neumann-Problems auf polyedrisch berandeten Gebieten und die in dieser Arbeit neu entwickelten Abschätzungen erlaubt es viele Konvergenzresultate für Neumann-Randsteuerungsprobleme zu verbessern. Unter anderem werden Probleme mit $L^2(\Gamma)$-Regularisierung betrachtet und verbesserte Abschätzungen für die numerische Approximation mit einer vollen Diskretisierung, dem Postprocessing-Zugang und der variationellen Diskretisierung hergeleitet.

Ferner wird auch ein neuer Zugang mit einer Energieregularisierung auf polygonal berandeten Gebieten betrachtet, wobei in diesem Fall die Konvergenzrate nur von den Innenwinkeln der Eckpunkte des Gebietes abhängt.

Das Ziel dieser Arbeit ist es stets zu untersuchen, welche Konvergenzrate auf quasi-uniformen Gittern erwartet werden kann, und inwiefern die bestmögliche Konvergenzrate durch lokale Netzverfeinerung wiederhergestellt werden kann, so dass numerische Berechnungsverfahren für Neumann-Randsteuerungsprobleme signifikant verbessert werden können.

Vorwort

Während der Anfertigung dieser Dissertation war ich als wissenschaftlicher Mitarbeiter am *Institut für Mathematik und Bauinformatik* der *Universität der Bundeswehr München* angestellt und war Mitglied im *Internationalen Graduierten- und Doktoratskolleg (IGDK) 1754*, wobei ich der Deutschen Forschungsgemeinschaft (DFG) für die Finanzierung sehr dankbar bin.

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CHAPTER 1

Introduction

The optimization of partial differential equations is a well-examined field in mathematics and the first contributions already appeared in the early 70's, see for instance the fundamental book of Lions [60] and the references therein. In particular during the last 20 years this topic has been studied more intensively as recent developments in computer technology allow an accurate computation of large-scale optimization problems. Optimal control of partial differential equations has a wide range of applications. Many physical phenomena can be modeled by boundary value problems and these models can be optimized regarding some control quantity which may be a parameter, a source term or the shape of the underlying geometry, just to mention a few possibilities.

An optimal control problem is an optimization problem in a function space setting and we denote by $U_{ad} \subset U$ the *set of admissible controls* and by Y the *state space*. The control and state are related to each other by means of a partial differential equation whose solution operator is denoted by $S\colon U_{ad} \to Y$. The kind of problems we are going to investigate in this thesis reads

$$J(y,u) \to \min! \quad \text{subject to} \quad y = Su \quad \text{and} \quad u \in U_{ad}. \tag{1.1}$$

An application we want to emphasize is the optimal control in solid mechanics. There exist a couple of models which describe the relation between forces acted onto and the stress and deformation of a solid figure. The question of interest is, how to adjust the forces in order to achieve a prescribed deformation. This has been intensively studied in [47] for linear elasticity models and in [35, 48] for elastoplastic models. A possible control quantity is the force acted on some part of the boundary. In elasticity this is modeled as a Neumann boundary condition and hence, optimal control problems of this kind are also called *Neumann boundary control problems* that we want to investigate in this thesis. A detailed survey on further applications can be found in the book [52].

In this thesis we consider especially linear-quadratic optimal control problems having the following structure. The target functional consists of a tracking term and a regularization term, namely

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_U, \tag{1.2}$$

where y_d is some given desired state one wants to reach as close as possible, and $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, the computational domain. The second part depending on the regularization parameter $\alpha > 0$ is required to ensure well-posedness of these problems. We exclude the case $\alpha = 0$ because *bang-bang* controls – controls which attain the values of the control bounds only – are obtained. For unconstrained problems with $U_{ad} = U$ the optimization problem is not even well-posed for $\alpha = 0$. In this thesis we investigate the usual case that $U = L^2(\Gamma)$, and the recently developed energy regularization approach where $U = H^{-1/2}(\Gamma)$ for Neumann control problems. Moreover, we restrict our considerations to the case that the *control–to–state mapping* S is the solution operator of the boundary value problem

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \partial_n y = u & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where $\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator. Due to technical restrictions problems involving control constraints are of interest in many applications. Thus, we assume that

$$u \in U_{ad} := \{u \in U\colon\ u_a \le u \le u_b \quad \text{a.e. on } \Gamma\}, \tag{1.4}$$

with certain control bounds $u_a, u_b \in \mathbb{R}$ satisfying $u_a < u_b$. The aim of this thesis is to prove error estimates on locally refined meshes for the numerical approximation of the optimal control problem (1.2)–(1.4). Let us give a brief overview of some closely related contributions. The numerical approximation of distributed control problems has been investigated by Sirch [84] who derived sharp estimates in $L^2(\Omega)$ and $L^\infty(\Omega)$ for polygonal domains as well as for three-dimensional prismatic domains with anisotropic meshes, and by G. Winkler [92] who proved estimates on general polyhedral domains using an isotropic mesh refinement strategy. In another thesis by Pfefferer [74] Neumann boundary control problems on polygonal domains are intensively studied and sharp error estimates with local mesh refinement are proven. The fundamental theory developed in these three dissertations forms the basis for our investigations.

In the first part of this thesis we will investigate the numerical solution of the state equation (1.3) using the *finite element method*. We restrict our considerations to continuous and piecewise linear finite elements on some triangulation $\mathcal{T}_h$ of the underlying domain Ω. We will emphasize the convergence behavior of the discrete solution when Ω has a polygonal or polyhedral boundary Γ. It is well known [44, 45, 54, 69, 87] that singularities at edges and corners occur which could result in a reduced convergence rate. For instance in a vicinity of corners of polygonal domains or edges at polyhedral domains the solution of (1.3) contains terms of the form

$$r^\lambda \cos(\lambda\varphi), \qquad \lambda := \pi/\omega,$$

where (r,φ) and (r,φ,z) are polar coordinates centered in the corner, or cylinderical coordinates centered in the edge, respectively. The singular exponent $\lambda = \lambda^c = \pi/\omega$ for corners in 2D or $\lambda = \lambda^e := \pi/\omega$ for edges in 3D depends solely on the opening angle ω of the corner or edge. At corners of polyhedral domains the corresponding singular exponents λ^c depend upon the solution of an eigenvalue problem for the Laplace-Beltrami operator and can in general not be determined analytically.

The regularity of the variational solution of (1.3) depends then solely on the number

$$\lambda := \begin{cases} \min\limits_{\text{corners } c} \lambda^c, & \text{in 2D,} \\ \min\limits_{\substack{\text{edges } e \\ \text{corners } c}} \{\lambda^e, 1/2 + \lambda^c\}, & \text{in 3D,} \end{cases} \tag{1.5}$$

and one can show [45] that $y \in H^{1+\lambda-\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$, provided that f and g are sufficiently regular. Consequently the finite element approximation y_h satisfies the estimate

$$\|y - y_h\|_{H^1(\Omega)} \le ch^{\min\{1,\lambda-\varepsilon\}} |y|_{H^{\min\{2,1+\lambda-\varepsilon\}}(\Omega)},$$

on quasi-uniform triangulations, see e. g. [23, 85]. Throughout this thesis $c > 0$ is a generic constant which is independent of the mesh size h, the solution y and the input data f and g, and might have another value at each occurrence.

In order to retain the best possible convergence rate many contributions, e. g. [16, 72, 76], investigated the finite element method on meshes which are refined locally in the vicinity of singular edges and corners. Throughout this thesis we will use the assumption that all elements $T \in \mathcal{T}_h$ satisfy

$$h_T := \operatorname{diam}(T) \sim \begin{cases} h^{1/\mu}, & \text{if } r_T = 0, \\ h r_T^{1-\mu}, & \text{if } r_T > 0, \end{cases}$$

where r_T denotes the distance to the singular points, and $\mu \in (0, 1]$ is some refinement parameter which has to be chosen appropriately. More precisely, upper bounds of this parameter are of interest which guarantee optimal convergence. To the best of our knowledge there are only contributions, e. g. [12, 61], that discuss error estimates on a pure isotropically refined mesh for the Dirichlet problem and certain mixed problems when the computational domain is polyhedral. The results in these references is that the estimate

$$\|y - y_h\|_{H^1(\Omega)} + h^{-1}\|y - y_h\|_{L^2(\Omega)} \le ch$$

holds when the refinement condition $\mu < \lambda$ is satisfied. In this thesis also a proof for the pure Neumann problem (1.3) on a general polyhedral domain is given and the same refinement criterion was used.

Having the application of boundary control problems in mind these finite element error estimates in $H^1(\Omega)$ or $L^2(\Omega)$ are not sufficient to show also discretization error estimates for the control problem. As we will see later at a certain point in the convergence proofs we have to insert a finite element error estimate on the boundary, more precisely in $L^2(\Gamma)$. The obvious strategies of using a trace theorem or the Aubin-Nitsche method yield in certain cases lower convergence rates than expected, which is the reason why many contributions about boundary control problems present suboptimal results. In a recent contribution Apel, Pfefferer and Rösch [9] developed an advanced strategy which relies basically on a domain decomposition technique already used to show local maximum norm estimates [80]. Under the assumption $\mu < 1/4 + \lambda/2$ the estimate

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{3/2} \tag{1.6}$$

has been shown for an arbitrary polygonal domain Ω, provided that $f \in C^{0,\sigma}(\overline{\Omega})$ and $g \equiv 0$. The new result in this thesis is an extension of the proof to the three-dimensional case. We will show that the estimate (1.6) remains true with the definition of the number λ from (1.5) in the three-dimensional case. This result has already been published in [10] but in this paper a slightly stronger refinement criterion at singular corners is used.

The second part of this thesis deals with error estimates for the numerical approximation of the optimal control problem (1.2)–(1.4). The choice $U = L^2(\Gamma)$ has already been investigated intensively in the literature. We will solve a discretized version of the optimality system to obtain an approximate solution $(\bar{y}_h, \bar{u}_h)$, and we are interested in the question how fast do these solutions converge towards the solution $(\bar{y}, \bar{u})$ of the continuous problem, i. e. for which numbers $\beta \in \mathbb{R}$, called the convergence rate, does the estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch^\beta$$

hold for all $h > 0$.

There exist a couple of discretization strategies and we will investigate three in detail. For instance the full discretization approach studied by Falk [41] and Gevici [43], who used a piecewise constant approximation of the control variable and a piecewise linear and continuous approximation of the state and adjoint state variable. In the latter reference the convergence rate $\beta = 1$ has been shown for arbitrary convex polygonal domains. This thesis presents an extension of this result to non-convex polygonal and polyhedral domains and we will observe that $\beta = 1$ is attained independent of the geometry and local mesh refinement. This is not the case for the state variable. We also derive estimates of the form

$$\|\bar{y} - \bar{y}_h\|_X \le ch^\beta,$$

in the norm of some certain Banach space X. While in case of $X = H^1(\Omega)$ the discrete state converges with rate $\beta = \min\{1, \lambda - \varepsilon\}$, $\varepsilon > 0$, which is the same rate like for the approximate solution of the boundary value problem, and with rate $\beta = 1$ on a locally refined mesh satisfying $\mu < \lambda$, the convergence rate in case of $X = L^2(\Omega)$ is only $\beta = \min\{2, 1/2 + \lambda\} - \varepsilon$, $\varepsilon > 0$, and $\beta = 2 - \varepsilon$ can be reached with $\mu < 1/4 + \lambda/2$. The same results were proved by Pfefferer [74] for polygonal domains and in the present thesis the results are extended to polyhedral domains.

The full discretization using piecewise constant controls is obviously not the best choice, as we only get the poor convergence rate $\beta = 1$. However, we are restricted due to the approximation order of the discrete control space, but not due to the regularity of the control which is more regular than $H^1(\Gamma)$. Hence, other approaches which exploit higher regularity of $\bar{u}$ are of interest.

One possible approach which has been introduced by Hinze [50] is the *variational discretization*. This technique relies on the fact that the necessary optimality condition yields a pointwise representation of the control in dependence of the adjoint state and it suffices to approximate only the state variables with piecewise linear elements. The control is not discretized explicitly, but can be described exactly by the nodal values of the discrete adjoint state and the control bounds, so it is discretized implicitly. There are many contributions studying error estimates for polygonal domains, but only a few dealing with polyhedral domains. We want to mention Hinze and Matthes [51] who proved $\beta = 3/2$ for convex polygonal and polyhedral domains on quasi-uniform meshes, but this rate is too pessimistic. The same rate was proven by Casas and Mateos [25] for semilinear Neumann boundary control problems on convex polygonal domains only. An extension to non-convex polygonal domains can be found in an article of Mateos and Rösch [62] who proved $\beta = \min\{2, 1 + \lambda/2, 1/2 + \lambda\} - \varepsilon$, $\varepsilon > 0$. The results in all these contributions are suboptimal as there were used suboptimal finite element error estimates in the $L^2(\Gamma)$-norm. We are indeed in the position to improve these results using the estimates proved in this thesis. The investigations made here are for polyhedral domains only and we prove the sharp error bound $\beta = \min\{2, 1/2 + \lambda\} - \varepsilon$ for quasi-uniform meshes and $\beta = 2 - \varepsilon$ for refined meshes according to $\mu < 1/4 + \lambda/2$. Exactly the same rate has already been proven for polygonal domains by Apel, Pfefferer and Rösch [9].

A third approach we are going to address is the *postprocessing* strategy first of all considered by Meyer and Rösch [67] for distributed control problems. The idea is to compute a fully discrete solution of the discrete optimality system using a piecewise constant control approximation again, and to compute another control $\tilde{u}_h$ in a postprocessing step by an evaluation of the pointwise representation of the optimality condition. The function $\tilde{u}_h$ is as in the *variational approach* piecewise linear, but it is not a finite element function. However, one observes that these functions possess better approximation properties. A first proof for Neumann boundary control problems can be found in a paper of Mateos and Rösch [62] who proved $\beta = \min\{2, 1 + \lambda/2, 1/2 + \lambda\} - \varepsilon$ for polygonal domains when a quasi-uniform family of triangulations is used. The improved rate $\beta = \min\{2, 1/2 + \lambda\} - \varepsilon$ which is sharp has been proved by Pfefferer [74], and for locally refined meshes the rate $\beta = 2 - \varepsilon$ was retained in a contribution of Apel, Pfefferer and Rösch [9] when the family of triangulations is refined according to $\mu < 1/4 + \lambda/2$. In the present thesis the same result is proved also for polyhedral domains.

We moreover investigate another regularization approach that was already suggested in Lion's book [60]. His idea was to assume only as much regularity for the control as it is required to obtain the existence of a corresponding state in $H^1(\Omega)$. This means for Neumann boundary control problems that the control is searched in the space $U = H^{-1/2}(\Gamma)$. To the best of our knowledge error estimates for this strategy have not been investigated before and we will discuss sharp estimates for polygonal domains depending on the opening angle at the corner points. Similar investigations can be found in a contribution of Of, Phan and Steinbach [71] where discretization error estimates for the closely related Dirichlet control problems in $H^{1/2}(\Gamma)$ are developed. With this regularization the optimal control $\bar{u}$ is less regular, and exhibits a similar behavior like the solution of a Dirichlet control problem in $L^2(\Gamma)$ [63], meaning that the control is drawn down to zero at convex corners and can tend to infinity at reentrant corners. As a consequence, the solution of the optimization problem possesses less regularity than for $U = L^2(\Gamma)$ and hence, the approximate solutions converge with a lower rate. An approximation with piecewise linear state and adjoint state, and a carefully designed control discretization which has to satisfy the Ladyschenskaya-Babuška-Brezzi stability condition, yields the convergence rate $\beta = \min\{3/2, \lambda\} - \varepsilon$, $\varepsilon > 0$, for the discretization error of the control in the $H^{-1/2}(\Gamma)$-norm and $\beta = \min\{1, \lambda - 1/2\} - \varepsilon$, $\varepsilon > 0$, in the $L^2(\Gamma)$-norm, when there are no control constraints. In particular, we show that the refinement criterion $\mu < 2\lambda/3$, which is necessary for all corners having interior angle larger than $120°$, guarantees the optimal convergence rate $\beta = 3/2 - \varepsilon$, $\varepsilon > 0$, in the $H^{-1/2}(\Gamma)$-norm. We further investigate problems involving control constraints where we may exploit higher regularity of the solution. Due to the behavior at reentrant corners the control can become active in a vicinity of these corners and is hence regular. The error estimates can be improved if this is the case. It should be noted that the results for the energy regularization approach have already been published in [14].

This thesis is structured as follows. In Chapter 2 we investigate regularity results for weak solutions of the boundary value problem (1.3). For sake of completeness we summarize well-known results in classical Sobolev spaces, but more important for us are results in weighted

Sobolev spaces which allow a more accurate description of singular parts contained in the solution. Based on these regularity results we derive finite element error estimates in Chapter 3. Initially, some local estimates for certain projection operators onto piecewise polynomial spaces are investigated. As a consequence, we are in the position to prove finite element error estimates in $H^1(\Omega)$, $L^2(\Omega)$ and $L^2(\Gamma)$ for polyhedral domains, and an estimate in $H^{1/2}(\Gamma)$ for polygonal domains. In this chapter the reader will find all estimates with and without local refinement. The application to Neumann boundary control problems with $L^2(\Gamma)$-regularization is considered in Chapter 4. There, estimates for the three already mentioned discretization strategies are stated and at the end of this chapter some numerical experiments are presented which confirm the theoretically predicted convergence behavior. Optimal control problems with the energy regularization approach are considered in Chapter 5. There, we also apply the new finite element error estimates to certain discretization strategies for the optimization problems and confirm the results in numerical experiments. We will distinguish in this chapter among problems without and with control constraints because better error estimates can be expected when constraints are present.

CHAPTER 2

Singularities in polygonal and polyhedral domains

The purpose of this section is to collect regularity results for the weak solution of the Neumann problem

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma, \end{aligned} \tag{2.1}$$

when $\Omega \in \mathbb{R}^n$, $n \in \{2, 3\}$, is a bounded domain with polygonal or polyhedral boundary Γ.

In Section 2.1 we will introduce several classical function spaces that are frequently used in the context of partial differential equations. Regularity results in classical Sobolev spaces are well-known [33, 44, 45], but we will need a more accurate description of the singular parts which occur in the solution of (2.1) in order to derive sharp finite element error estimates. Hence, we will discuss the asymptotics of weak solutions of the Neumann problem in Section 2.2. Based on these investigations one can think about adopted function spaces which allow a better description of the singularities. In this thesis we will use weighted Sobolev and Hölder spaces that are discussed in Section 2.3, and used for the numerical analysis in Chapter 3. Besides the regularity results in these spaces, also embedding theorems and trace spaces are presented.

2.1 Classical function spaces

In this section we introduce some notation used throughout this thesis, and recall basic knowledge concerning functional analysis that is used frequently later. Moreover, classical regularity results in Sobolev spaces are presented.

2.1.1 Notation and basic properties

First, some basic definitions are introduced, and we use a similar notation as in the introductory books of McLean [66], Steinbach [85] and Grossmann et. al. [46].

Throughout this thesis $n \in \mathbb{N}$ denotes the spatial dimension of the underlying domain, i. e. $\Omega \subset \mathbb{R}^n$, which is assumed to be Lipschitz. Note, that there are also polyhedral domains which are not Lipschitz, but we exclude these special cases for simplification purpose. By $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ we denote a multi-index and by

$$\frac{\partial^{\boldsymbol{\alpha}} u}{\partial x^{\boldsymbol{\alpha}}}(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x)$$

the $\boldsymbol{\alpha}$-th derivative of u.

The classical function spaces that are used to describe strong solutions of (2.1) are the spaces of k-times ($k \in \mathbb{N}_0$) continuously differentiable functions that we denote by $C^k(\overline{\Omega})$, equipped with the standard norm

$$\|u\|_{C^k(\overline{\Omega})} := \max_{|\boldsymbol{\alpha}|\le k} \sup_{x\in\Omega} \left| \frac{\partial^{\boldsymbol{\alpha}}}{\partial x^{\boldsymbol{\alpha}}} u(x) \right| .$$

Closely related are the *Hölder spaces* $C^{k,\gamma}(\overline{\Omega})$, where $\gamma \in (0,1)$ is the so-called Hölder exponent. These spaces contain all functions in $C^k(\overline{\Omega})$ where the Hölder constants

$$[u]_{\gamma,\Omega} := \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}}$$

of all k-th derivatives are finite. A norm of $C^{k,\gamma}(\overline{\Omega})$ is defined by

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \|u\|_{C^k(\overline{\Omega})} + \max_{|\boldsymbol{\alpha}|=k} \left[\frac{\partial^{\boldsymbol{\alpha}}}{\partial x^{\boldsymbol{\alpha}}} u \right]_{\gamma,\Omega} .$$

Let us now weaken the assumptions upon differentiability of solutions of (2.1). Usually one is interested in weak solutions that we describe by the spaces defined in the following. It is assumed that the reader is at least familiar with the integral definition in the sense of Lebesgue (see [40, Chapter 4]). The classical *Lebesgue spaces* $L^p(\Omega)$ for $p \in [1,\infty]$ are defined as the space of Lebesgue-measurable functions having finite norm

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p}, & \text{for } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|, & \text{for } p = \infty. \end{cases}$$

It is well-known that $L^p(\Omega)$ forms a Banach space. The corresponding dual space is $[L^p(\Omega)]^* = L^q(\Omega)$ where q is the dual exponent given by $p^{-1} + q^{-1} = 1$. Moreover, there holds the Hölder inequality [1, Lemma 1.14], i. e. for given $u \in L^p(\Omega)$ and $v \in [L^p(\Omega)]^*$ there hold $uv \in L^1(\Omega)$ as well as the estimate

$$\|uv\|_{L^1(\Omega)} \le c \|u\|_{L^p(\Omega)} \|v\|_{[L^p(\Omega)]^*}. \tag{2.2}$$

The space $L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(u,v)_\Omega := \int_\Omega u(x) v(x) \, \mathrm{d}x.$$

Next, we introduce appropriate spaces to describe also weak derivatives of functions on Ω. As the usual definition of differentiability is too strong for our purposes we introduce the concept of *weak differentiability*. The function $u \in L^p_{loc}(\Omega)$ (this means that $u \in L^p(B)$ for all $B \subset\subset \Omega$) possesses an $\boldsymbol{\alpha}$-th *weak derivative* ($\boldsymbol{\alpha} \in \mathbb{N}_0^n$) if there exists some $v \in L^p_{loc}(\Omega)$ satisfying

$$\int_\Omega u(x) \frac{\partial^{\boldsymbol{\alpha}}}{\partial x^{\boldsymbol{\alpha}}} \phi(x)\, dx = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega v(x)\phi(x)\, dx \qquad \forall \phi \in C_0^\infty(\Omega).$$

If this is the case we write $D^{\boldsymbol{\alpha}} u = v$ as the weak derivative is unique. It remains to define appropriate function spaces. For some given non-negative integer $k \in \mathbb{N}_0$ and a real number $p \in [1, \infty]$ we define the *Sobolev spaces*

$$W^{k,p}(\Omega) := \{v \in L^p(\Omega) \colon D^{\boldsymbol{\alpha}} v \in L^p(\Omega) \quad \forall |\boldsymbol{\alpha}| \le k\}.$$

This space is a Banach space and is equipped with the norm and semi-norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| \le k} \|D^{\boldsymbol{\alpha}} u\|^p_{L^p(\Omega)} \right)^{1/p}, \quad |u|_{W^{k,p}(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} u\|^p_{L^p(\Omega)} \right)^{1/p}, \quad p \in [1, \infty),$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\boldsymbol{\alpha}| \le k} \|D^{\boldsymbol{\alpha}} u\|_{L^\infty(\Omega)}, \qquad |u|_{W^{k,\infty}(\Omega)} := \sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} u\|_{L^\infty(\Omega)},$$

It is also possible to define Sobolev spaces $W^{k+\gamma,p}(\Omega)$ of non-integral order with some $\gamma \in (0, 1)$. To this end, we introduce the functional

$$[u]_{p,\gamma,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+\gamma p}}\, dx\, dy \right)^{1/p}, \tag{2.3}$$

which leads to the *Sobolev-Slobodetskij norm* defined by

$$\|u\|_{W^{k+\gamma,p}(\Omega)} := \left(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\boldsymbol{\alpha}| = k} [D^{\boldsymbol{\alpha}} u]^p_{p,\gamma,\Omega} \right)^{1/p}. \tag{2.4}$$

This norm induces the space $W^{k+\gamma,p}(\Omega)$.

As this thesis deals with polygonal and polyhedral domains Ω we can assume that a uniform cone property holds, and consequently the Hilbertian Sobolev spaces $W^{s,2}(\Omega)$ are equivalent to the spaces $H^s(\Omega)$ introduced in Definition 5.1 of [93] for all $s \in \mathbb{R}$, which can be concluded from Satz 5.4 in [93].

Let us briefly discuss some further properties of Sobolev and Hölder spaces. The following embedding results can be found in [1, Section 8].

Lemma 2.1.1 (Embedding theorem)**.** *Let Ω be a bounded domain with $\dim(\Omega) = n$.*

1. *Let $k_1 \in \mathbb{N}$, $p \in [1, \infty)$, $\gamma \in (0, 1)$ and $k_2 \in \mathbb{N}_0$ such that $k_1 - n/p \ge k_2 + \gamma$. Then there holds the continuous embedding*

$$W^{k_1,p}(\Omega) \hookrightarrow C^{k_2,\gamma}(\overline{\Omega}).$$

2. *Let $k_1, k_2 \in \mathbb{N}_0$ and $p_1, p_2 \in [1, \infty)$ such that $k_1 \ge k_2$ and $k_1 - n/p_1 \ge k_2 - n/p_2$. Then there holds the continuous embedding*

$$W^{k_1,p_1}(\Omega) \hookrightarrow W^{k_2,p_2}(\Omega).$$

3. *Let* $k_1, k_2 \in \mathbb{N}_0$ *and* $p_1, p_2 \in [1, \infty)$ *such that* $k_1 > k_2$ *and* $k_1 - n/p_1 > k_2 - n/p_2$*. Then there holds the compact embedding*

$$W^{k_1,p_1}(\Omega) \overset{c}{\hookrightarrow} W^{k_2,p_2}(\Omega).$$

It is also possible to define Sobolev spaces on submanifolds of Ω, whereas we will merely require them on the boundary Γ or some boundary edge or face. Since the boundary is not arbitrarily regular we cannot define Sobolev spaces of arbitrary order on the boundary of Lipschitz domains, but only of order $s \in [0, 1]$. Analogous to (2.3) and (2.4) we can then define a Sobolev-Slobodetskij norm on the boundary by replacing Ω by Γ and n by $n-1$. This induces the space $H^s(\Gamma)$ for some given $s \in [0, 1]$. A definition with order higher than 1 is only possible on the parts of the boundary which are smooth. Therefore, let $\{\Gamma^{(i)}\}_{j\in\mathcal{F}}$ denote the set of boundary edges if $n = 2$, or boundary faces if $n = 3$. The definition of the spaces $H^s(\Gamma^{(i)})$ is reasonable for all $s \geq 0$ and $i \in \mathcal{F}$ with the norm (2.4). Indeed, the use of these spaces is sufficient for finite element error estimates. For functions which are in $H^s(\Gamma^{(i)})$ for all $i \in \mathcal{F}$, we consequently write

$$H^s_{pw}(\Gamma) := \prod_{i\in\mathcal{F}} H^s(\Gamma^{(i)}).$$

The Sobolev spaces on the boundary introduced above can be used to describe traces of functions defined on Ω. A well-known result (see e. g. [29]) we will frequently use is

Lemma 2.1.2 (Trace theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$*,* $n = 2, 3$*, be a Lipschitz domain. For a given function* $u \in H^s(\Omega)$ *with* $s \in (1/2, 3/2)$ *the trace* $u|_\Gamma$ *belongs to* $H^{s-1/2}(\Gamma)$ *and the inequality*

$$\|u\|_{H^{s-1/2}(\Gamma)} \leq c\|u\|_{H^s(\Omega)}$$

holds.

Finally we will define the dual spaces of $H^s(\Omega)$ and $H^s(\Gamma)$. For arbitrary $s \in \mathbb{R}$ the dual space of $H^s(\Omega)$ is denoted by $[H^s(\Omega)]^*$, and a related norm is the standard operator norm

$$\|u\|_{[H^s(\Omega)]^*} := \sup_{\substack{v\in H^s(\Omega)\\ v\not\equiv 0}} \frac{\langle u, v\rangle_{[H^s(\Omega)]^*\times H^s(\Omega)}}{\|v\|_{H^s(\Omega)}}, \tag{2.5}$$

where

$$\langle u, v\rangle_{[H^s(\Omega)]^*\times H^s(\Omega)} := \int_\Omega u(x)v(x)\,\mathrm{d}x$$

is the duality pairing of $H^s(\Omega)$ with $[H^s(\Omega)]^*$.

Analogously, the dual space of $H^s(\Gamma)$ can be defined, but only for $s \in [0, 1]$ as the parametrization of the boundary has restricted regularity. It is known that the dual spaces of $H^s(\Gamma)$ are $[H^s(\Gamma)]^* = H^{-s}(\Gamma)$ defined e. g. in [66, page 98]. Throughout this thesis we will abbreviate the commonly used dual pairings by

$$\langle \cdot, \cdot\rangle_\Omega := \langle \cdot, \cdot\rangle_{[H^1(\Omega)]^*\times H^1(\Omega)},$$
$$\langle \cdot, \cdot\rangle_\Gamma := \langle \cdot, \cdot\rangle_{[H^{1/2}(\Gamma)]^*\times H^{1/2}(\Gamma)}.$$

2.1.2 Classical regularity results

This section is devoted to classical regularity results for weak solutions of (2.1). The weak formulation of (2.1) can be derived by multiplying the differential equation with a test function $v \in H^1(\Omega)$ and applying Green's first identity. As a consequence we obtain the variational problem: *Find* $y \in H^1(\Omega)$ *such that*

$$\int_\Omega \nabla y(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_\Omega y(x)v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(x)v(x)\,\mathrm{d}s_x \qquad \forall v \in H^1(\Omega). \tag{2.6}$$

One can write this problem in a more compact form using the bilinear form

$$a(\cdot,\cdot)\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

defined by

$$a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_\Omega u(x)v(x)\,\mathrm{d}x,$$

Then, (2.6) reads

Find $y \in H^1(\Omega)$:

$$a(y,v) = \langle f, v\rangle_\Omega + \langle g, v\rangle_\Gamma \qquad \forall v \in H^1(\Omega). \tag{2.7}$$

The existence of a unique solution $y \in H^1(\Omega)$ of problem (2.7) is guaranteed by the *Lax-Milgram Theorem* when $f \in [H^1(\Omega)]^*$ and $g \in H^{-1/2}(\Gamma)$. The function y is called *weak solution* as it does not necessarily satisfy the differentiability requirements to be a solution of the original problem (2.1).

In the following Theorem we collect some already known regularity results in the classical Sobolev spaces introduced in Section 2.1.1. We restrict our considerations to polygonal and polyhedral domains, whereas the exact definitions are postponed to Section 2.2.

Theorem 2.1.3. *Let* $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, *be a bounded polygonal or polyhedral domain. Assume that* $f \in L^2(\Omega)$ *and* $g \in H^{1/2}(\Gamma)$. *The variational solution* $y \in H^1(\Omega)$ *of* (2.7) *satisfies the following assertions:*

a) There exists some $t \in (0, 1/2]$ *such that*

$$y \in H^{3/2+t}(\Omega).$$

b) If Ω *is convex, assertion a) holds for some* $t = 1/2$.

Proof. a) The assertion for polygonal domains is proved in Corollary 2.4.4 of [45], see also Remark 2.4.5 for inhomogeneous boundary data. For polyhedral domains a proof for $g \equiv 0$ is also stated in Grisvard's book [45, Corollary 2.6.7]. For inhomogeneous Neumann conditions one can find a proof in the book of Dauge [33]. In Corollary 23.5 of this reference it was stated that some $\varepsilon > 0$ exists such that the family of operators

$$\mathcal{A}^s := (-\Delta + I, \partial_n)\colon H^{1+s}(\Omega) \to [H^{1-s}(\Omega)]^* \times H^{-1/2+s}(\Gamma)$$

is $[0,s]$-regular for all $0 < s \le 1/2 + \varepsilon$, $s \ne 1/2$. According to Definition 20.9 this means that if there is a function $u \in H^1(\Omega)$ satisfying $\mathcal{A}u = (f, g)$ with input data $f \in [H^{1-s}(\Omega)]^*$ and

$g \in H^{-1/2+s}(\Gamma)$, then there holds $u \in H^{1+s}(\Omega)$. We are in particular interested in this result for $s \in (1/2, 1/2+\varepsilon]$. With $s = t + 1/2$ we conclude the assertion.
b) The assertion follows in case of a polygonal domain again from Corollary 2.4.4 in [45]. For polyhedral domains this result is stated in the book of Maz'ya and Rossmann [65] who proved regularity results in weighted Sobolev spaces that we discuss in Section 2.3. The assertion follows from Theorem 8.1.10 in this book when setting $p = 2$ and all weights to zero, and taking into account that $\delta_+^{(k)} > 1$ and $\Lambda_j > 2$, see also Section 8.3.5 in this book. □

2.2 Singularities in polygonal and polyhedral domains

Before we introduce weighted Sobolev spaces we will get a deeper insight into corner and edge singularities in this section. Once we know the structure of the occurring singularities it is more intuitive to define these adjusted spaces.

Singularities in 2D

For two-dimensional problems we consider domains $\Omega \subset \mathbb{R}^2$ satisfying the following definition.

Definition 2.2.1. *A bounded domain $\Omega \subset \mathbb{R}^2$ is called* polygonal*, if the boundary Γ consists of $d \in \mathbb{N}$ straight edges meeting each other at an angle greater than zero and smaller than 2π.*

Moreover, we denote by

- *$\{\Gamma^{(j)}\}_{j\in\mathcal{C}}$ the set of edges, numerated counter-clockwise, where $\mathcal{C} := \{1, \dots, d\}$ is the corresponding index set,*
- *$\{x^{(j)}\}_{j\in\mathcal{C}}$ the set of corner points such that $x^{(j)}$ is the intersection of $\Gamma^{(j)}$ and $\Gamma^{(j+1)}$ ($\Gamma^{(d+1)} = \Gamma^{(1)}$ by convention).*

We associate to each corner $x^{(j)}$, $j \in \mathcal{C}$, polar coordinates (r_j, φ_j) centered in $x^{(j)}$ such that $\varphi_j = 0$ and $\varphi_j = \omega_j$ coincide with the boundary edges intersecting each other in $x^{(j)}$. To simplify the notation we omit the index j in what follows.

It is already well-known that singularities occur in the vicinity of the corner points. Let us briefly discuss how the singular parts in the solution can be determined. We basically follow the ideas from the fundamental contribution of Kondrat'ev [54] and the books of Grisvard [44], Nazarov and Plamenevsky [69], and Kozlov, Maz'ya and Rossmann [55]. Since the domain Ω coincides in a neighborhood of a corner $x^{(j)}$ with an infinite angle

$$\mathcal{K} := \{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \colon \varphi \in (0,\omega),\ r > 0\}, \qquad \omega \in (0, 2\pi),$$

we first consider the problem

$$-\Delta y = f \quad \text{in } \mathcal{K}, \qquad \partial_n y = 0 \quad \text{on } \partial\mathcal{K}.$$

In polar coordinates (r, φ) the angle $\mathcal{K}$ degenerates to a semi-infinite strip in the (r, φ)-layer. Note that we neglected the zero-order term since it is immaterial for the asymptotic behavior of the solution.

With a *Mellin transformation* the angle is transformed into a vertical line in the complex plane, where the problem is equivalent to a parameter-dependent ordinary differential equation depending only on the variable φ. A solution of this ordinary differential equation can be computed analytically and after the transformation back to $\mathcal{K}$ we get the singular decomposition of the solution. The *Mellin transformation* can be performed in two steps, namely

- apply the substitution $r = \mathrm{e}^{-t}$ to transform the problem to an infinite strip, and
- apply a partial Fourier-transformation in t to get a one-dimensional parameter-dependent boundary value problem.

In the following, this technique is applied to our model problem. After change to polar coordinates, the differential equation reads

$$-\left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \varphi^2}\right) = f.$$

Now we employ the substitution $r = e^{-t}$. The chain-rule and the property $\partial t/\partial r = -r^{-1}$ then imply

$$-\left(\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial \varphi^2}\right) = \mathrm{e}^{-2t} f. \tag{2.8}$$

Fortunately, the mixed terms vanish by this substitution. The second step of the Mellin transformation is a partial Fourier transformation:

$$\hat{y}(\xi, \varphi) = \mathcal{F}(y)(\xi, \varphi) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi t} y(t, \varphi)\mathrm{d}t.$$

Applying this transformation to equation (2.8) leads to

$$\xi^2 \hat{y}(\xi, \varphi) - \frac{\partial^2}{\partial \varphi^2}\hat{y}(\xi, \varphi) = \hat{F}(\xi, \varphi) := \mathcal{F}\left(\mathrm{e}^{-2\cdot} f(\cdot, \varphi)\right)(\xi, \varphi).$$

From this we finally obtain the parameter-dependent ordinary differential equation

$$\begin{aligned} -\hat{y}''(\varphi) + \xi^2 \hat{y}(\varphi) &= \hat{F}(\varphi) \qquad \forall \varphi \in (0, \omega) \\ \hat{y}'(0) = \hat{y}'(\omega) &= 0. \end{aligned} \tag{2.9}$$

The differential operator $\mathfrak{U}(\xi) = -d^2/d\varphi^2 + \xi^2$ is often referred to as *operator pencil* in the literature. The homogeneous problem (2.9) possesses only trivial solutions, except for some isolated parameters – the eigenvalues of $\mathfrak{U}$. By some computations the solution

$$\hat{y}_H(\xi, \varphi) = \begin{cases} \hat{c}_k \cosh\left(\mathrm{i}\frac{k\pi}{\omega}\varphi\right), & \text{if } \xi = \mathrm{i}\frac{k\pi}{\omega} \text{ for some } k \in \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases}$$

for the homogeneous problem is obtained, where $\hat{c}_k \in \mathbb{R}$, $k \in \mathbb{Z}$, are arbitrary constants. An application of the inverse Fourier transform yields

$$y_H(t, \varphi) = \mathcal{F}^{-1}(\hat{y}_H)(t, \varphi) = \sum_{k \in \mathbb{Z}} c_k \cos\left(\frac{k\pi}{\omega}\varphi\right) \mathrm{e}^{-\frac{k\pi}{\omega}t},$$

with $c_k := \hat{c}_k/\sqrt{2\pi}$, $k \in \mathbb{Z}$. Taking into account the substitution $r = e^{-t}$ finally leads to

$$y_H = \sum_{k\in\mathbb{Z}} c_k r^{\lambda_k} \cos(\lambda_k \varphi) \tag{2.10}$$

with the singular exponent $\lambda_k = k\pi/\omega$ and some constant $c_k \in \mathbb{R}$. Note that the singularities corresponding to $k < 0$ cannot occur since this would be a contradiction to $y_H \in H^1(\mathcal{K})$. The solution for $k = 0$ is constant and hence regular.

In order to find a solution of the inhomogeneous problem (2.9), one can use a Green's function representation of the solution and can determine the asymptotic behavior using the theorem on residues. For further details the interested reader is referred to [69, Chapter 5 and 7] and [45, Chapter 2], where the singular solutions

$$S_j^m(r_j, \varphi_j) := \begin{cases} r_j^{\lambda_{j,m}} \cos(\lambda_{j,m}\varphi_j), & \text{if } \lambda_{j,m} \notin \mathbb{Z}, \\ r_j^{\lambda_{j,m}} \left(\ln r_j \cos(\lambda_{j,m}\varphi_j) + \varphi_j \sin(\lambda_{j,m}\varphi_j)\right), & \text{if } \lambda_{j,m} \in \mathbb{Z}, \end{cases} \tag{2.11}$$

for $j \in \mathcal{C}$ and $m \in \mathbb{N}$ with singular exponents

$$\lambda_{j,m} = m\pi/\omega_j,$$

are derived.

With the technique described above one can finally show the asymptotic behavior of the solution of (2.1). For a detailed proof of the following result we refer e. g. to [44, Theorem 4.4.4.13], [45, Section 2.7], [69, Theorem 4.4].

Theorem 2.2.2. *Let $f \in L^q(\Omega)$ and $g \in W^{1-1/q,q}(\Gamma^{(j)})$ for all $j \in \mathcal{C}$, with some $q \in (1, \infty)$ be given. For all $j \in \mathcal{C}$ denote by*

$$\eta_j \in C_0^\infty(\mathbb{R}_+) \quad \text{with} \quad \eta_j(r_j) = \begin{cases} 1, & \text{if } r_j \le R_1, \\ 0, & \text{if } r_j \ge R_2, \end{cases}$$

arbitrary cut-off functions with constants $R_1, R_2 \in \mathbb{R}$ such that $0 < R_1 < R_2$. Assume that

$$2 - 2/q \neq \lambda_{j,m} \qquad \forall j \in \mathcal{C}, m \in \mathbb{N}.$$

Then, some constants $c_{j,m} \in \mathbb{R}$, $j \in \mathcal{C}$, $m \in \mathbb{N}$, and a function $y_R \in W^{2,q}(\Omega)$ exist, such that the solution of (2.1) can be decomposed into

$$y(x) = y_R(x) + \sum_{j=1}^{d} \sum_{\substack{m\in\mathbb{N} \\ \lambda_{j,m}<2-2/q}} c_{j,m}\,\eta_j(r_j) S_j^m(r_j, \varphi_j). \tag{2.12}$$

Proof. This assertion can be found in [45, page 82] and [69, Theorem 4.4] where the additional term $c_0 + c_{j,0} \ln r_j$ occurs in the solution, which is a consequence of the singularity S_j^0 (note that $\lambda_{j,0} = 0 \in \mathbb{Z}$). The constant term c_0 is regular and can be transferred into y_R. However, for $q > 1$ we get by embeddings that $f \in [H^1(\Omega)]^*$ and $g \in H^{-1/2}(\Gamma)$ and the Lax-Milgram Lemma yields $y \in H^1(\Omega)$. That the logarithmic term cannot occur follows directly from the fact that $\ln r_j \notin H^1(\Omega)$ which would be a contradiction. □

Actually, the regularity is restricted by the strongest singularities which correspond to the singular exponents $\lambda_j := \lambda_{j,1}$ for $j \in \mathcal{C}$. Hence, we summarize them to a vector

$$\vec{\lambda} := (\lambda_1, \ldots, \lambda_d)^\top.$$

Singularities in 3D

In contrast to the two-dimensional case we have to distinguish between singularities occurring at edges and corners in the three-dimensional case. Let us first introduce some notation.

Definition 2.2.3. *An open and bounded domain $\Omega \subset \mathbb{R}^3$ is called* polyhedral *if its boundary consists of plane boundary faces, which intersect each other at an angle greater than zero. We denote by*

- $\{\Gamma^{(i)}\}_{i\in\mathcal{F}}$, $\mathcal{F} := \{1,\dots,d^*\}$, *the set of boundary faces, by*
- $\{M_k\}_{k\in\mathcal{E}}$, $k \in \mathcal{E} := \{1,\dots,d\}$, *the set of edges, and by*
- $\{x^{(j)}\}_{j\in\mathcal{C}}$, $j \in \mathcal{C} := \{1,\dots,d'\}$, *the set of corners.*

The singularities occurring in a vicinity of an edge M_k, $k \in \mathcal{E}$, have principally the same structure as corner singularities in 2D. We introduce cylinder coordinates (r_k, φ_k, z_k) such that the z_k-axis coincides with M_k and $\varphi_k = 0$ and $\varphi_k = \omega_k$ correspond to the two faces meeting in M_k. It is known, e.g. from [45, Section 2.5], [78], that the edge singularities have always the structure

$$S_k^{e,m}(r_k, \varphi_k, z_k) := G(\psi_k^{e,m})(r_k, z_k) S_k^m(r_k, \varphi_k),$$

where S_k^m are the singular functions from (2.11) obtained already in the two-dimensional case, with the modification that the edge singular exponent is

$$\lambda_k^m = \lambda_k^{e,m} := m\pi/\omega_k, \qquad m = 1, 2, \dots .$$

Moreover, $\psi_k^{e,m} \in H^{1+\lambda_k^{e,m}}(\mathbb{R})$ is some function, and G is the operator which defines the convolution integral

$$G(\psi)(r_k, z_k) := \frac{r_k}{\pi} \int_{\mathbb{R}} \frac{\psi(z_k - t)}{r_k^2 + t^2}\, \mathrm{d}t.$$

In the following we denote the singular exponent related to the strongest singularity by $\lambda_k^e := \lambda_k^{e,1}$, and summarize all of them to a vector $\vec{\lambda}^e \in \mathbb{R}^d$.

Let us investigate the structure of the singularities occurring at the corners of the domain Ω. Similar investigations can be found in [18, 87, 91],[69, Chapter 10 § 2].

We introduce spherical coordinates $(\rho_j, \varphi_j, \theta_j)$ centered at the corner $x^{(j)}$, $j \in \mathcal{C}$. Without loss of generality we assume that $x^{(j)} = 0$, and to simplify the notation we omit the index j. First, we consider the boundary value problem in an infinite polyhedral cone

$$\mathcal{K} := \{\vec{x} \in \mathbb{R}^3 \colon x/|x| \in \mathcal{G}\},$$

where $\mathcal{G}$ is a polygonal domain chosen appropriately such that $\mathcal{K}$ and Ω coincide in a vicinity of the corner $x^{(j)}$. In order to transform the problem in $\mathcal{K}$ into a parameter-dependent boundary value problem we follow basically the steps described on page 13. The differential equation of the boundary value problem

$$-\Delta y = f \quad \text{in } \mathcal{K}, \qquad \partial_n y = 0 \quad \text{on } \partial K, \tag{2.13}$$

reads in spherical coordinates

$$-\left(\frac{\partial^2 y}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial y}{\partial \rho}+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial y}{\partial\theta}\right)+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 y}{\partial\varphi^2}\right)=f.$$

Again, we introduce the substitution $\rho = \mathrm{e}^{-t}$ and with the chain rule exploiting $\partial t/\partial\rho = -\rho^{-1} = -e^{t}$ we obtain

$$-\left(\frac{\partial^2 y}{\partial t^2}-\frac{\partial y}{\partial t}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 y}{\partial\varphi^2}\right)=e^{-2t}f.$$

In what follows we abbreviate the Laplace-Beltrami operator on the surface $\mathcal{G}$ by

$$\Delta_{\mathcal{G}} y := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 y}{\partial\varphi^2}.$$

With a partial Fourier transformation

$$\hat{y}(\xi,\varphi,\theta)=\mathcal{F}(y)(\xi,\varphi,\theta):=\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\xi t}y(t,\varphi,\theta)\,\mathrm{d}t,$$

we arrive at the parameter dependent boundary value problem

$$\begin{aligned}(\mathrm{i}\xi+\xi^2)\hat{y}-\Delta_{\mathcal{G}}\hat{y}=\tilde{F}:=\mathcal{F}(e^{-2t}f) &\qquad \text{in } \mathcal{G},\\ \partial_n\hat{y}=0 &\qquad \text{on } \partial\mathcal{G}.\end{aligned} \tag{2.14}$$

This boundary value problem on some subset of the unit sphere can be transformed to a plane problem. In Figure 2.1 this procedure is illustrated for the "Fichera" corner which denotes the corner at the intersection of three mutually orthogonal planes. When considering the planar problem one has to take care of the boundary conditions. For the domain illustrated in Figure 2.1b the original Neumann conditions are present at Γ_N only, whereas there are periodic boundary conditions at Γ_P. Moreover, all points on Γ_S^j, $j=1,2$, correspond to only one single point in the original domain.

We are interested in the parameters ξ for those the homogeneous problem has a non-trivial solution. This is the case when $\tilde{\xi} := -\mathrm{i}\xi-\xi^2$ is an eigenvalue of the problem

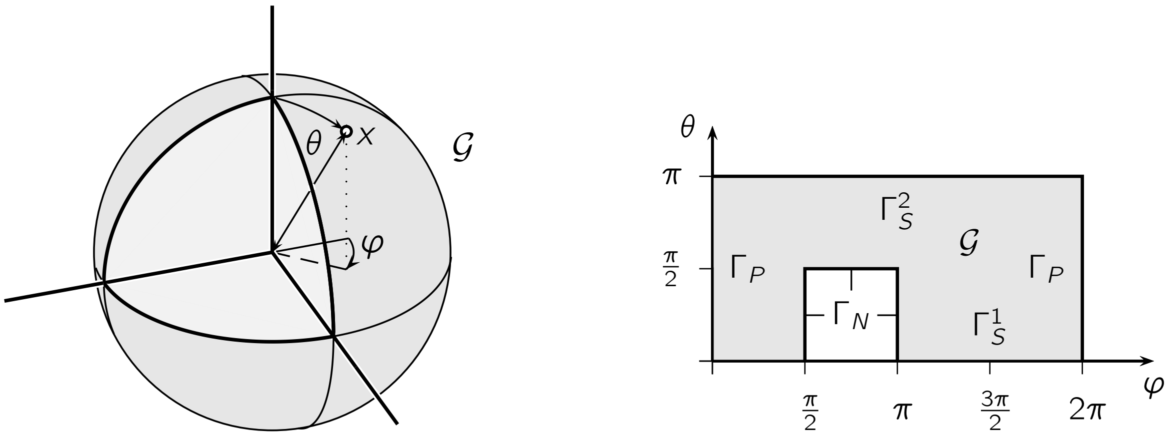
$$-\Delta_{\mathcal{G}}\hat{y}=\tilde{\xi}\hat{y} \quad \text{in} \quad \mathcal{G}, \qquad \partial_n\hat{y}=0 \quad \text{on} \quad \partial\mathcal{G}, \tag{2.15}$$

compare also [87, Equation (1.12)]. We denote the eigenvalues by $\tilde{\lambda}^{c,i}$, $i\in\mathbb{N}_0$, where $\tilde{\lambda}^{c,0}=0$ for the pure Neumann problem. The eigenvalue of interest is the second one because the related singular function contained in the solution of (2.13) will restrict the regularity. Due to the substitution $\tilde{\xi} := -\mathrm{i}\xi-\xi^2$ we set

$$\hat{\lambda}^{c,i}:=\mathrm{i}\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+\tilde{\lambda}^{c,i}}\right). \tag{2.16}$$

As a consequence, when $\xi=\hat{\lambda}^{c,i}$ for $i\in\mathbb{N}$, the problem (2.14) with $\tilde{F}\equiv 0$ possesses some non-trivial solution $\hat{y}_i(\varphi,\theta)$, the eigenfunction related to $\hat{\lambda}^{c,i}$. After an inverse Fourier transformation we get the solution of the homogeneous equation

$$y_H(t,\varphi,\theta)=\sum_{i=1}^{\infty}c_i e^{\mathrm{i}\hat{\lambda}^{c,i}t}\hat{y}_i(\varphi,\theta), \qquad c_i\in\mathbb{R},\ i\in\mathbb{N},$$



(a) The surface $\mathcal{G}$ in the original domain

(b) The surface $\mathcal{G}$ in the (φ, θ)-plane

Figure 2.1: The surface $\mathcal{G}$ for a "Fichera" corner.

and with the substitution $\rho = \mathrm{e}^{-t}$ as well as (2.16) we arrive at the representation

$$y_H(\rho, \varphi, \theta) = \sum_{i=1}^{\infty} c_i \rho^{\lambda^{c,i}} \hat{y}_i(\varphi, \theta) \qquad \text{with} \qquad \lambda^{c,i} := -\frac{1}{2} + \sqrt{\frac{1}{4} + \tilde{\lambda}^{c,i}}.$$

In a similar way it is possible to find a solution of the inhomogeneous problem. However, since this would exceed the scope of this thesis we refer to [45, Section 2.6], where the singular functions

$$S_j^{c,m}(\rho_j, \varphi_j, \theta_j) := \rho_j^{\lambda_j^{c,m}} F_j^m(\varphi, \theta),$$

with some smooth functions $F_j^m(\cdot, \cdot)$ are derived. The leading corner singular exponents are also summarized to a vector

$$\vec{\lambda}^c := (\lambda_1^{c,1}, \ldots, \lambda_{d'}^{c,1})^\top .$$

In summary it can be said, therefore, that the corner singularities contained in the solution of (2.1) depend solely upon the eigenvalues of the problem (2.15). The computation of these eigenvalues has been discussed intensively in the literature. In special cases it is possible to compute eigenvalues analytically. We want to mention e. g. Stephan and Whiteman [87] who derived $\tilde{\lambda}_j^{c,1} = 40/9$ ($\Rightarrow \lambda_j^{c,1} = 5/3$) for the three-dimensional L-shape domain. However, in general the eigenvalue $\lambda_j^{c,1}$ has to be computed approximately. Walden and Kellogg [91] and Beagles and Whiteman [18] present a discretization of (2.15) using a finite difference method. Pester [73] and Leguillon [58] used a finite element discretization instead. The discrete eigenvalue problem can be solved with Rayleigh quotient minimization. In the two latter references some computations for the "Fichera domain" are presented. For the Dirichlet problem the exponent $\lambda_j^{c,1} \approx 0.44$ was computed approximately. However, the eigenvalues depend on the type of boundary condition. For a pure Neumann boundary we have computed $\lambda_j^{c,1} \approx 0.84$ with the software package CoCoS [73]. We also refer to the summary in [31, Section 1].

2.3 Weighted Sobolev spaces

In this section we introduce weighted Sobolev spaces which allow us to describe the singular parts contained in the solution of a boundary value problem in a more accurate way. Let us first outline the basic idea behind these spaces. Imagine $y \in H^1(\Omega)$ is the unique solution of (2.7) in some non-convex polygonal domain $\Omega \subset \mathbb{R}^2$. Since at least one corner has interior angle larger than 180° the corresponding singular exponent is smaller than one. As a consequence, at least one second derivative is not square integrable in a ball B with radius R centered at this corner point. This can be simply checked by computing the integral of the second derivative for r, for which holds

$$\int_{B\cap\Omega} r(x)^{2(\lambda-2)}\,\mathrm{d}x = c\int_0^R r^{2(\lambda-2)+1}\,\mathrm{d}r = c\frac{r^{2(\lambda-1)}}{2(\lambda-1)}\bigg|_0^R = \infty,$$

since $\lambda - 1 < 0$. As a remedy, we introduce the additional weight function $r(x)^\beta$ with some $\beta \in \mathbb{R}$ chosen appropriately, such that $r^\beta D^{\boldsymbol{\alpha}} y \in L^2(\Omega)$ for all $|\boldsymbol{\alpha}| = 2$. A simple calculation yields

$$\int_{B\cap\Omega} r(x)^{2(\lambda-2+\beta)}\,\mathrm{d}x = c\int_0^R r^{2(\lambda-2+\beta)+1}\,\mathrm{d}r = c\frac{r^{2(\lambda+\beta-1)}}{2(\lambda+\beta-1)}\bigg|_0^R < \infty,$$

which holds under the condition $\beta > 1 - \lambda$. In the following we will define function spaces that contain such kind of weight functions.

2.3.1 Definition and basic properties

We begin with the definition of weighted Sobolev spaces for two-dimensional polygonal domains. The weights are the distance functions to the corner points that we define by

$$r_j(x) := |x - x^{(j)}|, \quad \forall j \in \mathcal{C}.$$

Definition 2.3.1. *Let $\Omega \in \mathbb{R}^2$ be a polygonal domain and denote by $\{U_j\}_{j\in\mathcal{C}}$, an open covering of Ω such that U_j contains only the corner $x^{(j)}$ but no other ones. For a non-negative integer $\ell \in \mathbb{N}_0$, a real number $p \in [1,\infty]$ and a vector $\vec{\alpha} \in \mathbb{R}^d$ the* weighted Sobolev space *$W^{\ell,p}_{\vec{\alpha}}(\Omega)$ is defined as the closure of $C^\infty(\bar{\Omega} \setminus \{x^{(0)}, \dots, x^{(d)}\})$ with respect to the norm*

$$\|v\|_{W^{\ell,p}_{\vec{\alpha}}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}|\le\ell}\sum_{j\in\mathcal{C}}\int_{U_j\cap\Omega} r_j(x)^{p\alpha_j}|D^{\boldsymbol{\alpha}} v(x)|^p\,\mathrm{d}x\right)^{1/p}, & \text{if } p \in [1,\infty), \\ \displaystyle\sum_{|\boldsymbol{\alpha}|\le\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in U_j\cap\Omega} r_j(x)^{\alpha_j}|D^{\boldsymbol{\alpha}} v(x)|, & \text{if } p = \infty. \end{cases} \tag{2.17}$$

A seminorm related to (2.17) is defined by

$$|v|_{W^{\ell,p}_{\vec{\alpha}}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}|=\ell}\sum_{j\in\mathcal{C}}\int_{U_j\cap\Omega} r_j(x)^{p\alpha_j}|D^{\boldsymbol{\alpha}} v(x)|^p\,\mathrm{d}x\right)^{1/p}, & \text{if } p \in [1,\infty), \\ \displaystyle\sum_{|\boldsymbol{\alpha}|=\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in U_j\cap\Omega} r_j(x)^{\alpha_j}|D^{\boldsymbol{\alpha}} v(x)|, & \text{if } p = \infty. \end{cases} \tag{2.18}$$

We will frequently use the norm (2.17) on a subset $\mathcal{G} \subset \Omega$, and in case of $\mathcal{G} \subset U_j$ we write

$$\|\cdot\|_{W^{\ell,p}_{\vec{\alpha}}(\mathcal{G})} = \|\cdot\|_{W^{\ell,p}_{\alpha_j}(\mathcal{G})}, \tag{2.19}$$

where the weight function is still the distance to the corner of Ω. The weights related to the other corners have no influence and can be neglected.

On polyhedral domains both weights for corners and edges are required. The corresponding weight functions are defined by

$$\rho_j(x) := |x - x^{(j)}| \quad \forall j \in \mathcal{C}, \qquad r_k(x) := \inf_{y \in M_k} |x - y| \quad \forall k \in \mathcal{E}.$$

Moreover, we define the index set

$$X_j := \{k \in \mathcal{E} \colon x^{(j)} \text{ is an endpoint of } M_k\}.$$

Definition 2.3.2. *Let $\Omega \in \mathbb{R}^3$ be a polyhedral domain and denote by $\{U_j\}_{j\in\mathcal{C}}$ an open covering of Ω such that U_j contains only the corner $x^{(j)}$ but no other ones. For a non-negative integer $\ell \in \mathbb{N}_0$, a real number $p \in [1, \infty]$ and vectors $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ the space $W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)$ is defined as the closure of $C^\infty(\bar{\Omega}\backslash\{x^{(1)}, \ldots, x^{(d')}\})$ with respect to the norm*

$$\|v\|_{W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)} := \begin{cases} \left(\sum_{|\boldsymbol{\alpha}|\le\ell} \sum_{j\in\mathcal{C}} \int_{\Omega\cap U_j} \rho_j(x)^{p(\beta_j-\ell+|\boldsymbol{\alpha}|)} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{p\delta_k} |D^{\boldsymbol{\alpha}} v(x)|^p \right)^{\frac{1}{p}}, & \text{if } p \in [1,\infty), \\ \sum_{|\boldsymbol{\alpha}|\le\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in\Omega\cap U_j} \rho_j(x)^{\beta_j-\ell+|\boldsymbol{\alpha}|} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{\delta_k} |D^{\boldsymbol{\alpha}} v(x)|, & \text{if } p = \infty. \end{cases} \tag{2.20}$$

Analogous to (2.18) we introduce a corresponding seminorm by

$$|v|_{W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)} := \begin{cases} \left(\sum_{|\boldsymbol{\alpha}|=\ell} \sum_{j\in\mathcal{C}} \int_{\Omega\cap U_j} \rho_j(x)^{p\beta_j} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{p\delta_k} |D^{\boldsymbol{\alpha}} v(x)|^p \right)^{\frac{1}{p}}, & \text{if } p \in [1,\infty), \\ \sum_{|\boldsymbol{\alpha}|=\ell} \max_{j\in\mathcal{C}} \operatorname*{ess\,sup}_{x\in\Omega\cap U_j} \rho_j(x)^{\beta_j} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{\delta_k} |D^{\boldsymbol{\alpha}} v(x)|, & \text{if } p = \infty. \end{cases} \tag{2.21}$$

Trace spaces

Let $\ell \in \mathbb{N}$ and $p \in [1, \infty]$ be arbitrary. For polygonal domains $\Omega \subset \mathbb{R}^2$ the space $W^{\ell-1/p,p}_{\vec{\alpha}}(\Gamma^{(j)})$, $j \in \mathcal{C}$, is the trace space of $W^{\ell,p}_{\vec{\alpha}}(\Omega)$ and is equipped with the norm

$$\|u\|_{W^{\ell-1/p,p}_{\vec{\alpha}}(\Gamma^{(j)})} := \inf\left\{ \|v\|_{W^{\ell,p}_{\vec{\alpha}}(\Omega)} \colon v \in W^{\ell,p}_{\vec{\alpha}}(\Omega) \text{ and } v|_{\Gamma^{(j)}} = u \right\}. \tag{2.22}$$

Analogously we can define a trace space of weighted Sobolev spaces for polyhedral domains $\Omega \subset \mathbb{R}^3$. By $W^{\ell-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma^{(i)})$, $i \in \mathcal{F}$, we denote the trace space of $W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)$ on the boundary face $\Gamma^{(i)}$, with the naturally induced norm

$$\|u\|_{W^{\ell-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma^{(i)})} := \inf\left\{\|v\|_{W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)} : v \in W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega) \text{ and } v|_{\Gamma^{(i)}} = u\right\}.$$

Moreover we write

$$u \in W^{\ell-1/p,p}_{\vec{\alpha}}(\Gamma) \;:\Longleftrightarrow\; u \in W^{\ell-1/p,p}_{\vec{\alpha}}(\Gamma^{(j)}) \qquad \forall j \in \mathcal{C}.$$

Analogously we define the global trace space $W^{\ell-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma)$.

Embeddings

It remains to collect some auxiliary results about the weighted Sobolev spaces introduced above. In the following we denote by G either the domain Ω or its boundary Γ. The integer n is the dimension of G. The next Lemma can also be found in [74, Lemma 2.29].

Lemma 2.3.3. *The following embeddings hold:*

a) *Assume that $1 \le p < q \le \infty$, and that the weight vectors $\vec{\alpha}, \vec{\alpha}' \in \mathbb{R}^d$ satisfy the inequality*

$$\alpha'_j + \frac{n}{q} < \alpha_j + \frac{n}{p} \qquad \forall j \in \mathcal{C}.$$

Then, for all $\ell \in \mathbb{N}_0$ the continuous embedding

$$W^{\ell,q}_{\vec{\alpha}'}(G) \hookrightarrow W^{\ell,p}_{\vec{\alpha}}(G)$$

holds.

b) *Let $\ell \in \mathbb{N}_0$ and $p \in [1,\infty]$. Assume that the weight vectors $\vec{\alpha}, \vec{\alpha}' \in \mathbb{R}^d$ satisfy the inequality*

$$\alpha'_j \le 1 + \alpha_j, \quad \alpha'_j > -\frac{n}{p} \qquad \forall j \in \mathcal{C}.$$

Then, the continuous embedding

$$W^{\ell+1,p}_{\vec{\alpha}'}(G) \hookrightarrow W^{\ell,p}_{\vec{\alpha}}(G)$$

holds. If $\alpha'_j < 1 - \alpha_j$ for all $j \in \mathcal{C}$ the embedding is even compact.

A similar result holds for the weighted Sobolev spaces on polyhedral domains introduced in Definition 2.3.2. Proofs for the results of the following Lemma can be found in [65, Lemma 8.1.1 and Lemma 8.1.2].

Lemma 2.3.4. *The following embeddings hold:*

a) *Let* $1 \le q < p \le \infty$ *and* $\ell \in \mathbb{N}_0$. *Assume that the vectors* $\vec{\beta}, \vec{\beta}' \in \mathbb{R}^{d'}$ *and* $\vec{\delta}, \vec{\delta}' \in \mathbb{R}^d$ *satisfy*

$$\begin{aligned} \beta_j' + \frac{n}{p} &< \beta_j + \frac{n}{q} && \forall j \in \mathcal{C}, \\ 0 < \delta_k' + \frac{n-1}{p} &< \delta_k + \frac{n-1}{q} && \forall k \in \mathcal{E}. \end{aligned}$$

Then, the continuous embedding

$$W^{\ell,p}_{\vec{\beta}',\vec{\delta}'}(G) \hookrightarrow W^{\ell,q}_{\vec{\beta},\vec{\delta}}(G)$$

holds.

b) *Let* $p \in [1,\infty)$ *and* $\ell \in \mathbb{N}_0$ *be given. Assume that the vectors* $\vec{\beta}, \vec{\beta}' \in \mathbb{R}^{d'}$ *and* $\vec{\delta}, \vec{\delta}' \in \mathbb{R}^d$ *satisfy*

$$\begin{aligned} &\beta_j' \le 1 + \beta_j && \forall j \in \mathcal{C}, \\ &\delta_k' \le 1 + \delta_k, \quad \delta_k, \delta_k' > -\frac{n-1}{p} && \forall k \in \mathcal{E}. \end{aligned}$$

Then, the continuous embedding

$$W^{\ell+1,p}_{\vec{\beta}',\vec{\delta}'}(G) \hookrightarrow W^{\ell,p}_{\vec{\beta},\vec{\delta}}(G)$$

holds. In case of $\beta_j' < 1 + \beta_j$, $j \in \mathcal{C}$, *and* $\delta_k' < 1 + \delta_k$, $k \in \mathcal{E}$, *the embedding is even compact.*

2.3.2 Regularity Results

We are now in the position to formulate some regularity results for the solution of the variational problem posed in Section 2.1.2

Find $y \in H^1(\Omega)$:

$$a(y,v) = \langle f, v\rangle_\Omega + \langle g, v\rangle_\Gamma \qquad \forall v \in H^1(\Omega), \tag{2.23}$$

using the weighted Sobolev spaces introduced in Section 2.3.1.

Polygonal Domains

First, we discuss a regularity result in weighted Sobolev spaces for polygonal domains $\Omega \subset \mathbb{R}^2$. A proof of the following result using already our notation can be found in [74, Lemma 3.10].

Theorem 2.3.5. *Let* $f \in W^{0,2}_{\vec{\alpha}}(\Omega)$ *and* $g \in W^{1/2,2}_{\vec{\alpha}}(\Gamma)$. *Then, the solution* y *of* (2.23) *belongs to* $W^{2,2}_{\vec{\alpha}}(\Omega)$ *if the weight vector* $\vec{\alpha} \in \mathbb{R}^d$ *satisfies*

$$\begin{aligned} 1 - \lambda_j < \alpha_j &< 1 \qquad \text{if} \quad 1 - \lambda_j \ge 0, \\ 0 \le \alpha_j &< 1 \qquad \text{if} \quad 1 - \lambda_j < 0, \end{aligned} \tag{2.24}$$

for all $j \in \mathcal{C}$. *Furthermore, there holds the* a priori *estimate*

$$\|y\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} \le c \left(\|f\|_{W^{0,2}_{\vec{\alpha}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\alpha}}(\Gamma)} \right).$$

For the proof of optimal finite element error estimates on the boundary the regularity in $W^{2,2}_{\vec{\alpha}}(\Omega)$ is not sufficient. In this case we want to exploit regularity in $W^{2,\infty}_{\vec{\beta}}(\Omega)$ with an appropriate weight $\vec{\beta} \in \mathbb{R}^d$. The following result is proven in [9].

Theorem 2.3.6. *Assume that $g \equiv 0$ and $f \in C^{0,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$. Then, the solution y of (2.23) belongs to $W^{2,\infty}_{\vec{\beta}}(\Omega)$ with a weight vector $\vec{\beta} \in \mathbb{R}^d$ having components satisfying*

$$\begin{aligned} 2 - \lambda_j < \beta_j < 2 \qquad & \text{if} \quad 2 - \lambda_j \geq 0, \\ 0 \leq \beta_j < 2 \qquad & \text{if} \quad 2 - \lambda_j < 0, \end{aligned}$$

for all $j \in \mathcal{C}$. Furthermore, there holds the a priori *estimate*

$$\|y\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)} \leq c \|f\|_{C^{0,\sigma}(\overline{\Omega})}.$$

Polyhedral Domains

Regularity results for the solution of problem (2.23) in weighted Sobolev spaces when $\Omega \subset \mathbb{R}^3$ is a polyhedral domain, are proven e. g. in [2, 31, 65, 95]. We will exploit the following result for finite element error estimates later.

Theorem 2.3.7. *Let functions $f \in W^{0,p}_{\vec{\beta},\vec{\delta}}(\Omega)$ and $g \in W^{1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma)$ with $p \in (1,\infty)$ be given. Assume that the edge and corner weights $\vec{\delta} \in \mathbb{R}^d_+$ and $\vec{\beta} \in \mathbb{R}^{d'}_+$ satisfy*

$$\begin{aligned} 2 - 2/p - \min\{2, \lambda^e_k\} < \delta_k < 2 - 2/p \qquad & \forall k \in \mathcal{E}, \\ 2 - 3/p - \min\{1, \lambda^c_j\} < \beta_j < 3 - 3/p \qquad & \forall j \in \mathcal{C}. \end{aligned}$$

Then, the solution $y \in H^1(\Omega)$ of the variational problem (2.23) satisfies

$$D^{\boldsymbol{\alpha}} y \in W^{1,p}_{\vec{\beta},\vec{\delta}}(\Omega) \qquad \forall |\boldsymbol{\alpha}| = 1.$$

Moreover, the a priori *estimate*

$$\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,p}_{\vec{\beta},\vec{\delta}}(\Omega)} + \|y\|_{L^p(\Omega)} \leq c \left(\|f\|_{W^{0,p}_{\vec{\beta},\vec{\delta}}(\Omega)} + \|g\|_{W^{1-1/p,p}_{\vec{\beta},\vec{\delta}}(\Gamma)} \right) \tag{2.25}$$

holds.

Proof. The desired assertion is stated in Theorem 8.1.10 of [65] under the additional assumption that $\lambda = -1$ and $\lambda = 0$ are the only eigenvalues of the problem

$$\begin{aligned} -\Delta_{\mathcal{G}_j} v = \lambda(\lambda+1) v \qquad & \text{in } \mathcal{G}_j, \\ \partial_n v = 0 \qquad & \text{on } \partial\mathcal{G}_j, \end{aligned}$$

that are contained in the strip $-1 \leq Re\,\lambda \leq 0$. Note, that this eigenvalue problem is the same as (2.15) when inserting the definition

$$\lambda := -\frac{1}{2} + \sqrt{\frac{1}{4} + \tilde{\xi}},$$

compare also [56, Equation (2.3.3)]. That this strip indeed contains only the eigenvalues 0 and -1 in our situation, and, that algebraic and geometric multiplicity are equal, has been discussed in [56, Section 2.3.1].

It remains to prove the *a priori* estimate (2.25) which is not directly stated in [65], but in the following, we outline how this estimate can be concluded. To this end, introduce the space

$$\mathcal{H} := \left\{ v \in L^p(\Omega) \colon D^{\boldsymbol{\alpha}} v \in W^{1,p}_{\vec\beta,\vec\delta} \quad \forall |\boldsymbol{\alpha}| = 1 \right\}$$

with the naturally induced norm as stated on the left-hand side of (2.25), and the operator

$$\mathcal{A} := \begin{pmatrix} -\Delta + I \\ \partial_n \end{pmatrix} : \mathcal{H} \to W^{0,p}_{\vec\beta,\vec\delta}(\Omega) \times W^{1-1/p,p}_{\vec\beta,\vec\delta}(\Gamma).$$

It is easy to confirm that the operator $\mathcal{A}$ is linear and bounded since the estimates

$$\begin{aligned} \|-\Delta u\|_{W^{0,p}_{\vec\beta,\vec\delta}(\Omega)} &\le c|u|_{W^{2,p}_{\vec\beta,\vec\delta}(\Omega)}, \\ \|u\|_{W^{0,p}_{\vec\beta,\vec\delta}(\Omega)} &\le c\|u\|_{L^p(\Omega)}, \\ \|\partial_n u\|_{W^{1-1/p,p}_{\vec\beta,\vec\delta}(\Gamma)} &\le c \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} u\|_{W^{1,p}_{\vec\beta,\vec\delta}(\Omega)}, \end{aligned}$$

hold for arbitrary $u \in \mathcal{H}$. More precisely, the first estimate follows directly from the norm definition (2.20), the second one from a trivial embedding taking into account that $\vec\beta, \vec\delta \ge 0$, and the third one from the definition of the trace space on page 19. We also confirm that $\mathcal{A}$ is bijective which is equivalent to the existence and uniqueness of a solution in $\mathcal{H}$ and follows from the first part of this theorem and the Lax-Milgram Lemma. From the *bounded inverse theorem* [37, Theorem 3.7] we conclude that the inverse mapping $\mathcal{A}^{-1}$ is also continuous which is equivalent to (2.25). □

The above theorem excludes $p = \infty$, but we will require this case in order to derive optimal finite element error estimates in the $L^2(\Gamma)$-norm. To overcome this issue we apply regularity results in weighted Hölder spaces.

Theorem 2.3.8. *Let a function $f \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$ be given and assume that $g \equiv 0$. Moreover, assume that the weights $\vec\delta \in \mathbb{R}^d_+$ and $\vec\beta \in \mathbb{R}^{d'}_+$ satisfy*

$$\begin{aligned} &2 - \lambda^e_k < \delta_k < 2 && \forall k \in \mathcal{E}, \\ &2 - \lambda^c_j < \beta_j && \forall j \in \mathcal{C}. \end{aligned}$$

Then, the solution $y \in H^1(\Omega)$ of (2.23) satisfies

$$D^{\boldsymbol{\alpha}} y \in W^{1,\infty}_{\vec\beta,\vec\delta}(\Omega) \qquad \forall |\boldsymbol{\alpha}| = 1.$$

Proof. Let us first introduce the weighted Hölder spaces defined in [65, Section 8.2]. We denote by $U_{j,k} := \{x \in U_j \cap \Omega \colon r_k(x) < 3\rho_j(x)/2\}$, $k \in X_j$, a covering of U_j. Furthermore, a Hölder exponent $\sigma \in (0,1)$, a non-negative integer $\ell \in \mathbb{N}_0$ and some weights $\vec\beta \in \mathbb{R}^{d'}$, $\vec\delta \in \mathbb{R}^d$ with $\delta_k \ge 0$ $(k \in \mathcal{E})$ are given. To each edge we associate the integer $m_k := [\delta_k - \sigma] + 1$. The weighted Hölder space $C^{\ell,\sigma}_{\vec{\beta},\vec{\delta}}(\Omega)$ denotes the space of ℓ times continuously differentiable functions on $\tilde{\Omega} := \overline{\Omega} \setminus (\bigcup_{k\in\mathcal{E}} \overline{M}_k)$ with finite norm

$$\begin{aligned} \|u\|_{C^{\ell,\sigma}_{\vec{\beta},\vec{\delta}}(\Omega)} := & \sum_{j=1}^{d'} \sum_{|\boldsymbol{\alpha}|\le\ell} \sup_{x\in U_j} \rho_j(x)^{\beta_j-\ell-\sigma+|\boldsymbol{\alpha}|} \prod_{k\in X_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\max\{0,\delta_k-\ell-\sigma+|\boldsymbol{\alpha}|\}} |(D^{\boldsymbol{\alpha}}u)(x)| \\ & + \sum_{j=1}^{d'} \sum_{k\in X_j} \sum_{|\boldsymbol{\alpha}|=\ell-m_k} \sup_{\substack{x,y\in U_{j,k} \\ |x-y|<\rho_j(x)/2}} \rho_j(x)^{\beta_j-\delta_k} \frac{|(D^{\boldsymbol{\alpha}}u)(x)-(D^{\boldsymbol{\alpha}}u)(y)|}{|x-y|^{m_k+\sigma-\delta_k}} \\ & + \sum_{j=1}^{d'} \sum_{|\boldsymbol{\alpha}|=\ell} \sup_{\substack{x,y\in U_j \\ |x-y|<\rho_j(x)/2}} \rho_j^{\beta_j}(x) \prod_{k\in X_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\delta_k} \frac{|(D^{\boldsymbol{\alpha}}u)(x)-(D^{\boldsymbol{\alpha}}u)(y)|}{|y-x|^{\sigma}}. \end{aligned} \tag{2.26}$$

We introduce weights $\vec{\beta}' := \vec{\beta}+\sigma$ and $\vec{\delta}' := \vec{\delta}+\sigma$ and observe that the inequalities

$$\|f\|_{C^{0,\sigma}_{\vec{\beta}',\vec{\delta}'}(\Omega)} \le c\|f\|_{C^{0,\sigma}_{\vec{\sigma},\vec{0}}(\Omega)} \le c\|f\|_{C^{0,\sigma}(\overline{\Omega})}$$

hold, where the first inequality is a consequence of the embedding theorem [65, Lemma 8.2.1] which holds for arbitrary $\beta'_j \ge \sigma$ ($j\in\mathcal{C}$) and $\delta'_k \ge 0$ ($k\in\mathcal{E}$), and the second one can be confirmed when inserting $\beta_j=\sigma$ and $\delta_k=0$ in the definition of the norm (2.26).

The assumptions upon $\vec{\beta}$ and $\vec{\delta}$ imply that

$$\begin{aligned} 2-\lambda^e_k < \delta'_k-\sigma < 2 \qquad & k\in\mathcal{E}, \\ 2-\lambda^c_j < \beta'_j-\sigma \qquad & j\in\mathcal{C}, \end{aligned} \tag{2.27}$$

and with the regularity result from [64, Theorem 5.1 and Remark 5.1] we obtain $D^{\boldsymbol{\alpha}}y \in C^{1,\sigma}_{\vec{\beta}',\vec{\delta}'}(\Omega)$ for all $|\boldsymbol{\alpha}|=1$. It remains to show that

$$\|D^{\boldsymbol{\alpha}}u\|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(\Omega)} \le c\|D^{\boldsymbol{\alpha}}u\|_{C^{1,\sigma}_{\vec{\beta}',\vec{\delta}'}(\Omega)} \qquad \forall|\boldsymbol{\alpha}|=1.$$

It suffices to bound the $W^{1,\infty}_{\vec{\beta},\vec{\delta}}(\Omega)$-norm by the first row in the norm definition (2.26). Obviously, when inserting $\beta_j=\beta'_j-\sigma$, the corner weights coincide. Inserting $\delta_k=\delta'_k-\sigma\ge 0$ yields for the edge weights

$$\left(\frac{r_k}{\rho_j}\right)^{\delta_k} = \left(\frac{r_k}{\rho_j}\right)^{\delta'_k-\sigma} \le c\left(\frac{r_k}{\rho_j}\right)^{\max\{0,\delta'_k-\sigma-1+|\boldsymbol{\alpha}|\}},$$

where we exploited $\delta'_k-\sigma \ge \max\{0,\delta'_k-\sigma-1+|\boldsymbol{\alpha}|\}$ for all $|\boldsymbol{\alpha}|\le 1$. Consequently, we have shown (2.27) and the assertion can be concluded. □

CHAPTER 3

Finite element error estimates

In this chapter the numerical approximation of the Neumann boundary value problem discussed in the last chapter using the *Finite Element Method* is considered. Our aim is to derive sharp discretization error estimates in dependence of the geometry of the underlying domain Ω.

We will show that the singularities discussed in Section 2.2 have also influence on the convergence rate of the finite element method. If the singular exponents are smaller than a certain bound we cannot expect an optimal convergence rate, and our aim is to investigate how the convergence rate and the singular exponents are then related to each other.

Special emphasis will be put on local mesh refinement which is a well known technique used to compensate reduced convergence rates. As we know the structure of the occurring singularities in advance, it is possible to derive *a priori* mesh refinement conditions in such a way that we can determine the refinement parameter once we know the singular exponents. This idea is indeed very old and has been intensively investigated in the literature, e. g. [6, 12, 15, 42, 72, 81]. In all these contributions error estimates in the $H^1(\Omega)$- and $L^2(\Omega)$-norm are proved. However, in the context of boundary control problems finite element error estimates on the boundary Γ are of interest. The novelties in this chapter are a finite element error estimate in $H^{1/2}(\Gamma)$ on polygonal domains, and the extension of the $L^2(\Gamma)$-estimate proved in [9] for polygonal domains to polyhedral domains.

This chapter is structured as follows. In Section 3.1 we introduce the finite element approximation of the solution of (2.23). Error estimates for local projection operators, e. g. interpolation operators, are a prerequisite that we investigate in Section 3.2. The new results in this thesis are tailored local estimates which exploit the regularity in weighted Sobolev spaces stated in Section 2.3. The main results are the finite element error estimates that we are able to formulate in Section 3.3 for quasi-uniform meshes, and in Section 3.4 for locally refined meshes.

3.1 Discretization of boundary value problems

In order to transform the variational problem (2.23) into a finite-dimensional problem a finite-dimensional space $V_h \subset H^1(\Omega)$ is introduced which leads to the Galerkin formulation: *Find* $y_h \in V_h$ *such that*

$$a(y_h, v_h) = \langle f, v_h\rangle_\Omega + \langle g, v_h\rangle_\Gamma \qquad \forall v_h \in V_h. \tag{3.1}$$

The finite element method belongs to the class of Galerkin methods and has the additional property that V_h possesses a basis consisting of piecewise polynomial functions having small support. A prerequisite for the construction of such a basis is a decomposition of the domain Ω into elements $T \in \mathcal{T}_h$ such that

$$\Omega = \operatorname{int} \bigcup_{T\in\mathcal{T}_h} \overline{T}, \qquad T_1 \cap T_2 = \emptyset \text{ for all } T_1, T_2 \in \mathcal{T}_h \text{ with } T_1 \neq T_2.$$

Throughout this thesis triangular elements for two-dimensional, and tetrahedral elements for three-dimensional problems are considered. The index h is a mesh parameter denoting the maximal diameter of all elements in $\mathcal{T}_h$. Moreover, it is assumed that the triangulation $\mathcal{T}_h$ is feasible, i. e. that the intersection of two different element closures is either empty, a common vertex, a common edge, or – for tetrahedral meshes – a common face (compare also [23, Definition 3.3.11]). In other words the presence of hanging nodes will be avoided. The resulting decomposition $\mathcal{T}_h$ of Ω is called *triangulation* or *finite element mesh*.

For each $T \in \mathcal{T}_h$ we denote the diameter of the smallest ball containing T and the diameter of largest ball contained in T by h_T and ρ_T, respectively. Throughout this thesis we deal with *shape regular* families of triangulations $\{\mathcal{T}_h\}_{h>0}$ only, i. e. some constants $h_0 > 0$ and $\kappa > 0$ exist such that

$$\frac{\rho_T}{h_T} \geq \kappa \qquad \forall T \in \mathcal{T}_h,$$

is satisfied for all $h \in (0, h_0]$.

Further, we define by $\mathcal{N}_h$ the set of nodes of $\mathcal{T}_h$, and by $\mathcal{N}_T$ the set of nodes belonging to the element $T \in \mathcal{T}_h$. The induced boundary mesh defined by

$$\mathcal{E}_h := \{\operatorname{int}(\overline{T} \cap \Gamma) : T \in \mathcal{T}_h\},$$

forms a conforming triangulation of the boundary Γ consisting of intervals (2D) or triangles (3D).

In this thesis only continuous and piecewise linear finite elements are considered, i. e. the space of ansatz and test functions is

$$V_h := \left\{v_h \in C(\overline{\Omega}) : v_h \text{ is affine linear on all } T \in \mathcal{T}_h\right\}. \tag{3.2}$$

As V_h is finite-dimensional each element in V_h can be represented as a linear combination of basis functions. Usually, the nodal basis

$$\{\varphi_n \in V_h : \varphi_n(\tilde{n}) = \delta_{n,\tilde{n}} \text{ for all } \tilde{n}, n \in \mathcal{N}_h\}, \quad \text{where} \quad \delta_{n,\tilde{n}} := \begin{cases} 1, & \text{if } \tilde{n} = n, \\ 0, & \text{if } \tilde{n} \neq n, \end{cases}$$

is used. The solution of (3.1) can then be written in the form

$$y_h(x) = \sum_{n\in\mathcal{N}_h} y_h(n)\varphi_n(x),$$

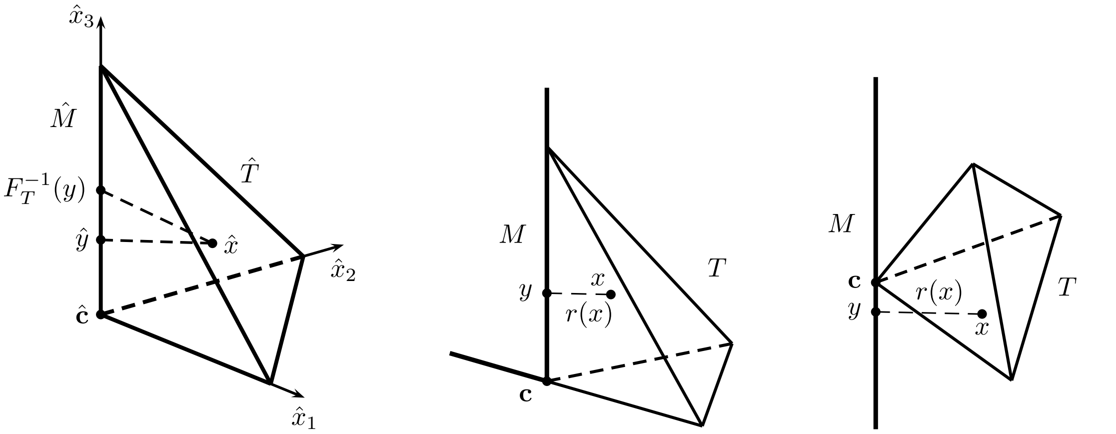
and due to the linear structure of the problem (3.1) it remains to test the variational formulation with basis functions only. This leads to the finite element formulation

$$\sum_{n\in\mathcal{N}_h} y_h(n)a(\varphi_n,\varphi_{\tilde{n}}) = \langle f,\varphi_{\tilde{n}}\rangle_\Omega + \langle g,\varphi_{\tilde{n}}\rangle_\Gamma \qquad \forall \tilde{n}\in\mathcal{N}_h, \tag{3.3}$$

which is a linear equation system for the $\text{card}(\mathcal{N}_h)$ unknown nodal values $y_h(n)$.

3.2 Error estimates for projection operators

This section is devoted to error estimates for several projection operators onto spaces of piecewise polynomials on $\mathcal{T}_h$ or $\mathcal{E}_h$, which have a local representation on each element or some patch of elements. The estimates we prove will exploit the accurate regularity results in weighted Sobolev spaces.



(a) Reference element

(b) Element touching an edge and a corner

(c) Element touching an edge in a single point

Figure 3.1: The reference element $\hat{T}$ and the different positions of the original element T.

A well-known idea is to transform the tetrahedron/triangle to a reference element where all estimates and embeddings depend only on the geometry of this element, but not on $\mathcal{T}_h$. In what follows we investigate the reference transformation in detail. To this end, let $\hat{T}$ be the standard reference triangle/tetrahedron having vertices

$$\mathcal{N}_{\hat{T}} := \begin{cases} \{(0,0)^\top,(1,0)^\top,(0,1)^\top\}, & \text{for } n=2,\\ \{(0,0,0)^\top,(1,0,0)^\top,(0,1,0)^\top,(0,0,1)^\top\}, & \text{for } n=3.\end{cases}$$

We denote by F_T the affine linear transformation from $\hat{T}$ to a world element $T\in\mathcal{T}_h$. This transformation allows us to associate to each function $u\colon T\to\mathbb{R}$ another one on the reference element, namely $\hat{u}\colon\hat{T}\to\mathbb{R}$ defined by

$$\hat{u}(\hat{x}) := u(F_T(\hat{x})).$$

From [27, Theorem 15.1] it is known that the estimates

$$|u|_{W^{\ell,p}(T)} \le c h_T^{-\ell} |T|^{1/p} |\hat{u}|_{W^{\ell,p}(\hat{T})}, \tag{3.4}$$

$$|\hat{u}|_{W^{\ell,p}(\hat{T})} \le c h_T^{\ell} |T|^{-1/p} |u|_{W^{\ell,p}(T)}, \tag{3.5}$$

hold for all $u \in W^{\ell,p}(T)$, $\ell \in \mathbb{N}_0$, $p \in [1,\infty]$.

Next, we introduce weighted Sobolev spaces on a reference setting. The definition is analogous to $W^{\ell,p}_{\vec{\alpha}}(\Omega)$ or $W^{\ell,p}_{\vec{\beta},\vec{\delta}}(\Omega)$ with slight modifications of the weight functions. First, we make the following general assumption.

Assumption 1. *Each element $T \in \mathcal{T}_h$ touches at most one corner of Ω. For three-dimensional problems it is assumed that at most one edge of an element T is contained in an edge of Ω.*

This assumption is not too restrictive as elements violating this condition only have to be refined, e. g. by bisection. However, this assumption is not essential for the results in this thesis, but it simplifies the notation significantly as it suffices to introduce only one corner and one edge weight on the reference element.

For planar problems we introduce the space $W^{\ell,p}_{\alpha}(\hat{T})$ with $\ell \in \mathbb{N}_0$, $p \in [1,\infty]$ and $\alpha \in \mathbb{R}$, which is defined analogous to Definition 2.3.1 with the modification that there is only one weight, namely

$$\hat{r}(\hat{x}) := |\hat{x}|.$$

Note that it is possible to define F_T such that $F_T(0) = x^{(j)}$ if $x^{(j)}$, $j \in \mathcal{C}$, is also a corner of T. A norm in $W^{\ell,p}_{\alpha}(\hat{T})$ is defined by

$$\|\hat{u}\|_{W^{\ell,p}_{\alpha}(\hat{T})} := \begin{cases} \left(\sum_{|\boldsymbol{\alpha}| \le \ell} \int_{\hat{T}} \hat{r}(\hat{x})^{p\alpha} |\hat{D}^{\boldsymbol{\alpha}} \hat{u}(\hat{x})|^p \mathrm{d}\hat{x} \right)^{1/p}, & \text{if } p \in [1,\infty), \\ \sum_{|\boldsymbol{\alpha}| \le \ell} \operatorname*{ess\,sup}_{\hat{x} \in \hat{T}} \hat{r}(\hat{x})^{\alpha} |\hat{D}^{\boldsymbol{\alpha}} \hat{u}(\hat{x})|, & \text{if } p = \infty. \end{cases}$$

A seminorm is defined analogous to (2.18) when taking the sum only over all $|\boldsymbol{\alpha}| = \ell$.

For three-dimensional problems we need the weight functions

$$\hat{\rho}(\hat{x}) := |\hat{x}|, \qquad \hat{r}(\hat{x}) := \inf_{\hat{y} \in \hat{M}} |\hat{x} - \hat{y}|,$$

where $\hat{M} := \{t\boldsymbol{e}_3 \colon t \in (0,1)\}$ is the reference edge. If Assumption 1 did not hold, e. g. when two edges of T were contained in edges of Ω, we would have to introduce two edge weights on the reference setting. This is in principal possible but we exclude this for simplification purposes. In order to maintain consistency with the notation we assume that the reference transformation $F_T \colon \hat{T} \to T$ is defined in such a way that the following properties hold:

- If T has an edge which is contained in an edge M_k, $k \in \mathcal{E}$, we assume that $F_T(\hat{M}) \subset M_k$.
- If T has a corner which is also a corner $x^{(j)}$, $j \in \mathcal{C}$, of Ω, we assume that $F_T(0) = x^{(j)}$.
- If T touches an edge M_k, $k \in \mathcal{E}$, in a single point, we assume that $F_T(0) \in M_k$.

Analogous to Definition 2.3.2 we define for $\ell \in \mathbb{N}_0$, $p \in [1,\infty]$ and $\beta, \delta \in \mathbb{R}$ the space $W^{\ell,p}_{\beta,\delta}(\hat{T})$ equipped with the norm

$$\|\hat{u}\|_{W^{\ell,p}_{\beta,\delta}(\hat{T})} := \begin{cases} \left(\sum_{|\alpha|\le\ell} \int_{\hat{T}} \hat{\rho}(\hat{x})^{p(\beta-\ell+|\alpha|)} \left(\frac{\hat{r}}{\hat{\rho}}(\hat{x})\right)^{p\delta} |\hat{D}^\alpha \hat{u}(\hat{x})|^p \mathrm{d}\hat{x} \right)^{1/p}, & \text{if } p \in [1,\infty), \\ \sum_{|\alpha|\le\ell} \operatorname*{ess\,sup}_{\hat{x}\in\hat{T}} \hat{\rho}(\hat{x})^{\beta-\ell+|\alpha|} \left(\frac{\hat{r}}{\hat{\rho}}(\hat{x})\right)^{\delta} |\hat{D}^\alpha \hat{u}(\hat{x})|. & \text{if } p = \infty. \end{cases}$$

Let us briefly discuss the relation between the weights in the reference setting and the original weights. For two-dimensional problems it is clear that

$$r(x) \sim h_T \hat{r}(\hat{x}) \qquad \forall x \in T, \tag{3.6}$$

when T touches the corner $x^{(j)}$.

For three-dimensional problems we have to distinguish among certain situations. For the case illustrated in Figure 3.1b that one edge of T is contained in an edge M of Ω, we define the points

$$y = \arg\min_{z\in M} |x - z|, \qquad \hat{y} = \arg\min_{\hat{z}\in\hat{M}} |\hat{x} - \hat{z}|.$$

Note that y and $F_T(\hat{y})$ are in general different points. For the transformation from $\hat{T}$ to T we have to exploit the property

$$r(x) = |x - y| \sim h_T |\hat{x} - F_T^{-1}(y)| \sim h_T |\hat{x} - \hat{y}| = h_T \hat{r}(\hat{x}), \tag{3.7}$$

where the second equivalence holds due to the assumed *shape regularity* of $\mathcal{T}_h$. Moreover, if T touches also the corner $\boldsymbol{c} := x^{(j)}$, we observe that

$$\rho(x) = |x - \boldsymbol{c}| \sim h_T |\hat{x} - \hat{\boldsymbol{c}}| = h_T \hat{\rho}(\hat{x}), \tag{3.8}$$

since $\boldsymbol{c} = F_T(\hat{\boldsymbol{c}})$. The case illustrated in Figure 3.1c where T touches the edge M only in a single point $\boldsymbol{c}$ is treated slightly different. Due to the assumed shape regularity the angles between the edge M and the edges of T touching $\boldsymbol{c}$ are bounded from below by a constant independent of h. Exploiting this fact yields the property

$$r(x) = |x - y| \sim |x - \boldsymbol{c}| \sim h_T |\hat{x} - \hat{\boldsymbol{c}}| = h_T \hat{\rho}(\hat{x}), \tag{3.9}$$

and the edge weight becomes a corner weight in the reference setting. Note, that we did not consider weights related to corners and edges that are not touched by T, since these weights are not needed for our analysis.

The technique we use to derive local estimates is in most cases the same. We introduce some polynomial $w \in \mathcal{P}_k$ of degree $k \in \mathbb{N}_0$ which is preserved by the projection operator, apply stability properties for this operator and insert results of polynomial approximation theory. In classical Sobolev spaces the Deny-Lions type arguments (also known as Bramble-Hilbert-Lemma) from Dupont and Scott [38] are sufficient for our purposes. In weighted Sobolev spaces we will use the following analogue.

Lemma 3.2.1. *Let a positive integer $\ell \in \mathbb{N}$ and some $q \in (1,\infty)$ be given. There exists some polynomial $p \in \mathcal{P}_{\ell-1}(\hat{T})$ such that the following results hold.*

a) $(n=2)$. Let some function $v \in W^{\ell,q}_{\vec{\alpha}}(\hat{T})$ with weights $\vec{\alpha} \in \mathbb{R}^d$ be given such that

$$-2/q < \alpha_j < 3 - 2/q \qquad \forall j \in \mathcal{C}. \tag{3.10}$$

Then,

$$\|v - p\|_{W^{\ell,q}_{\vec{\alpha}}(\hat{T})} \le c|v|_{W^{\ell,q}_{\vec{\alpha}}(\hat{T})}.$$

b) $(n=3)$. Let $v \in W^{\ell,q}_{\vec{\beta},\vec{\delta}}(\hat{T})$ be some function with weights $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ satisfying

$$\begin{aligned} \beta_j &< 4 - 3/q \qquad \forall j \in \mathcal{C}, \\ -2/p < \delta_k &< 3 - 2/q \qquad \forall k \in \mathcal{E}. \end{aligned} \tag{3.11}$$

Then,

$$\|v - p\|_{W^{\ell,q}_{\vec{\beta},\vec{\delta}}(\hat{T})} \le c|v|_{W^{\ell,q}_{\vec{\beta},\vec{\delta}}(\hat{T})}.$$

The constant c depends solely on $\hat{T}$.

Proof. The assertion for $n=2$ can be found in [74, Lemma 2.30] and [13, Lemma 2.2] where the latter reference contains a detailed proof, but the assumptions upon the weights are too restrictive. To show the extension to $n=3$ we merely mimic the ideas therein. Actually, it suffices to show that the norm equivalence

$$\|u\|_{W^{\ell,q}_{\vec{\beta},\vec{\delta}}(\hat{T})} \sim |u|_{W^{\ell,q}_{\vec{\beta},\vec{\delta}}(\hat{T})} + \sum_{|\boldsymbol{\alpha}|\le \ell-1} \left| \int_{\hat{T}} D^{\boldsymbol{\alpha}} u \right|, \tag{3.12}$$

holds, and to insert $u = v - p$ with some $p \in \mathcal{P}_{\ell-1}(\hat{T})$ such that the second term on the right-hand side vanishes. The proof of (3.12) presented in [13] can be extended to the case $n=3$ since the key steps are the embeddings

$$W^{\ell,q}_{\vec{\beta},\vec{\delta}}(\hat{T}) \overset{c}{\hookrightarrow} W^{\ell-1,q}_{\vec{\beta},\vec{\delta}}(\hat{T}) \quad \text{and} \quad W^{1,q}_{\vec{\beta},\vec{\delta}}(\hat{T}) \hookrightarrow W^{1,1}_{\vec{1},\vec{1}}(\hat{T}) \hookrightarrow L^1(\hat{T}). \tag{3.13}$$

The first embedding is stated in part two of Lemma 2.3.4 and holds under the assumption $-2/q < \delta_k$ for all $k \in \mathcal{E}$. The second embedding in (3.13) is also a consequence of Lemma 2.3.4 and holds under the assumption (3.11). □

The local estimates we are going to derive in the remainder of this section will always depend upon the position of the element T. Therefore, we introduce the quantities

$$r_{j,T} := \inf_{x\in T} |x^{(j)} - x|, \qquad r_T := \min_{j\in\mathcal{C}} r_{j,T},$$

for $j \in \mathcal{C}$ if Ω is a polygonal domain, and

$$\rho_{j,T} := \inf_{x\in T} |x^{(j)} - x|, \qquad r_{k,T} := \inf_{\substack{x\in T \\ y\in M_k}} |y - x|, \qquad r_T := \min_{k\in\mathcal{E}} r_{k,T},$$

for $j \in \mathcal{C}$ and $k \in \mathcal{E}$ if Ω is a polyhedral domain. In all subsequent estimates the generic constant c is independent of these quantities.

3.2.1 The nodal interpolant I_h

The nodal interpolant maps a given function u to a discrete function in V_h such that the interpolant coincides with u in the nodes of $\mathcal{T}_h$. Hence, for an arbitrary continuous function we can define the nodal interpolant by

$$I_h \colon C(\overline{\Omega}) \to V_h, \qquad [I_h u](x) := \sum_{n \in \mathcal{N}_h} u(n)\varphi_n(x). \tag{3.14}$$

This operator can also be defined element-wise which makes the derivation of global estimates easier. The definition (3.14) is equivalent to

$$[I_h u](x) = \sum_{n \in \mathcal{N}_T} u(n)\varphi_n(x) \qquad \text{if } x \in \overline{T}.$$

It is also possible to define the nodal interpolant on the reference element by

$$[\hat{I}_h \hat{u}](\hat{x}) = \sum_{\hat{n} \in \mathcal{N}_{\hat{T}}} \hat{u}(\hat{n})\hat{\varphi}_{\hat{n}}(\hat{x}) \qquad \text{if } \hat{x} \in \operatorname{cl} \hat{T}.$$

As a direct consequence of this definition one can show the following stability property:

Lemma 3.2.2 (Stability of I_h)**.** *Let some function $u \in C(\operatorname{cl} \hat{T})$ be given. Then, the stability estimate*

$$|\hat{I}_h \hat{u}|_{W^{k,p}(\hat{T})} \le c \|\hat{u}\|_{L^\infty(\hat{T})}$$

holds for arbitrary $k \in \{0, 1\}$, $p \in [1, \infty]$.

Proof. With the definition of $\hat{I}_h$, the triangle inequality and the Hölder inequality (2.2) we obtain for arbitrary $|\boldsymbol{\alpha}| \le 1$ that

$$\|D^{\boldsymbol{\alpha}} \hat{I}_h \hat{u}\|^p_{L^p(\hat{T})} \le \sum_{\hat{n} \in \mathcal{N}_{\hat{T}}} \|\hat{u}(\hat{n}) D^{\boldsymbol{\alpha}} \varphi_{\hat{n}}\|^p_{L^p(\hat{T})} \le \|\hat{u}\|^p_{L^\infty(\hat{T})} \sum_{\hat{n} \in \mathcal{N}_{\hat{T}}} \int_{\hat{T}} (D^{\boldsymbol{\alpha}} \hat{\varphi}_{\hat{n}}(\hat{x}))^p \, \mathrm{d}\hat{x}.$$

Moreover, one observes that

$$\int_{\hat{T}} (D^{\boldsymbol{\alpha}} \hat{\varphi}_{\hat{n}}(\hat{x}))^p \, \mathrm{d}\hat{x} \le c \quad \text{with} \quad c = c(\hat{T}),$$

taking the definition of the basis functions $\hat{\varphi}_{\hat{n}}$ into account. □

Due to the stability in in the maximum norm the nodal interpolant is a suitable choice when deriving error estimates in $L^\infty(\Omega)$.

Lemma 3.2.3. *Let some function $u \in C(\overline{T})$ on an element $T \in \mathcal{T}_h$ with $T \subset U_j$ for some $j \in \mathcal{C}$ be given. Assume that $|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T)} \le c$, with weights satisfying*

$$0 \le \beta_j < 2, \qquad 0 \le \delta_k < 5/3, \quad \forall k \in X_j.$$

Then the error estimate

$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^2 |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T)} \cdot \begin{cases} \rho_{j,T}^{-\beta_j} \prod\limits_{k\in X_j} \left(\dfrac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k}, & \text{if } \rho_{j,T} > 0,\ r_{k,T} > 0\ (\forall k \in X_j), \\ h_T^{-\delta_k} \rho_{j,T}^{\delta_k - \beta_j}, & \text{if } \rho_{j,T} > 0,\ r_{k,T} = 0, \\ h_T^{-\beta_j}, & \text{if } \rho_{j,T} = 0, \end{cases} \tag{3.15}$$

holds. Furthermore, let $\kappa := \max\{\beta_j, \max_{k\in X_j} \delta_k\}$ *denote the largest weight. Then the estimate above simplifies to*

$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^2 |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T)} \cdot \begin{cases} r_T^{-\kappa}, & \text{if } \quad r_T > 0, \\ h_T^{-\kappa}, & \text{if } \quad r_T = 0. \end{cases} \tag{3.16}$$

Proof. We consider first the case that T is away from the singular points which implies $u \in W^{2,\infty}(T)$. Hence, a standard interpolation error estimate can be applied and introducing the weights afterwards yields

$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^2 \rho_{j,T}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T)}. \tag{3.17}$$

Consider now the case that $r_T = 0$. In this case, standard interpolation error estimates cannot be applied. We introduce a first-order polynomial $w \in \mathcal{P}_1(T)$ and apply the transformation to the reference element, the stability estimate of Lemma 3.2.2, as well as the embedding $W^{1,p}(\hat T) \hookrightarrow L^\infty(\hat T)$ for some $p > 3$, and obtain

$$\|u - I_h u\|_{L^\infty(T)} \le \|\hat u - \hat w\|_{L^\infty(\hat T)} + \|\hat I_h(\hat u - \hat w)\|_{L^\infty(\hat T)} \le c\|\hat u - \hat w\|_{W^{1,p}(\hat T)}. \tag{3.18}$$

First, we consider the case that T touches the edge M_k, $k \in \mathcal{E}$, and is away from the corner point, i.e. $\rho_{j,T} > 0$. We distinguish among the cases whether T touches M_k in an edge of T (see Figure 3.1b) or in a single point (see Figure 3.1c). In the first case we use the embedding $W^{2,p}_{1,1}(\hat T) \hookrightarrow W^{1,p}(\hat T)$ from Lemma 2.3.4, the Bramble-Hilbert type argument in weighted Sobolev spaces presented in Lemma 3.2.1, and the embedding $W^{0,\infty}_{\delta_k,\delta_k}(\hat T) \hookrightarrow W^{0,p}_{1,1}(\hat T)$ which holds for $\delta_k < 5/3$ and the choice $p = 3+\varepsilon$ when ε is chosen sufficiently small, and arrive at

$$\|u - I_h u\|_{L^\infty(T)} \le c\|\hat u - \hat w\|_{W^{2,p}_{1,1}(\hat T)} \le c|\hat u|_{W^{2,p}_{1,1}(\hat T)} \le c|\hat u|_{W^{2,\infty}_{\delta_k,\delta_k}(\hat T)} \tag{3.19}$$

For the transformation back to the original element T we take (3.7) into account and finally get

$$\|u - I_h u\|_{L^\infty(T)} \le c h_T^{2-\delta_k} \rho_{j,T}^{\delta_k - \beta_j} |u|_{W^{2,\infty}_{\vec\beta,\vec\delta}(T)}, \tag{3.20}$$

where we already inserted the remaining weights in the last step, and used the fact that $\rho_{j,T} > 0$. If T touches M_k only in a single point we replace in (3.19) the spaces $W^{2,p}_{1,1}(\hat T)$ by $W^{2,p}_{1,0}(\hat T)$ and $W^{2,\infty}_{\delta_k,\delta_k}(\hat T)$ by $W^{2,\infty}_{\delta_k,0}(\hat T)$ and use (3.9) instead of (3.7). It is easy to confirm that we arrive at (3.20) again.

Let now T touch additionally the corner $x^{(j)}$ and let an edge of T be contained in M_k, $k \in X_j$. The other edges M_ℓ, $\ell \in X_j \setminus \{k\}$, meeting in $x^{(j)}$ can be neglected, as T touches them only

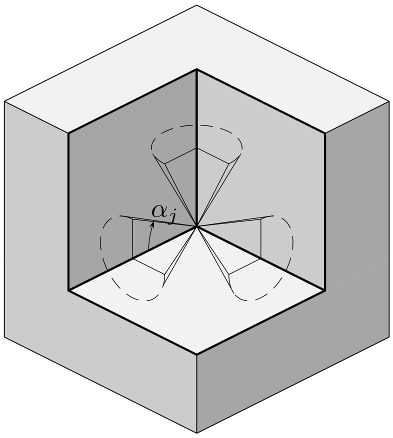


Figure 3.2: Definition of the cones $C_k^{\alpha_j}$ at a reentrant corner.

in $x^{(j)}$. From (3.8) and (3.9) we conclude for these edges $1 = \hat{\rho}/\hat{\rho} \sim r_\ell/\rho_j$. We consider again (3.18), employ the embedding from Lemma 2.3.4 with $\beta_j < 2$ and $\delta_k < 5/3$, and obtain using (3.7) and (3.8) the estimate

$$\|\hat{u} - \hat{w}\|_{W^{1,p}(\hat{T})} \le c|\hat{u}|_{W^{2,p}_{1,1}(\hat{T})} \le c|\hat{u}|_{W^{2,\infty}_{\beta_j,\delta_k}(\hat{T})} \le ch_T^{2-\beta_j}|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T)}.$$

In the last step we merely inserted the remaining weights which are bounded on T. If no edge of T coincides with an edge of Ω we can derive the same estimate when replacing $W^{2,p}_{1,1}(\hat{T})$ and $W^{2,\infty}_{\beta_j,\delta_k}(\hat{T})$ by $W^{2,p}_{1,0}(\hat{T})$ and $W^{2,\infty}_{\beta_j,0}(\hat{T})$, respectively. The edge weights can be inserted afterwards where we exploit that r_k/ρ_j, $k \in X_j$ is bounded from below by a positive constant within T which is a consequence of the assumed shape-regularity. After insertion into (3.18) we arrive at

$$\|u - I_h u\|_{L^\infty(T)} \le ch_T^{2-\beta_j}|u|_{W^{2,\infty}_{\vec{\beta},\vec{\delta}}(T)},$$

and have proved the estimate (3.15) completely.

Let us now investigate how we can deduce (3.16) from (3.15). For the case $r_T > 0$ we simplify the factors $r_{k,T}$ and $\rho_{j,T}$ appropriately. To this end, we introduce the following definitions. We denote the interior angle between the edges M_k and M_ℓ by $\alpha_{k,\ell}$ and write $\alpha_j := \frac{1}{4}\min_{k,\ell \in X_j} \alpha_{k,\ell}$ for the quarter of the minimal angle between all edges having an endpoint in $x^{(j)}$. We define some cones $C_k^{\alpha_j}$, $k \in X_j$, also illustrated in Figure 3.2, by

$$C_k^{\alpha_j} := \{x \in U_j \cap \Omega \colon r_k(x)/\rho_j(x) \le \sin\alpha_j\}.$$

Outside of the cone $C_k^{\alpha_j}$, the angular distance $r_k(x)/\rho_j(x)$ is then bounded from below by a constant depending only on the angles between the edges. If $T \cap C_k^{\alpha_j} = \emptyset$ for all $k \in X_j$, the angular distances to all edges are bounded from below by $\sin\alpha_j$ and we consequently get

$$\rho_{j,T}^{-\beta_j} \prod_{k \in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k} \le c\rho_{j,T}^{-\beta_j} \le cr_T^{-\kappa},$$

provided that $\delta_k \ge 0$, $k \in X_j$. In case of $T \cap C_k^{\alpha_j} \ne \emptyset$ for some $k \in X_j$ we have $r_T = r_{k,T}$. Since the angular distances to the other edges M_ℓ, $\ell \in X_j \backslash \{k\}$, are again bounded from below we arrive at

$$\rho_{j,T}^{-\beta_j} \prod_{\ell \in X_j} \left(\frac{r_{\ell,T}}{\rho_{j,T}}\right)^{-\delta_\ell} \leq c \rho_{j,T}^{\delta_k - \beta_j} r_{k,T}^{-\delta_k}.$$

In case of $\delta_k \geq \beta_j$ we have $\rho_{j,T}^{\delta_k-\beta_j} \leq c$, otherwise we exploit $r_{k,T} \leq \rho_{j,T}$ and arrive at $\rho_{j,T}^{\delta_k-\beta_j} r_{k,T}^{-\delta_k} \leq c r_T^{-\beta_j}$. Hence, for both cases we get

$$\rho_{j,T}^{-\beta_j} \prod_{k \in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k} \leq c r_T^{-\kappa}. \tag{3.21}$$

Inserting this into (3.17) yields the desired estimate (3.16) for $r_T > 0$. To obtain the estimate (3.16) in case of $r_T = 0$ we show that

$$h_T^{-\delta_k} \rho_{j,T}^{\delta_k - \beta_j} \leq c h_T^{-\kappa} \tag{3.22}$$

holds, which follows trivially in case of $\delta_k \geq \beta_j$, and otherwise, this is a consequence of $\rho_{j,T} \geq c h_T$. □

It is also possible to define the nodal interpolant on the boundary Γ based on its triangulation $\mathcal{E}_h$. The space of continuous and piecewise linear functions on Γ is denoted by

$$V_h^\partial := \{v_h \in C(\Gamma) \colon v_h \text{ is affine linear on all } E \in \mathcal{E}_h\},$$

and we investigate the interpolant I_h^∂ defined by

$$I_h^\partial \colon C(\Gamma) \to V_h^\partial, \qquad [I_h^\partial u](x) := \sum_{n \in \mathcal{N}_h \cap \Gamma} u(n) \varphi_n(x) \qquad \forall x \in \Gamma. \tag{3.23}$$

In the following Lemma we derive a local interpolation error estimate for two-dimensional problems. Consequently, Γ is a one-dimensional manifold in $\mathbb{R}^2$.

Lemma 3.2.4. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let $E \in \mathcal{E}_h$ be a boundary element such that $E \subset U_j \cap \Gamma$ for some $j \in \mathcal{C}$. Assume that $u \in W^{2,p}_\gamma(E)$ with arbitrary $p \in [2, \infty]$ and*

$$0 \leq \gamma < \begin{cases} 2 - 1/p, & \text{if } \ell = 0, \\ 3/2 - 1/p, & \text{if } \ell = 1, \end{cases}$$

For $\ell \in \{0, 1\}$ the local estimate

$$\|u - I_h^\partial u\|_{H^\ell(E)} \leq c h_E^{2-\ell} |E|^{1/2-1/p} |u|_{W^{2,p}_\gamma(E)} \begin{cases} r_E^{-\gamma}, & \text{if } r_E > 0, \\ h_E^{-\gamma}, & \text{if } r_E = 0, \end{cases}$$

holds, where $r_E := \operatorname{dist}(E, x^{(j)})$.

Proof. In case of $r_E > 0$ we may exploit higher regularity, namely $u \in W^{2,p}(E)$. An application of standard estimates and insertion of the weight afterwards yields

$$\|u - I_h^\partial u\|_{H^\ell(E)} \leq c h_E^{2-\ell} |E|^{1/2-1/p} |u|_{W^{2,p}(E)} \leq c h_E^{2-\ell} |E|^{1/2-1/p} r_E^{-\gamma} |u|_{W^{2,p}_\gamma(E)}.$$

Otherwise, if $r_E = 0$, we apply the transformation to the reference interval $\hat{E} := (0,1)$, insert a first-order polynomial $\hat{w} \in \mathcal{P}_1(\hat{E})$, exploit the stability of I_h^∂ in $L^\infty(\hat{E})$ (see Lemma 3.2.2), and the embeddings

$$W_\delta^{2,2}(\hat{E}) \hookrightarrow W_1^{2,1+\varepsilon}(\hat{E}) \hookrightarrow W^{1,1+\varepsilon}(\hat{E}) \hookrightarrow L^\infty(\hat{E}) \hookrightarrow L^2(\hat{E}), \quad \text{if } \ell = 0 \text{ with } \delta = 3/2 - \varepsilon,$$
$$W_\delta^{2,2}(\hat{E}) \hookrightarrow H^1(\hat{E}) \hookrightarrow L^\infty(\hat{E}), \quad \text{if } \ell = 1 \text{ with } \delta = 1.$$

Then we arrive at

$$\begin{aligned} \|u - I_h^\partial u\|_{H^\ell(E)} &\le c h_E^{-\ell} |E|^{1/2} \left(\|\hat{u} - \hat{w}\|_{H^\ell(\hat{E})} + \|\hat{u} - \hat{w}\|_{L^\infty(\hat{E})} \right) \\ &\le c h_E^{-\ell} |E|^{1/2} \|\hat{u} - \hat{w}\|_{W_\delta^{2,2}(\hat{E})}. \end{aligned}$$

With the Deny-Lions-type argument in weighted Sobolev spaces from Lemma 3.2.1, and the embedding $W_\gamma^{0,p}(\hat{E}) \hookrightarrow W_\delta^{0,2}(\hat{E})$ which holds for $\gamma < \delta + 1/2 - 1/p$, we get

$$\|\hat{u} - \hat{w}\|_{W_\delta^{2,2}(\hat{E})} \le c |\hat{u}|_{W_\gamma^{2,p}(\hat{E})},$$

and the assertion follows from the transformation back to E exploiting the property (3.6). □

3.2.2 The quasi-interpolant Z_h

The quasi-interpolant Z_h first introduced by Scott and Zhang [82] has two significant advantages over the nodal interpolant. On the one hand it is applicable for non-smooth functions $u \in W^{\ell,p}(\Omega)$ with

$$\ell \ge 1 \quad \text{if} \quad p = 1, \qquad \ell > 1/p \quad \text{if} \quad p > 1, \tag{3.24}$$

and, on the other hand that it possesses better stability properties as we will see in Lemma 3.2.5. We follow the definition from [82] and associate a set $\sigma_n \subset \Omega$ to each node $n \in \mathcal{N}_h$ according to the following rules:

a) If $n \in \mathcal{N}_h$ is an interior node, then $\sigma_n := T \in \mathcal{T}_h$ such that $n \in \mathcal{N}_T$.

b) If $n \in \mathcal{N}_h$ is a boundary node, then $\sigma_n := E \in \mathcal{E}_h$ such that $n \in \overline{E}$ and $E \subset \Gamma$.

Let $\Pi_{\sigma_n} \colon L^1(\sigma_n) \to \mathcal{P}_1(\sigma_n)$ denote the $L^2(\sigma_n)$-projection onto the space of first-order polynomials on σ_n. Note that this projection is well-defined for functions $u \in W^{\ell,p}(\Omega)$ with ℓ, p satisfying (3.24). The operator Z_h is then defined by

$$Z_h \colon W^{\ell,p}(\Omega) \to V_h, \qquad [Z_h u](x) := \sum_{n \in \mathcal{N}_h} (\Pi_{\sigma_n} u)(n) \varphi_n(x). \tag{3.25}$$

In contrast to I_h it is not possible to define Z_h locally on a single element $T \in \mathcal{T}_h$, but on some patch S_T defined by

$$S_T := \operatorname{int} \bigcup_{\substack{T' \in \mathcal{T}_h \\ \overline{T}' \cap \overline{T} \neq \emptyset}} \overline{T'}.$$

For estimates on a reference setting we introduce the patch $S_{\hat{T}} := F_T^{-1}(S_T)$. Note, that the patch $S_{\hat{T}}$ has diameter $h_{S_{\hat{T}}} = \mathcal{O}(1)$ and contains a ball of radius $\rho_{S_{\hat{T}}} = \mathcal{O}(1)$.

The following stability estimate for Z_h is proved in [82]:

Lemma 3.2.5. *Let some function $u \in W^{\ell,p}(S_T)$ with $\ell \in \mathbb{N}$ and $p \in [1,\infty]$ be given satisfying (3.24). Then, the stability estimate*

$$\|\hat{Z}_h\hat{u}\|_{W^{k,q}(\hat{T})} \leq c\|\hat{u}\|_{W^{\ell,p}(S_{\hat{T}})}. \tag{3.26}$$

holds for all $k \in \mathbb{N}_0$ and $q \in [1,\infty]$.

With this stability estimate we can prove an interpolation error estimate based on the technique we already used in the proof of Lemma 3.2.3.

Lemma 3.2.6. *Let $T \in \mathcal{T}_h$ with $T \subset U_j$ for some $j \in \mathcal{C}$. Assume that the function $u \in H^1(S_T)$ satisfies $|u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \leq c$ for some $p \in (6/5,\infty]$ and weights satisfying*

$$0 \leq \beta_j < 5/2 - 3/p, \qquad 0 \leq \delta_k < 5/3 - 2/p, \quad \forall k \in X_j.$$

Then, for $\ell \in \{0,1\}$, the interpolation error estimate

$$|u - Z_h u|_{H^\ell(T)} \leq$$
$$c h_T^{2-\ell}|T|^{1/2-1/p}|u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \cdot \begin{cases} \rho_{j,T}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k}, & \text{if } \rho_{j,S_T} > 0,\ r_{k,S_T} > 0\ (\forall k \in X_j), \\ h_T^{-\delta_k}\rho_{j,T}^{\delta_k-\beta_j}, & \text{if } r_{k,S_T} = 0,\ \rho_{j,S_T} > 0, \\ h_T^{-\beta_j}, & \text{if } \rho_{j,S_T} = 0, \end{cases} \tag{3.27}$$

holds. Moreover, with $\kappa := \max\{\beta_j, \max_{k\in X_j}\delta_k\}$ this estimate simplifies to

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell}|T|^{1/2-1/p}|u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \cdot \begin{cases} r_T^{-\kappa}, & \text{if } r_{S_T} > 0, \\ h_T^{-\kappa}, & \text{if } r_{S_T} = 0. \end{cases} \tag{3.28}$$

Proof. If the patch S_T is away from the singular points we have higher regularity, more precisely $u \in W^{2,p}(S_T)$, and hence, we may apply the interpolation error estimates in classical Sobolev spaces from [82], and introduce the weights afterwards. This leads to

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell}|T|^{1/2-1/p}\rho_{j,T}^{-\beta_j}\prod_{k\in X_j}\left(\frac{r_{k,T}}{\rho_{j,T}}\right)^{-\delta_k}|u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)} \tag{3.29}$$

If $r_{S_T} = 0$ we introduce a first-order polynomial $w \in \mathcal{P}_1(S_T)$ and obtain with the triangle inequality

$$|u - Z_h u|_{H^\ell(T)} \leq |u - w|_{H^\ell(T)} + |Z_h(u - w)|_{H^\ell(T)}. \tag{3.30}$$

For the first part of (3.30) the transformation to the reference element and the trivial embedding $H^1(\hat{T}) \hookrightarrow L^2(\hat{T})$ yield

$$|u - w|_{H^\ell(T)} \leq c\,|T|^{1/2}h_T^{-\ell}|\hat{u} - \hat{w}|_{H^\ell(\hat{T})} \leq c\,|T|^{1/2}h_T^{-\ell}\|\hat{u} - \hat{w}\|_{H^1(\hat{T})}.$$

For the second part on the right-hand side of (3.30) we apply an inverse inequality, the transformation to the reference element, and the stability of Z_h in $H^1(S_{\hat{T}})$ (compare Lemma 3.2.5), and obtain

$$\begin{aligned}|Z_h(u-w)|_{H^\ell(T)} &\le ch_T^{-\ell}\|Z_h(u-w)\|_{L^2(T)}\\ &\le c|T|^{1/2}h_T^{-\ell}\|\hat{Z}_h(\hat{u}-\hat{w})\|_{L^2(\hat{T})}\\ &\le c|T|^{1/2}h_T^{-\ell}\|\hat{u}-\hat{w}\|_{H^1(S_{\hat{T}})}.\end{aligned}$$

Together with the embedding $W^{2,6/5}(S_{\hat{T}}) \hookrightarrow H^1(S_{\hat{T}})$, the estimate (3.30) simplifies to

$$|u-Z_hu|_{H^\ell(T)} \le ch_T^{-\ell}|T|^{1/2}\|\hat{u}-\hat{w}\|_{W^{2,6/5}(S_{\hat{T}})}. \tag{3.31}$$

Now, we employ a Deny-Lions type argument, e. g. the version from Theorem 3.2 in [38] where the estimate depends only on $h_{S_{\hat{T}}}$ and $\rho_{S_{\hat{T}}}$. After the transformation back to S_T we arrive at

$$|u-Z_hu|_{H^\ell(T)} \le ch_T^{-\ell}|T|^{1/2}|\hat{u}|_{W^{2,6/5}(S_{\hat{T}})} \le ch_T^{2-\ell}|T|^{1/2-5/6}|u|_{W^{2,6/5}(S_T)}, \tag{3.32}$$

where $|S_T| \sim |T|$ was exploited in the last step. Henceforth, we have to distinguish among the cases whether S_T touches a corner or only a single edge.

We first consider the case that S_T touches the edge M_k for some $k \in X_j$, but is away from the corners. The Hölder inequality with $q := 5p/6$ and $1/q + 1/q' = 1$ yields

$$\begin{aligned}|u|^{6/5}_{W^{0,6/5}(S_T)} &= \int_{S_T} r_k(x)^{6\delta_k/5}|u(x)|^{6/5}r_k(x)^{-6\delta_k/5}\mathrm{d}x\\ &\le \left(\int_{S_T} r_k(x)^{p\delta_k}|u(x)|^p\mathrm{d}x\right)^{6/(5p)}\left(\int_{S_T} r_k(x)^{-q'6\delta_k/5}\mathrm{d}x\right)^{1/q'}.\end{aligned} \tag{3.33}$$

The second integral can be integrated exactly in cylinderical coordinates (r_k, φ_k, z_k) around M_k and is bounded if $2 - q'6\delta_k/5 > 0$. This condition is equivalent to $\delta_k < 5/3 - 2/p$ when inserting the definition of q and q'. As S_T is contained in a cylindrical sector around M_k having length and radius proportional h_T there exist constants $c_i > 0$, $i \in \{1,2,3\}$, such that

$$\begin{aligned}\left(\int_{S_T} r_k(x)^{-q'6\delta_k/5}\mathrm{d}x\right)^{5/(6q')} &\le \left(\int_{c_1}^{c_1+c_2h_T}\int_0^{c_3h_T} r_k^{1-q'6\delta_k/5}\mathrm{d}r_k\mathrm{d}z_k\right)^{5/(6q')}\\ &\le ch_T^{-\delta_k}|T|^{5/6-1/p},\end{aligned} \tag{3.34}$$

where we used $1/q' = 1 - 6/(5p)$ and $|T| \sim h_T^3$ in the last step. Inserting (3.33) with (3.34) into (3.32) leads to

$$|u-Z_hu|_{H^\ell(T)} \le ch_T^{2-\ell-\delta_k}|T|^{1/2-1/p}\rho_{j,T}^{\delta_k-\beta_j}|u|_{W^{2,p}_{\vec{\beta},\vec{\delta}}(S_T)}, \tag{3.35}$$

where we already inserted the remaining weights and exploited that S_T is away from the corner.

Let now S_T contain also the corner $\boldsymbol{c} := x^{(j)}$. Analogous to (3.33) we derive an embedding into an appropriate weighted Sobolev space and obtain using the Hölder inequality with $q := 5p/6$

$$
\begin{aligned}
|v|^{6/5}_{W^{0,6/5}(S_T)} &= \int_{S_T} \rho_j(x)^{6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{6\delta_k/5} |v(x)|^{6/5} \rho_j(x)^{-6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{-6\delta_k/5} \mathrm{d}x \\
&\le c \left(\int_{S_T} \rho_j(x)^{p\beta_j} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{p\delta_k} |v(x)|^p \mathrm{d}x \right)^{6/(5p)} \\
&\quad \times \left(\int_{S_T} \rho_j(x)^{-q'6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{-q'6\delta_k/5} \mathrm{d}x \right)^{1/q'} . \qquad (3.36)
\end{aligned}
$$

We introduce spherical coordinates $(\rho_j, \varphi_k, \vartheta_k)$, which are centered at $\boldsymbol{c}$ and coincide with the edge M_k for $\vartheta_k = 0$. This definition implies that $r_k/\rho_j = \sin(\vartheta_k)$. The integrals over ϑ_k are bounded by a constant independent of h_T under the condition $-q'6\delta_k/5 > -2$ which is implied by $-2/p < \delta_k < 5/3 - 2/p$ for all $k \in X_j$. Hence, a constant $c_1 > 0$ exists such that the second integral in (3.36) can be simplified to

$$
\begin{aligned}
\left(\int_{S_T} \rho_j(x)^{-q'6\beta_j/5} \prod_{k\in X_j} \left(\frac{r_k}{\rho_j}(x)\right)^{-q'6\delta_k/5} \mathrm{d}x \right)^{6/(5q')} &\le c \left(\int_0^{c_1 h_T} \rho_j^{2-q'6\beta_j/5} \mathrm{d}\rho_j \right)^{6/(5q')} \\
&\le c h_T^{-\beta_j} |T|^{5/6-1/p}, \qquad (3.37)
\end{aligned}
$$

provided that $-q'6\beta_j/5 > -3$ which is equivalent to the assumption $\beta_j < 5/2 - 3/p$. As a consequence, we get from (3.32) using (3.36) and (3.37) the estimate

$$
|u - Z_h u|_{H^\ell(T)} \le c h_T^{2-\ell-\beta_j} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec\beta,\vec\delta}(S_T)}, \qquad \text{if } \rho_{j,S_T} = 0,
$$

and have proven (3.27) completely.

The estimate (3.28) follows directly from (3.27) using the arguments from the proof of Lemma 3.2.3. We merely have to apply estimate (3.21) if $r_{S_T} > 0$, and (3.22) if $r_{S_T} = 0$, respectively. □

3.2.3 The $L^2(\Gamma)$-projection P_h^∂

In this section the $L^2(\Gamma)$-projection onto the finite-dimensional space

$$
U_h := \{ v_h \in L^\infty(\Gamma) \colon v_h \text{ is constant on each } E \in \mathcal{E}_h \} \qquad (3.38)
$$

is studied. Remember that $\mathcal{E}_h$ is a conforming triangulation of the boundary Γ. In this thesis only the two-dimensional case $\Omega \subset \mathbb{R}^2$ is considered. The $L^2(\Gamma)$-projection P_h^∂ defined by

$$
P_h^\partial \colon L^1(\Gamma) \to U_h, \qquad \int_\Gamma (u - P_h^\partial u) v_h \, \mathrm{d}s_x = 0 \quad \forall v_h \in U_h, \qquad (3.39)
$$

has also a local representation given by

$$
[P_h^\partial v](x) = |E|^{-1} \int_E u(t) \, \mathrm{d}s_t \qquad \text{if } x \in E,
$$

and preserves functions in U_h, i. e. $v_h = P_h^\partial v_h$ for all $v_h \in U_h$. On the reference element $\hat{E}$ we have accordingly

$$\hat{P}_h^\partial \hat{u} := |\hat{E}|^{-1} \int_{\hat{E}} \hat{u}(\hat{t}) \mathrm{d}s_{\hat{t}}.$$

From this definition the following stability estimate directly follows.

Lemma 3.2.7 (Stability of P_h^∂)**.** *Let some function $\hat{u} \in L^q(\hat{E})$ with $q \in [1, \infty]$ be given. Then the estimate*

$$\|\hat{P}_h^\partial \hat{u}\|_{L^p(\hat{E})} \le c\|\hat{u}\|_{L^q(\hat{E})}$$

holds for arbitrary $p \in [1, \infty]$.

As a consequence one can derive the following local estimate:

Lemma 3.2.8. *Let $E \in \mathcal{E}_h$ such that $E \subset U_j$ for some $j \in \mathcal{C}$, and a function $u \in W_\delta^{1,p}(E)$ with $p \in [2, \infty]$ and $\delta \in [0, 3/2 - 1/p)$ be given. Then, the estimate*

$$\|u - P_h^\partial u\|_{L^2(E)} \le c h_E |E|^{1/2-1/p} |u|_{W_\delta^{1,p}(E)} \cdot \begin{cases} r_E^{-\delta}, & \text{if} \quad r_E > 0, \\ h_E^{-\delta}, & \text{if} \quad r_E = 0, \end{cases}$$

holds.

Proof. The desired estimate in case of $r_E > 0$ is a direct consequence of the standard estimate

$$\|u - P_h^\partial u\|_{L^2(E)} \le c h_E |E|^{1/2-1/p} |u|_{W^{1,p}(E)}$$

and the property $|u|_{W^{1,p}(E)} \le r_E^{-\delta} |u|_{W_\delta^{1,p}(E)}$ which follows from Definition 2.3.1.

For elements touching a singular corner we introduce a constant $w \in \mathcal{P}_0$ and exploit the fact that $P_h^\partial w = w$. With the transformation to the reference element $\hat{E} := (0, 1)$, the stability estimate from Lemma 3.2.7, the embedding $W_1^{1,2}(E) \hookrightarrow L^2(E)$ from Lemma 2.3.3, and the Deny-Lions type argument from Lemma 3.2.1, we get

$$\begin{aligned} \|u - P_h^\partial u\|_{L^2(E)} &\le \|u - w\|_{L^2(E)} + \|P_h^\partial (u - w)\|_{L^2(E)} \le c|E|^{1/2} \|\hat{u} - \hat{w}\|_{L^2(\hat{E})} \\ &\le c|E|^{1/2} \|\hat{u} - \hat{w}\|_{W_1^{1,2}(\hat{E})} \le c|E|^{1/2} |\hat{u}|_{W_1^{1,2}(\hat{E})}. \end{aligned} \tag{3.40}$$

From the embedding stated in Lemma 2.3.3 we conclude

$$|\hat{u}|_{W_1^{1,2}(\hat{E})} \le c |\hat{u}|_{W_\delta^{1,p}(\hat{E})},$$

which holds for all $\delta < 3/2 - 1/p$. The desired estimate follows after the transformation back to E taking (3.6) into account. □

3.2.4 The midpoint interpolant R_h^∂

Another projection onto the set U_h defined in (3.38) is the *midpoint interpolant* sometimes also called 0-*interpolant*. We will require this interpolation operator to derive error estimates for the state variable of Neumann boundary control problems in $L^2(\Omega)$ on three-dimensional domains (see Section 4.2.1). Thus, we assume in this section that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain.

For each boundary element $E \in \mathcal{E}_h$ we denote the corresponding barycenter by x_E and define the interpolation operator R_h^∂ by

$$R_h^\partial\colon C(\Gamma) \to U_h, \qquad [R_h^\partial u]|_E := u(x_E) \quad E \in \mathcal{E}_h. \tag{3.41}$$

The corresponding analogue on the reference triangle $\hat{E}$ is defined by

$$\hat{R}_h^\partial \hat{u} = \hat{u}(x_{\hat{E}}), \quad \text{where} \quad x_{\hat{E}} := (1/3, 1/3),$$

for arbitrary $\hat{u} \in C(\operatorname{cl} \hat{E})$. As this interpolant is based on point evaluations of u (like the nodal interpolant I_h^∂ considered in Section 3.2.1) we have only stability in $L^\infty(E)$.

Lemma 3.2.9 (Stability of R_h^∂)**.** *Let some function $\hat{u} \in C(\operatorname{cl} \hat{E})$ be given. Then, the stability estimate*

$$\|\hat{R}_h^\partial \hat{u}\|_{L^p(\hat{E})} \le c\|\hat{u}\|_{L^\infty(\hat{E})}$$

holds for arbitrary $p \in [1, \infty]$.

From the definition of R_h^∂ we immediately get that R_h^∂ preserves piecewise constant functions, i. e. $R_h^\partial w = w$ for all $w \in \mathcal{P}_0(E)$. However, due to the choice of the points x_E we even get the relation

$$\int_E [R_h^\partial w](x)\, \mathrm{d}s_x = \int_E w(x)\, \mathrm{d}s_x \qquad \text{for all } w \in \mathcal{P}_1(E), \tag{3.42}$$

which allows us to derive error estimates with a rate higher than one provided that the function we want to approximate is sufficiently regular. We will exploit this fact in the following Lemma.

Lemma 3.2.10. *Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain and $\mathcal{E}_h$ a triangulation of its boundary. Let $E \in \mathcal{E}_h$ be an arbitrary boundary element with $E \subset U_j \cap \Gamma$ for some $j \in \mathcal{C}$. To each choice of weight vectors $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ we associate the number $\kappa := \max\{\beta_j, \max_{k \in X_j} \delta_k\}$. The following assertions hold:*

a) If $|u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)} \le c$ with $\vec{\beta} \in [0, 3/2)^{d'}$ and $\vec{\delta} \in [0, 1)^d$, then the estimate

$$\left| \int_E (u(x) - R_h^\partial u)\, \mathrm{d}s_x \right| \le c h_E^2 |E|^{1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)} \cdot \begin{cases} r_E^{-\kappa}, & \text{if } r_E > 0, \\ h_E^{-\kappa}, & \text{if } r_E = 0, \end{cases} \tag{3.43}$$

holds.

b) If $|u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)} \le c$ with $\vec{\beta} \in [0, 1)^{d'}$ and $\vec{\delta} \in [0, 1/2)^d$, the estimates

$$\|u - R_h^\partial u\|_{L^\infty(E)} \le c h_E |u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)} \cdot \begin{cases} \rho_{j,E}^{-\beta_j} \prod_{k \in X_j} \left(\frac{r_{k,E}}{\rho_{j,E}}\right)^{-\delta_k}, & \text{if } \rho_{j,E} > 0,\ r_{k,E} > 0\ \forall k \in X_j, \\ h_E^{-\delta_k} \rho_{j,E}^{\delta_k - \beta_j}, & \text{if } r_{k,E} = 0,\ \rho_{j,E} > 0, \\ h_E^{-\beta_j}, & \text{if } \rho_{j,E} = 0, \end{cases} \tag{3.44}$$

and in particular

$$\|u - R_h^\partial u\|_{L^\infty(E)} \le c h_E |u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)} \cdot \begin{cases} r_E^{-\kappa}, & \text{if } r_E > 0, \\ h_E^{-\kappa}, & \text{if } r_E = 0, \end{cases} \tag{3.45}$$

hold.

Proof. a) We apply the transformation to the reference triangle $\hat{E}$, introduce a first-order polynomial $\hat{w} \in \mathcal{P}_1$, apply the stability estimate from Lemma 3.2.9, and the embedding $W^{2,1+\varepsilon}(\hat{E}) \hookrightarrow L^\infty(\hat{E})$ which holds for arbitrary $\varepsilon > 0$, and use the Bramble-Hilbert Lemma. Applying the described technique step-by-step yields

$$\begin{aligned}\left|\int_E (u(x) - R_h^\partial u)\,\mathrm{d}s_x\right| &\le c|E|\left|\int_{\hat{E}} (\hat{u}(\hat{x}) - \hat{R}_h^\partial \hat{u})\mathrm{d}s_{\hat{x}}\right| \\ &\le c|E|\left(\left|\int_{\hat{E}}(\hat{u}(\hat{x}) - \hat{w}(\hat{x}))\mathrm{d}s_{\hat{x}}\right| + \left|\int_{\hat{E}} \hat{R}_h^\partial(\hat{u} - \hat{w})\mathrm{d}s_{\hat{x}}\right|\right) \\ &\le c|E|\|\hat{u} - \hat{w}\|_{L^\infty(\hat{E})} \le c|E|\|\hat{u} - \hat{w}\|_{W^{2,1+\varepsilon}(\hat{E})} \\ &\le c|E||\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})}. \end{aligned} \tag{3.46}$$

If $r_E > 0$ we use the trivial embedding $L^2(\hat{E}) \hookrightarrow L^{1+\varepsilon}(\hat{E})$ (note that we can chose $\varepsilon \in (0, 1)$), apply the transformation back to E and introduce the weights which yields

$$\begin{aligned}|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} &\le c h_E^2 |E|^{-1/2} |u|_{H^2(E)} \\ &\le c h_E^2 |E|^{-1/2} \rho_{j,E}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,E}}{\rho_{j,E}}\right)^{-\delta_k} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.\end{aligned}$$

Inserting also the simplification (3.21) yields together with (3.46) the estimate (3.43) in case of $r_E > 0$.

If $r_E = 0$ we have lower regularity and hence reuse the technique from the proof of Lemma 3.2.3, in particular the steps (3.19) and (3.20). If one edge of E is contained in the edge M_k and E is away from the corner we get with the embedding from Lemma 2.3.4, the property (3.7), and the fact that $\rho_{j,E} > 0$, the estimate

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \le c|\hat{u}|_{W^{2,2}_{\delta_k,\delta_k}(\hat{E})} \le c h_E^{2-\delta_k} |E|^{-1/2} \rho_{j,E}^{\delta_k - \beta_j} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

If E touches the edge only in a single point we apply (3.9) instead of (3.7) and get

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \le c|\hat{u}|_{W^{2,2}_{\delta_k,0}(\hat{E})} \le c h_E^{2-\delta_k} |E|^{-1/2} \rho_{j,E}^{\delta_k - \beta_j} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

If E touches additionally the corner $x^{(j)}$ and has an edge contained in M_k, we get with (3.7) and (3.8)

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \le c|\hat{u}|_{W^{2,2}_{\beta_j,\delta_k}(\hat{E})} \le c h_E^{2-\beta_j} |E|^{-1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

If E touches the corner $x^{(j)}$, but the edges M_k, $k \in X_j$, only in $x^{(j)}$, the property (3.8) yields

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \le c|\hat{u}|_{W^{2,2}_{\beta_j,0}(\hat{E})} \le c h_E^{2-\beta_j} |E|^{-1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

In all four cases the embeddings hold for $0 \le \beta_j < 3/2$ and $0 \le \delta_k < 1$ when we choose $\varepsilon > 0$ sufficiently small. Using also the simplification (3.22) we arrive at

$$|\hat{u}|_{W^{2,1+\varepsilon}(\hat{E})} \le |\hat{u}|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(\hat{E})} \le c h_E^{2-\kappa} |E|^{-1/2} |u|_{W^{2,2}_{\vec{\beta},\vec{\delta}}(E)}.$$

Together with (3.46) the estimate (3.43) follows for $r_E = 0$.

b) To show the estimate in the $L^\infty(E)$-norm we use again the transformation to a reference element, insert a polynomial $\hat{w} \in \mathcal{P}_0$, apply the stability estimate from Lemma 3.2.9, the embedding $W^{1,2+\varepsilon}(\hat{E}) \hookrightarrow L^\infty(\hat{E})$ and the Bramble-Hilbert Lemma. Then we arrive at

$$\|u - R_h^\partial u\|_{L^\infty(E)} \le c\|\hat{u} - \hat{w}\|_{L^\infty(\hat{E})} \le c|\hat{u}|_{W^{1,2+\varepsilon}(E)}. \tag{3.47}$$

The case $r_E > 0$ is easy since $u \in W^{1,\infty}(E)$. After transformation to E and insertion of the weights we get

$$|\hat{u}|_{W^{1,2+\varepsilon}(\hat{E})} \le ch_E|u|_{W^{1,\infty}(E)} \le ch_E\rho_{j,E}^{-\beta_j} \prod_{k\in X_j} \left(\frac{r_{k,E}}{\rho_{j,E}}\right)^{-\delta_k} |u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)}, \tag{3.48}$$

If $r_E = 0$ we proceed as in the proof of part a), and derive the estimate

$$|\hat{u}|_{W^{1,2+\varepsilon}(\hat{E})} \le ch_E|u|_{W^{1,\infty}_{\vec{\beta},\vec{\delta}}(E)} \begin{cases} h_E^{-\delta_k}\rho_{j,E}^{\delta_k-\beta_j}, & \text{if } r_{k,E} = 0,\ \rho_{j,E} > 0, \\ h_E^{-\beta_j}, & \text{if } \rho_{j,E} = 0. \end{cases} \tag{3.49}$$

where we have to distinguish among the possible positions of E, and use the embeddings $W^{1,\infty}_{\delta_k,\delta_k}(\hat{E}) \hookrightarrow W^{1,2+\varepsilon}(\hat{E})$, $W^{1,\infty}_{\delta_k,0}(\hat{E}) \hookrightarrow W^{1,2+\varepsilon}(\hat{E})$, $W^{1,\infty}_{\beta_j,\delta_k}(\hat{E}) \hookrightarrow W^{1,2+\varepsilon}(\hat{E})$ or $W^{1,\infty}_{\beta_j,0}(\hat{E}) \hookrightarrow W^{1,2+\varepsilon}(\hat{E})$ which hold under our assumptions upon β_j and δ_k, $k \in X_j$, compare also Lemma 2.3.4. Inserting (3.48) and (3.49) into (3.47) leads to the estimate (3.44). From the simplification (3.21) we conclude the estimate (3.45). □

3.3 Error estimates for quasi-uniform meshes

In this section we summarize some *a priori* error estimates for the solution y_h of (3.3) when the finite element mesh is not refined locally. Error estimates on locally refined meshes are considered in Section 3.4. We call a shape regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ *quasi-uniform* if some $h_0 > 0$ exists such that

$$h_T \sim h \qquad \forall T \in \mathcal{T}_h \tag{3.50}$$

is satisfied for all $h \in (0, h_0]$.

The convergence rate of the finite-element method on quasi-uniform meshes will depend upon the corner and edge singularity with the strongest influence. Therefore we define the singular exponent of the dominating singularity by

$$\lambda := \begin{cases} \min\limits_{j\in\mathcal{C}} \lambda_j, & \text{for } n = 2, \\ \min\limits_{j\in\mathcal{C},k\in\mathcal{E}} \{1/2 + \lambda_j^c, \lambda_k^e\}, & \text{for } n = 3. \end{cases} \tag{3.51}$$

Note that $\lambda > 1/2$ for arbitrary polygonal and polyhedral domains according to Definition 2.2.1 and 2.2.3, respectively.

Theorem 3.3.1. *Assume that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform.*

- *<u>$n = 2$:</u> Let $f \in W^{0,2}_{\vec{\alpha}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\alpha}}(\Gamma)$ with a weight vector $\vec{\alpha} \in \mathbb{R}^d$ defined by*

$$\alpha_j := \max\{0, 1 - \lambda_j + \varepsilon\} \qquad\qquad j \in \mathcal{C}.$$

- *$\underline{n=3}$: Let $f \in W^{0,2}_{\vec\beta,\vec\delta}(\Omega)$ and $g \in W^{1/2,2}_{\vec\beta,\vec\delta}(\Gamma)$ with weight vectors $\vec\beta \in \mathbb{R}^{d'}$ and $\vec\delta \in \mathbb{R}^{d}$ defined by*

$$\begin{aligned} \beta_j &:= \max\{0, 1/2 - \lambda_j^c + \varepsilon\} && j \in \mathcal{C}, \\ \delta_k &:= \max\{0, 1 - \lambda_k^e + \varepsilon\} && k \in \mathcal{E}. \end{aligned}$$

Then, for $\ell \in \{0,1\}$ the error estimates

$$\|y - y_h\|_{H^\ell(\Omega)} \le c h^{(2-\ell)\min\{1,\lambda-\varepsilon\}} \cdot \begin{cases} |y|_{W^{2,2}_{\vec\alpha}(\Omega)}, & \text{if } n = 2, \\ |y|_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)}, & \text{if } n = 3, \end{cases}$$

hold for sufficiently small $\varepsilon > 0$.

Proof. The assertion for the two-dimensional case is proved in [74, Corollary 3.39]. Analogously, we can derive the assertion for the three-dimensional case using the Lemma of Céa and the local interpolation error estimates from Lemma 3.2.6. This yields

$$\|y - y_h\|^2_{H^1(\Omega)} \le c \sum_{T \in \mathcal{T}_h} \|y - Z_h y\|^2_{H^1(T)} \le c h^{2(1-\kappa)} |y|^2_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)},$$

under the assumption that $\varepsilon > 0$ chosen sufficiently small such that the $\beta_j < 1$, $j \in \mathcal{C}$, and $\delta_k < 2/3$, $k \in \mathcal{E}$. Inserting the definition of the weights $\vec\beta$ and $\vec\delta$ leads to

$$h^{1-\kappa} = h^{\min\{1,\lambda-\varepsilon\}},$$

from which we conclude the estimate for $\ell = 1$. The estimate in the $L^2(\Omega)$-norm follows from the Aubin-Nitsche method. Therefore, let $w \in H^1(\Omega)$ be the weak solution of the dual problem

$$-\Delta w + w = y - y_h \quad \text{in } \Omega, \qquad \partial_n w = 0 \quad \text{on } \Gamma.$$

With the Galerkin orthogonality and the estimate already derived in the $H^1(\Omega)$-norm we get

$$\begin{aligned} \|y - y_h\|^2_{L^2(\Omega)} &= (y - y_h, y - y_h) = a(w, y - y_h) \\ &\le c \|w - Z_h w\|_{H^1(\Omega)} \|y - y_h\|_{H^1(\Omega)} \\ &\le c h^{2\min\{1,\lambda-\varepsilon\}} |w|_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)} |y|_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)}. \end{aligned} \tag{3.52}$$

From the regularity result stated in Theorem 2.3.7 and the embedding $L^2(\Omega) \hookrightarrow W^{0,2}_{\vec\beta,\vec\delta}(\Omega)$ which follows from Lemma 2.3.4 as $\vec\beta, \vec\delta \ge 0$, we obtain

$$|w|_{W^{2,2}_{\vec\beta,\vec\delta}(\Omega)} \le c \|y - y_h\|_{W^{0,2}_{\vec\beta,\vec\delta}(\Omega)} \le c \|y - y_h\|_{L^2(\Omega)}.$$

Inserting this estimate into (3.52) and dividing by $\|y - y_h\|_{L^2(\Omega)}$ yields the desired estimate for $\ell = 0$. □

Note that the convergence rate one in $H^1(\Omega)$-norm and two in $L^2(\Omega)$-norm can be expected for arbitrary convex domains.

Especially in the context of error estimates for boundary control problems finite element error estimates on the boundary are of interest. As a rule, one expects lower convergence rates than for the finite element error measured in $L^2(\Omega)$-norm, as we will see in the following theorem.

Theorem 3.3.2. *Let $f \in C^{0,\sigma}(\overline{\Omega})$ and $g \equiv 0$. Assume that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform.*

- *$\underline{n=2}$: We define weight vectors $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^d$ by*

$$\alpha_j := \max\{0, 1-\lambda_j+\varepsilon\}, \qquad \beta_j := \max\{0, 2-\lambda_j+\varepsilon\} \qquad \forall j \in \mathcal{C},$$

and

$$\eta := \|y\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)}.$$

- *$\underline{n=3}$: We define weight vectors $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\delta}, \vec{\varrho} \in \mathbb{R}^d$ by*

$$\alpha_j := \max\{0, \frac{1}{2}-\lambda_j^c+\varepsilon\}, \qquad \beta_j := \max\{0, 2-\lambda_j^c+\varepsilon\} \qquad \forall j \in \mathcal{C},$$
$$\delta_k := \max\{0, 1-\lambda_k^e+\varepsilon\}, \qquad \varrho_k := \max\{0, 2-\lambda_k^e+\varepsilon\} \qquad \forall k \in \mathcal{E},$$

and

$$\eta := \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|y\|_{L^\infty(\Omega)}.$$

Then, the error estimate

$$\|y-y_h\|_{L^2(\Gamma)} \le c h^{\min\{2,1/2+\lambda-\varepsilon\}} |\ln h|^{3/2} \eta$$

holds, provided that $\varepsilon > 0$ is sufficiently small.

Proof. The proof of this assertion for polygonal domains can be found in [74, Corollary 3.49]. The proof for polyhedral domains is postponed to the end of Section 3.4 as we will first prove Theorem 3.4.2 where an error estimate in $L^2(\Gamma)$ with local mesh refinement is stated. Once we have this estimate it requires only several slight modifications of the proof. □

In the following theorem we will prove also an estimate in the $H^{1/2}(\Gamma)$-norm, but consider only two-dimensional domains.

Theorem 3.3.3. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let the assumptions of Theorem 3.3.2 be satisfied. Additionally, introduce the weight vector $\vec{\gamma} \in \mathbb{R}^d$ whose components are defined by*

$$\gamma_j := \max\{0, 3/2-\lambda_j+\varepsilon\}.$$

Then, the estimate

$$\|y-y_h\|_{H^{1/2}(\Gamma)} \le c h^{\min\{3/2,\lambda-\varepsilon\}} |\ln h|^{3/2} \left(\|y\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)} + |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)} \right)$$

holds for sufficiently small $\varepsilon > 0$.

Proof. We obtain using the triangle inequality and an inverse inequality the estimate

$$\begin{aligned} \|y-y_h\|_{H^{1/2}(\Gamma)} &\le c \left(\|y - I_h^\partial y\|_{H^{1/2}(\Gamma)} + h^{-1/2} \|I_h^\partial y - y_h\|_{L^2(\Gamma)} \right) \\ &\le c \left(\|y - I_h^\partial y\|_{H^{1/2}(\Gamma)} + h^{-1/2} \|y - I_h^\partial y\|_{L^2(\Gamma)} + h^{-1/2} \|y - y_h\|_{L^2(\Gamma)} \right). \end{aligned} \tag{3.53}$$

To handle the terms depending on the interpolation error we exploit the regularity

$$y \in W^{2,\infty}_{\vec{\beta}'}(\Gamma) \hookrightarrow W^{2,2}_{\vec{\gamma}}(\Gamma), \quad \text{where} \quad \beta'_j := \max\{0, 2-\lambda_j+\varepsilon/2\}, \qquad j \in \mathcal{C}, \tag{3.54}$$

which follows from Theorem 2.3.6 and the embedding of Lemma 2.3.3. From the local estimate presented in Lemma 3.2.4 we conclude the global estimate

$$\|y - I_h^\partial y\|_{H^\ell(\Gamma)} \leq c h^{2-\ell-\max_{j\in\mathcal{C}}\gamma_j} |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)} \leq c h^{1/2-\ell+\min\{3/2,\lambda-\varepsilon\}} |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}, \tag{3.55}$$

for $\ell \in \{0,1\}$. Note that we have to choose $\varepsilon > 0$ sufficiently small such that $\gamma_j < 1$, $j \in \mathcal{C}$. With an interpolation argument we conclude the validity of this estimate also for $\ell = 1/2$. Inserting (3.55) for $\ell = 0$ and $\ell = 1/2$ as well as the estimate from Theorem 3.3.2 into (3.53) leads to the assertion. □

Remark 3.3.4. *The best possible convergence rate up to logarithmic factors that we can expect is two in the $L^2(\Gamma)$-norm and $3/2$ in the $H^{1/2}(\Gamma)$-norm. Obviously, in both norms the optimal rate is attained when $\lambda > 3/2$. This holds for polygonal domains when the interior angles of all corners are less than $120°$. If one or more corners have larger interior angle the use of local mesh refinement is a possibility to preserve the optimal convergence rates. This will be discussed in Section 3.4.*

3.4 Error estimates for locally refined meshes

The aim of this section is to improve the convergence rates presented in the foregoing section using local mesh refinement. We make some additional assumptions for the finite element meshes. The number $\mu \in (0,1]$ denotes the refinement parameter and $R > 0$ the refinement radius. We assume that some $h_0 > 0$ exists such that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ satisfies the condition

$$h_T \sim \begin{cases} h^{1/\mu}, & \text{if } r_T = 0, \\ h r_T^{1-\mu}, & \text{if } 0 < r_T < R, \\ h, & \text{if } r_T \geq R, \end{cases} \qquad \forall T \in \mathcal{T}_h, \tag{3.56}$$

for all $h \in (0, h_0]$, where r_T is the distance to the singular points, i. e.

$$r_T := \min_{j\in\mathcal{C}} \inf_{x\in T} |x - x^{(j)}|$$

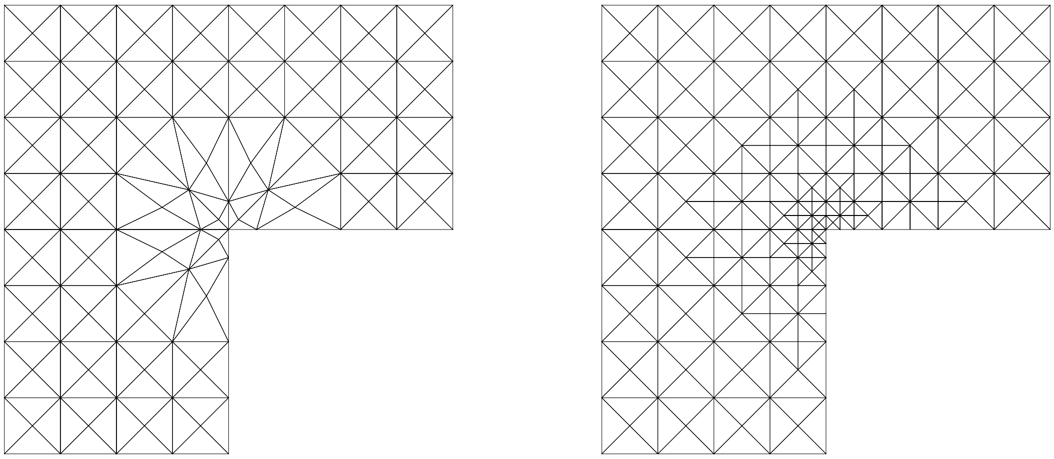
for two-dimensional polygonal domains, and

$$r_T := \min_{k\in\mathcal{E}} \inf_{\substack{x\in T \\ y\in M_k}} |x - y|$$

for three-dimensional polyhedral domains. Moreover, we assume throughout this thesis that $\mu > 1/3$ for three-dimensional problems because otherwise, the relation $h \sim |\mathcal{N}_h|^{-1/3}$ would not hold, see also the discussion in [12].

For planar problems it is also possible to introduce a different refinement parameter μ_j for each corner $x^{(j)}$, $j \in \mathcal{C}$, since the singularities are of local nature. The distance of T to the corner $x^{(j)}$ is denoted by

$$r_{j,T} := \inf_{x\in T} |x - x^{(j)}|,$$



(a) Refinement by mesh grading (b) Refinement by bisection

Figure 3.3: Locally refined polygonal domain with $R = 0.5$ and $\mu = 0.5$.

and the mesh criterion we will use later reads

$$h_T \sim \begin{cases} h^{1/\mu_j}, & \text{if } r_{j,T} = 0, \\ h r_{j,T}^{1-\mu_j}, & \text{if } 0 < r_{j,T} < R, \\ h, & \text{if } r_{j,T} \geq R. \end{cases} \tag{3.57}$$

For three-dimensional problems the use of different refinement parameters has e. g. been considered in [61, 4, 12] for pure isotropic refinement and in [5, 6, 24] for anisotropic refinement towards edges and isotropic refinement towards corners. However, in order to keep the proofs in this section as simple as possible we do not consider these advanced strategies and use (3.56) in the three-dimensional case.

3.4.1 Two-dimensional problems

Before we derive error estimates on locally refined meshes let us briefly discuss how meshes satisfying (3.57) can be generated. The strategy illustrated in Figure 3.3a is called *mesh grading* meaning that all vertices within some ball around $x^{(j)}$ with radius $R > 0$ are moved closer towards the singular corner. By change of the coordinate system we assume that $x^{(j)} = 0$. To all nodes $n \in \mathcal{N}_h$ with $0 < |n - x^{(j)}| < R$ we apply the coordinate transformation $(x_n, y_n) \to (x_n^*, y_n^*)$ by the formula

$$\begin{pmatrix} x_n^* \\ y_n^* \end{pmatrix} := \left(\frac{r}{R}\right)^{1/\mu - 1} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad \text{with} \quad r := \sqrt{x_n^2 + y_n^2}.$$

One can show [3, Section 19.2] that the resulting triangulation satisfies (3.57).

Another possibility, which allows also hierarchical meshes, is to use local bisection algorithms. The mesh illustrated in Figure 3.3b has been generated by the newest-vertex bisection method described by Bänsch [17]. All cells that either violate the condition (3.57) or are non-conform, are refined through the edge opposite to the newest vertex. This procedure is repeated until the refinement criterion is satisfied.

The error estimates presented in this section are independent of the refinement strategy as we will merely use the condition (3.57). First, an error estimate in $L^2(\Omega)$ and $H^1(\Omega)$ on polygonal domains is presented whose proof can be found in [8, Lemma 4.1] or [16, Theorem 5.1].

Theorem 3.4.1. *Assume that the input data satisfy $f \in W^{0,2}_{\vec\alpha}(\Omega)$ and $g \in W^{1/2}_{\vec\alpha}(\Gamma)$, and that the mesh criterion (3.57) holds such that weight vector $\vec\alpha \in \mathbb{R}^d_+$ and refinement parameters satisfy*

$$1 - \lambda_j < \alpha_j \le 1 - \mu_j \qquad (\Rightarrow \mu_j < \lambda_j).$$

Then, the error estimate

$$\|y - y_h\|_{H^\ell(\Omega)} \le ch^{2-\ell}|y|_{W^{2,2}_{\vec\alpha}(\Omega)} \le ch^{2-\ell}\left(\|f\|_{W^{0,2}_{\vec\alpha}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\alpha}(\Omega)}\right)$$

holds for $\ell \in \{0, 1\}$.

The following error estimate in the $L^2(\Gamma)$-norm can be found in [74, Theorem 3.48].

Theorem 3.4.2. *Let $f \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$ and $g \equiv 0$. Assume that the mesh condition (3.57) holds, and that the weight vectors $\vec\alpha, \vec\beta \in \mathbb{R}^d_+$ and the refinement parameters satisfy*

$$1 - \lambda_j < \alpha_j \le 1 - \mu_j \quad \text{and} \quad 2 - \lambda_j < \beta_j \le 5/2 - 2\mu_j$$

for all $j \in \mathcal{C}$. Then, the error estimate

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\left(\|y\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec\beta}(\Omega)}\right)$$

holds.

Combining this theorem with the regularity results from Section 2.3 leads to the following implication.

Corollary 3.4.3. *Let $f \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. The error estimate*

$$\|y - y_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\|f\|_{C^{0,\sigma}(\overline{\Omega})}$$

holds, if one of the following assumptions are satisfied:

1. *The interior angles of all corners of Ω are smaller than $120°$ and the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform ($\mu = 1$).*
2. *The corners having interior angle larger than $120°$ are refined locally according to*

$$\mu_j < \frac{1}{4} + \frac{\lambda_j}{2} \qquad \forall j \in \mathcal{C}.$$

In the following we will derive a finite element error estimate in the $H^{1/2}(\Gamma)$-norm as we already did in Theorem 3.3.3 for quasi-uniform meshes, but for locally refined meshes we have to use a slightly different technique. In the proof of Theorem 3.3.3 we applied an inverse inequality in step (3.53). This is not possible on refined meshes since a global inverse estimate on non quasi-uniform meshes would give us the factor $h_{E_{min}}^{-1}$ which would lead to a suboptimal convergence rate. Moreover we inserted interpolation error estimates in $L^2(\Gamma)$- and $H^{1/2}(\Gamma)$-norm, where the latter one was shown using an interpolation argument between global estimates in $L^2(\Gamma)$ and $H^1(\Gamma)$. This technique is also not applicable for refined meshes, because the bounds for the refinement parameter depend also on the norm in that we want to show an error estimate.

We consider first the required global interpolation error estimates.

Lemma 3.4.4. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\mathcal{E}_h$ a triangulation of its boundary. Let some function $y \in W^{2,2}_{\vec{\gamma}}(\Gamma)$ with $\vec{\gamma} \in \mathbb{R}^d$ be given. Assume that the mesh condition (3.57) holds such that the inequality*

$$0 \le \gamma_j \le \frac{3}{2} - \frac{3}{2}\mu_j, \qquad \forall j \in \mathcal{C}$$

is satisfied. Then, the interpolation error estimate

$$\left(\sum_{E \in \mathcal{E}_h} h_E^{-1} \|y - I_h^\partial y\|^2_{L^2(E)} \right)^{1/2} + |y - I_h^\partial y|_{H^{1/2}(\Gamma)} \le c h^{3/2} |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}$$

holds.

Proof. From the local estimates for some $E \subset U_j$ derived in Lemma 3.2.4 we obtain

$$h_E^{-1/2} \|y - I_h^\partial y\|_{L^2(E)} \le c \begin{cases} h^{(3/2-\gamma_j)/\mu_j} |y|_{W^{2,2}_{\gamma_j}(E)}, & \text{if } r_{j,E} = 0, \\ h^{3/2} r_{j,E}^{3/2(1-\mu_j)-\gamma_j} |y|_{W^{2,2}_{\gamma_j}(E)}, & \text{if } r_{j,E} > 0, \end{cases}$$

provided that $\gamma_j \in [0, 3/2)$. It is easy to show that the refinement condition $\gamma_j \le 3/2 - 3\mu_j/2$ leads to the desired estimate.

An error estimate in $H^{1/2}(\Gamma)$ has been proved by von Petersdorff [89]. From Theorem 3.10 in this reference we get the estimate

$$\|y - I_h^\partial y\|_{\tilde{H}^{1/2}(\Gamma_j)} \le c h^{3/2} |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma_j)}, \tag{3.58}$$

where $\tilde{H}^{1/2}(\Gamma_j)$ denotes the closure of the space $C_0^\infty(\Gamma_j)$ with respect to the norm $\|\cdot\|_{H^{1/2}(\mathbb{R})}$, see also [66] for a detailed discussion on this space. To show (3.58) von Petersdorff uses the assumption $\mu_j < 2\lambda_j/3$ as well as the choice $\gamma_j = 3(1-\mu_j)/2$. However, when tracing through the proof one easily verifies that the assertion remains true for some choice $0 \le \gamma_j \le 3(1-\mu_j)/2$.

To obtain a global estimate we use [89, Lemma 3.2] and obtain

$$|y - I_h^\partial y|_{H^{1/2}(\Gamma)} \le \|y - I_h^\partial y\|_{\tilde{H}^{1/2}(\Gamma)} \le c \sum_{j \in \mathcal{C}} \|y - I_h^\partial y\|_{\tilde{H}^{1/2}(\Gamma_j)},$$

which leads together with (3.58) to the desired estimate. □

With these interpolation error estimates we can improve the results of Theorem 3.3.3 and derive a bound for the refinement parameter such that optimal convergence rate in the $H^{1/2}(\Gamma)$-norm is guaranteed.

Theorem 3.4.5. *Let $f \in C^{0,\sigma}(\overline{\Omega})$, $\sigma \in (0,1)$ be given, and assume that $g \equiv 0$. Moreover, assume that the family of triangulations is locally refined according to (3.57) with $\mu_j > 1/3$, $j \in \mathcal{C}$, and that the weight vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma} \in \mathbb{R}^d_+$ satisfy*

$$\begin{aligned} 1 - \lambda_j < \alpha_j &\leq 1 - \mu_j, \\ 2 - \lambda_j < \beta_j &\leq 2 - \frac{3}{2}\mu_j, \\ 3/2 - \lambda_j < \gamma_j &\leq \frac{3}{2} - \frac{3}{2}\mu_j, \end{aligned} \tag{3.59}$$

for all $j \in \mathcal{C}$. Then, the error estimate

$$\|y - y_h\|_{H^{1/2}(\Gamma)} \leq ch^{3/2}|\ln h|^{3/2}\left(\|y\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)} + |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}\right) \tag{3.60}$$

holds.

Proof. Firstly, one confirms that y possesses the regularity demanded by the right-hand side of (3.60) which is a consequence of Theorem 2.3.5, Theorem 2.3.6 as well as the argument (3.54). By introducing the intermediate function $I_h^\partial y$ and applying the triangle inequality we get

$$|y - y_h|_{H^{1/2}(\Gamma)} \leq |y - I_h^\partial y|_{H^{1/2}(\Gamma)} + |I_h^\partial y - y_h|_{H^{1/2}(\Gamma)}.$$

For the second term we apply the inverse estimate from [32, Theorem 4.1] for meshes that are not quasi-uniform, introduce y as intermediate function and arrive at

$$|y - y_h|_{H^{1/2}(\Gamma)} \leq |y - I_h^\partial y|_{H^{1/2}(\Gamma)} + \left(\sum_{E \in \mathcal{E}_h} h_E^{-1}\left(\|y - I_h^\partial y\|^2_{L^2(E)} + \|y - y_h\|^2_{L^2(E)}\right)\right)^{1/2}. \tag{3.61}$$

The terms depending on the interpolation error $y - I_h^\partial y$ have been discussed in Lemma 3.4.4. Hence, we get

$$\left(\sum_{E \in \mathcal{E}_h} h_E^{-1}\|y - I_h^\partial y\|^2_{L^2(E)}\right)^{1/2} + |y - I_h^\partial y|_{H^{1/2}(\Gamma)} \leq ch^{3/2}|y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}. \tag{3.62}$$

It remains to derive an estimate for the finite element error on the right-hand side of (3.61). We will adopt the technique used in [9], where a finite element error estimate in $L^2(\Gamma)$ is proved, to our situation. We first consider the error in a vicinity of a corner point $x^{(j)}$, $j \in \mathcal{C}$. Therefore, we introduce the domains

$$\Omega^j_{R/n} := \{x \in \Omega \colon r_j(x) < R/n\}, \qquad \Gamma^j_{R/n} := \partial\Omega^j_{R/n} \cap \Gamma, \tag{3.63}$$

$$\tilde{\Omega}_{R/n} := \Omega \setminus \bigcup_{j \in \mathcal{C}} \Omega^j_{R/n}, \qquad \tilde{\Gamma}_{R/n} := \partial\tilde{\Omega}_{R/n} \cap \Gamma, \tag{3.64}$$

where $r_j(x) := |x - x^{(j)}|$. The radius $R > 0$ is assumed to be sufficiently small such that all corners have distance larger than $2R$ from each other, but appropriate scaling arguments allow us to set $R = 1$ without loss of generality.

In what follows we fix an arbitrary corner $\boldsymbol{c} := x^{(j)}$, $j \in \mathcal{C}$, and omit the index j to simplify the notation. In order to derive an error estimate in the vicinity of $\boldsymbol{c}$ we use the idea of Schatz and Wahlbin [80] and introduce a dyadic decomposition of Ω_R, namely

$$\Omega_i := \{x \in \Omega \colon d_{i+1} < r_{\boldsymbol{c}}(x) < d_i\}, \qquad r_{\boldsymbol{c}}(x) := |x - \boldsymbol{c}|,$$

with radii $d_i = 2^{-i}$, for $i = 0, 1, \ldots, I$, and $d_{I+1} = 0$. The inner-most ring has radius $d_I = c_I h^{1/\mu}$ with some $c_I > 0$ independent of h which implies $I \sim |\ln h|$. Moreover, denote by $\Gamma_i := \partial\Omega_i \cap \Gamma$ the boundary segments, which form in the same way a decomposition of Γ_R. We also define the patches of Ω_i by

$$\Omega_i^{(k)} := \operatorname{int}\left(\bar{\Omega}_{\max\{0,i-k\}} \cup \ldots \cup \bar{\Omega}_i \cup \ldots \cup \bar{\Omega}_{\min\{I,i+k\}}\right)$$

and write $\Omega_i' = \Omega_i^{(1)}$, $\Omega_i'' = \Omega_i^{(2)}$. For elements $T \in \mathcal{T}_h$ contained in or touching Ω_i we observe

$$h_T \sim \begin{cases} h d_i^{1-\mu}, & \text{for } i = 0, 1, \ldots, I-1, \\ d_I \sim h d_I^{1-\mu}, & \text{for } i = I, \end{cases}$$

which means that $\mathcal{T}_h$ is quasi-uniform within each Ω_i. Using this property and the decomposition of $\Gamma_{R/4}$ we get

$$\sum_{\substack{E \in \mathcal{E}_h \\ E \subset \Gamma_{R/4}}} h_E^{-1} \|y - y_h\|_{L^2(E)}^2 \le c h^{-1} \sum_{i=2}^{I} d_i^{-(1-\mu)} \|y - y_h\|_{L^2(\Gamma_i)}^2. \tag{3.65}$$

Let us consider the term within the sum on the right-hand side of (3.65). For $i = 2, \ldots, I-2$ we can use the Hölder inequality with $|\Gamma_i| \sim d_i$, a trace Theorem and get

$$d_i^{-(1-\mu)} \|y - y_h\|_{L^2(\Gamma_i)}^2 \le c d_i^{\mu} \|y - y_h\|_{L^\infty(\Omega_i)}^2. \tag{3.66}$$

Now, we can apply the local maximum norm estimate [90, Theorem 10.1] which reads in our situation

$$\|y - y_h\|_{L^\infty(\Omega_i)} \le c \left(|\ln h| \|y - I_h y\|_{L^\infty(\Omega_i')} + d^{-1} \|y - y_h\|_{L^2(\Omega_i')}\right), \tag{3.67}$$

where $d := \operatorname{dist}(\partial\Omega_i \setminus \Gamma, \partial\Omega_i' \setminus \Gamma)$. Due to our construction of Ω_i and its patches one easily confirms that $d \sim d_i$. Inserting (3.67) into (3.66) yields

$$d_i^{-(1-\mu)} \|y - y_h\|_{L^2(\Gamma_i)}^2 \le c \left(|\ln h|^2 d_i^{\mu} \|y - I_h y\|_{L^\infty(\Omega_i')}^2 + d_i^{\mu-2} \|y - y_h\|_{L^2(\Omega_i')}^2\right). \tag{3.68}$$

In order to derive a similar estimate in case of $i = I-1, I$ we use a slightly different technique. We introduce $I_h y$ as intermediate function, apply the discrete trace theorem

$$\|v_h\|_{L^2(\Gamma_i)} \le c h^{-1/(2\mu)} \|v_h\|_{L^2(\Omega_i')} \le c d_I^{-1/2} \|v_h\|_{L^2(\Omega_i')}$$

from [9, Lemma 3.11], and using again the Hölder inequality and the trace theorem in L^∞ we obtain

$$\begin{aligned} d_i^{-(1-\mu)} \|y - y_h\|_{L^2(\Gamma_i)}^2 &\le c \left(d_i^{-(1-\mu)} \|y - I_h^{\partial} y\|_{L^2(\Gamma_i)}^2 + d_I^{\mu-2} \|I_h y - y_h\|_{L^2(\Omega_i')}^2\right) \\ &\le c \left(d_i^{\mu} \|y - I_h y\|_{L^\infty(\Omega_i')}^2 + d_I^{\mu-2} \|I_h y - y_h\|_{L^2(\Omega_i')}^2\right) \\ &\le c \left(d_i^{\mu} \|y - I_h y\|_{L^\infty(\Omega_i')}^2 + d_I^{\mu-2} \|y - y_h\|_{L^2(\Omega_i')}^2\right). \end{aligned} \tag{3.69}$$

Inserting now (3.68) and (3.69) into (3.65) yields

$$\sum_{\substack{E\in\mathcal{E}_h\\ E\subset\Gamma_{R/4}}} h_E^{-1}\|y-y_h\|^2_{L^2(E)} \le ch^{-1}\left(|\ln h|^2\sum_{i=1}^{I} d_i^{\mu}\|y-I_hy\|^2_{L^\infty(\Omega_i)} + \|\gamma^{\mu/2-1}(y-y_h)\|^2_{L^2(\Omega_{R/2})}\right), \tag{3.70}$$

where $\gamma(x) := d_I + r_c(x)$. Note that $d_i \sim d_I + r_c(x)$ for all $x\in\Omega_i$, since $d_i \sim d_{i-1}$.

Let us discuss the terms depending on the interpolation error. We apply Lemma 3.7 of [9] which yields together with (3.59) the estimate

$$d_i^{\mu/2}\|y-I_hy\|_{L^\infty(\Omega_i)} \le c\begin{cases} h^2 d_i^{2-3\mu_j/2-\beta_j}|y|_{W^{2,\infty}_{\beta_j}(\Omega_i')} \le ch^2|y|_{W^{2,\infty}_{\beta_j}(\Omega_i')}, & \text{if } i=1,\dots,I-2,\\ h^{1/2+(2-\beta_j)/\mu_j}|y|_{W^{2,\infty}_{\beta_j}(\Omega_i')} \le ch^2|y|_{W^{2,\infty}_{\beta_j}(\Omega_i')}, & \text{if } i=I-1,I,\end{cases}$$

where we used the property $d_I^{\mu/2} \sim h^{1/2}$ to show the latter case. Summing up over all Ω_i, $i=1,\dots,I\sim|\ln h|$, leads to

$$h^{-1}|\ln h|^2\sum_{i=1}^{I} d_i^{\mu}\|y-I_hy\|^2_{L^\infty(\Omega_i)} \le ch^3|\ln h|^3|y|^2_{W^{2,\infty}_{\beta_j}(\Omega_R)}. \tag{3.71}$$

An estimate for the weighted finite element error on the right-hand side of (3.70) has been derived in [74, Lemma 3.61]. In this lemma we have to set $\tau := 1-\mu/2$. The criterion $\beta_j \le 3-\tau-2\mu_j$ is then equivalent to our assumption (3.59), and the result in our situation reads

$$\|\gamma^{\mu/2-1}(y-y_h)\|_{L^2(\Omega_{R/2})} \le c\left(h^2|\ln h|^{1/2}|y|_{W^{2,\infty}_{\beta_j}(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)}\right).$$

Now, we insert the global error estimate from Theorem 3.4.1 and arrive at

$$h^{-1}\|\gamma^{\mu/2-1}(y-y_h)\|^2_{L^2(\Omega_{R/2})} \le ch^3\left(|y|^2_{W^{2,2}_{\vec\alpha}(\Omega)} + |\ln h||y|^2_{W^{2,\infty}_{\vec\beta}(\Omega)}\right). \tag{3.72}$$

It remains to derive an error estimate on elements away from corner points. With the definition (3.64) we get

$$\sum_{\substack{E\in\mathcal{E}_h\\ E\cap\tilde\Gamma_{R/4}\neq\emptyset}} h_E^{-1}\|y-y_h\|^2_{L^2(E)} \le ch^{-1}\|y-y_h\|^2_{L^2(\tilde\Gamma_{R/8})}. \tag{3.73}$$

Using the Hölder inequality, the local maximum norm estimate (3.67) with $\tilde\Omega_{R/8}$ and $\tilde\Omega_{R/16}$ instead of Ω_i and Ω_i' (this yields $d=1/16$), standard interpolation error estimates as well as the global error estimate from Theorem 3.4.1, we arrive at

$$\begin{aligned}\|y-y_h\|_{L^2(\tilde\Gamma_{R/8})} &\le \|y-y_h\|_{L^\infty(\tilde\Omega_{R/8})}\\ &\le c\left(|\ln h|\|y-I_hy\|_{L^\infty(\tilde\Omega_{R/16})} + \|y-y_h\|_{L^2(\tilde\Omega_{R/16})}\right)\\ &\le ch^2\left(|y|_{W^{2,2}_{\vec\alpha}(\Omega)} + |\ln h||y|_{W^{2,\infty}_{\vec\beta}(\Omega)}\right).\end{aligned} \tag{3.74}$$

Collecting (3.61), (3.62), (3.70), (3.71), (3.72), (3.73) and (3.74) leads to the desired estimate. □

Corollary 3.4.6. *Let $f \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$ and $g \equiv 0$. Then, the error estimate*

$$\|y - y_h\|_{H^{1/2}(\Gamma)} \leq ch^{3/2}|\ln h|^{3/2}\|f\|_{C^{0,\sigma}(\overline{\Omega})}$$

holds, if one of the following assumptions are satisfied:

1. *The family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform ($\mu = 1$) and the interior angles of all corners of Ω are smaller than $120°$.*
2. *All corners of Ω having interior angle larger than $120°$ are refined locally according to (3.57) with refinement parameter*

$$\mu_j < 2\lambda_j/3 \qquad \forall j \in \mathcal{C}.$$

Proof. To conclude the assertion from Theorem 3.4.5 we first show that

$$y \in W^{2,\infty}_{\vec{\beta}}(\Omega) \cap W^{2,2}_{\vec{\alpha}}(\Omega) \cap W^{2,2}_{\vec{\gamma}}(\Gamma),$$

where $\vec{\alpha}, \vec{\beta}, \vec{\gamma} \in \mathbb{R}^d$ are weight vectors defined by

$$\alpha_j = \max\{0, 1 - \lambda_j + \varepsilon\}, \quad \beta_j = \max\{0, 2 - \lambda_j + \varepsilon/2\}, \quad \gamma_j = \max\{0, \frac{3}{2} - \lambda_j + \varepsilon\},$$

and $\varepsilon > 0$ is assumed to be sufficiently small. The regularity in $W^{2,\infty}_{\vec{\beta}}(\Omega)$ follows from Theorem 2.3.6. With the embeddings stated in Lemma 2.3.3 we moreover get

$$W^{2,\infty}_{\vec{\beta}}(\Omega) \hookrightarrow W^{2,2}_{\vec{\alpha}}(\Omega), \qquad W^{2,\infty}_{\vec{\beta}}(\Omega) \hookrightarrow W^{2,\infty}_{\vec{\beta}}(\Gamma) \hookrightarrow W^{2,2}_{\vec{\gamma}}(\Gamma). \tag{3.75}$$

It is simple to confirm that the assumption $\mu_j < 2\lambda_j/3$, $j \in \mathcal{C}$, and the definitions of $\vec{\alpha}, \vec{\beta}, \vec{\gamma} \in \mathbb{R}^d$ imply the conditions (3.59), and the desired estimate directly follows from Theorem 3.4.5 after taking the a-priori estimate from Theorem 2.3.6 into account. □

Remark 3.4.7. *We observe that the optimal convergence rate in the $H^1(\Omega)$- and $L^2(\Omega)$-norm is achieved on quasi-uniform meshes when $\lambda > 1$, i. e. when all corners have interior angle smaller than $180°$. This is not the case for the estimates on the boundary in the $L^2(\Gamma)$- and $H^{1/2}(\Gamma)$-norm as optimal convergence on quasi-uniform meshes is guaranteed only for $\lambda > 3/2$ meaning that the interior angles of all corners are smaller than $120°$. Thus, even for convex domains local mesh refinement is necessary to retain optimal convergence rates. The upper bounds for the refinement parameters are illustrated in Figure 3.4.*

3.4.2 Three-dimensional problems

We consider now error estimates for three-dimensional problems using the refinement condition (3.56). Note that we use the same grading towards all corners and edges, and hence, we have only one refinement parameter, namely $\mu \in (0, 1]$. Using different refinement parameters for different edges and corners is theoretically possible, see e. g. [61] for error estimates in $H^1(\Omega)$ and $L^2(\Omega)$, but the proof of the error estimate on the boundary presented in Theorem 3.4.14 would be too technical.

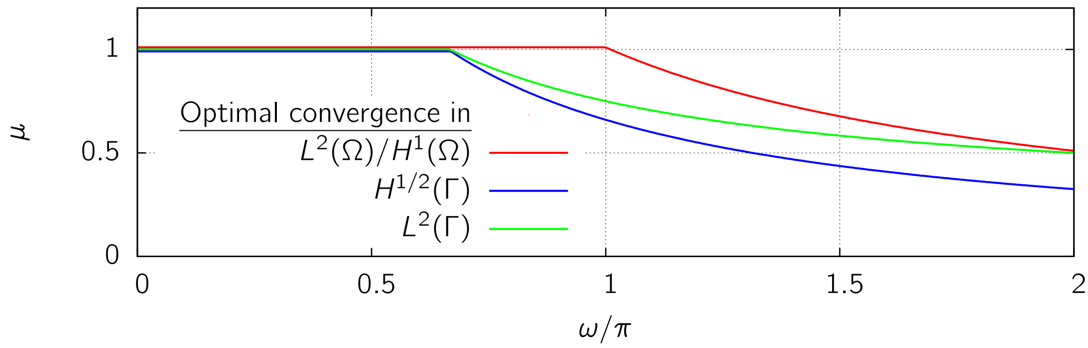


Figure 3.4: Illustration of the upper bounds for the refinement parameter μ derived in Theorems 3.4.1, 3.4.2 and 3.4.5.

For an arbitrary initial triangulation $\mathcal{T}_h^0$ one can for instance use the refinement strategy described by Bey [21] to generate a sequence of meshes $\{\mathcal{T}_h^k\}_{k=0}^K$ such that $\mathcal{T}_h^K$ satisfies the refinement condition (3.56). Bey's algorithm is applicable for an arbitrary marking strategy, but we will mark elements violating (3.56). In the k-th loop of this algorithm all marked tetrahedra $T \in \mathcal{T}_h^k$ are refined regularly, meaning that a new node is inserted at the midpoint of each edge of T, and T is decomposed into eight smaller tetrahedra. Note that the octahedron which is bounded by the six new nodes is decomposed into four tetrahedra, whereas three different choices are possible depending on which diagonal is chosen, and there exist strategies which avoid that subsequent meshes degenerate. To generate a conforming closure neighboring elements have to be refined appropriately. Each unrefined element which has at least one edge which is marked for refinement is also refined according to the following rules:

1. If one or two edges are marked, the element is refined by single or double bisection.
2. If three edges on the same face are marked, this face is refined regularly into four triangles and each of them forms the base surface of a new tetrahedron having its apex in the vertex opposite to the refined face.
3. If neither of the two rules can be applied the element is also refined regularly.

This strategy is repeated until no hanging nodes exist. The resulting triangulation is denoted by $\mathcal{T}_h^{k+1}$. If there are still elements violating the refinement condition (3.56) the procedure described above is repeated for $k = k+1$. If an element which has been generated by rule 1 or 2 is marked for refinement the refinement is revoked and the parent element is also refined regularly. This prevents that the elements degenerate. We will use this strategy in the numerical experiments in Section 4.3. There exist also other refinement strategies, for instance the newest-edge bisection algorithm from [17].

Let us now discuss finite element error estimates on meshes which are refined according to (3.56). We are again interested in upper bounds for the refinement parameter μ and begin with an improved finite element error estimate in the domain.

Theorem 3.4.8. *Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain. The family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is assumed to satisfy the condition (3.56). Moreover, let the mesh refinement parameter μ and* *the weights* $\vec{\alpha} \in \mathbb{R}_+^{d'}$ *and* $\vec{\delta} \in \mathbb{R}_+^{d}$ *fulfill the inequalities*

$$\begin{aligned} 1/2 - \lambda_j^c &< \alpha_j \le 1 - \mu, & \forall j \in \mathcal{C}, \\ 1 - \lambda_k^e &< \delta_k \le 1 - \mu, & \forall k \in \mathcal{E}. \end{aligned} \tag{3.76}$$

Then, for arbitrary input data $f \in W^{0,2}_{\vec{\alpha},\vec{\delta}}(\Omega)$, $g \in W^{1/2,2}_{\vec{\alpha},\vec{\delta}}(\Gamma)$ *the* a priori *error estimate*

$$\|y - y_h\|_{H^\ell(\Omega)} \le ch^{2-\ell}|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} \le ch^{2-\ell}\left(\|f\|_{W^{0,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\alpha},\vec{\delta}}(\Gamma)}\right) \tag{3.77}$$

holds for $\ell \in \{0, 1\}$.

Proof. As a consequence of Céa's Lemma and the decomposition of the domain Ω we obtain

$$\|y - y_h\|^2_{H^1(\Omega)} \le \sum_{T \in \mathcal{T}_h} \|y - Z_h y\|^2_{H^1(T)}. \tag{3.78}$$

It remains to apply the interpolation error estimates from Lemma 3.2.6 and to adjust the refinement parameter such that the desired convergence rate is achieved. The largest weight is denoted by $\kappa := \max_{j\in\mathcal{C},k\in\mathcal{E}}\{\alpha_j, \delta_k\}$. In case of $r_{S_T} = 0$ we have $h_T = h^{1/\mu}$ and with $\mu \le 1 - \kappa$ we get

$$\|y - Z_h y\|_{H^1(T)} \le ch^{(1-\kappa)/\mu}|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(S_T)} \le ch|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(S_T)}.$$

Otherwise, if $r_{S_T} > 0$, the mesh condition yields $h_T = hr_T^{1-\mu}$ and consequently

$$\|y - Z_h y\|_{H^1(T)} \le chr_T^{1-\mu-\kappa}|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(S_T)} \le ch|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(S_T)}.$$

Inserting these local estimates into (3.78) leads to

$$\|y - y_h\|_{H^1(\Omega)} \le ch|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} \le ch\left(\|f\|_{W^{0,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)}\right). \tag{3.79}$$

Here, we also applied the regularity result of Theorem 2.3.7 whose conditions are satisfied under our assumptions upon $\vec{\alpha}$ and $\vec{\delta}$. To show the estimate for $\ell = 0$ we can repeat the arguments used in the proof of Theorem 3.3.1. □

If we assume slightly better regularity of the input data in classical Sobolev spaces we get by the embeddings from Lemma 2.3.4 the following simplified version of Theorem 3.4.8. Recall that the singular exponent belonging to the strongest singularity is

$$\lambda := \min_{\substack{j\in\mathcal{C}\\k\in\mathcal{E}}}\{1/2 + \lambda_j^c, \lambda_k^e\}, \tag{3.80}$$

and the bound for the refinement parameter from (3.76) depends solely on this number.

Corollary 3.4.9. *Let functions* $f \in L^2(\Omega)$ *and* $g \in H^{1/2}(\Gamma)$ *be given. The error estimate*

$$\|y - y_h\|_{H^\ell(\Omega)} \le ch^{2-\ell}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\right)$$

holds, provided that one of the following assumptions holds:

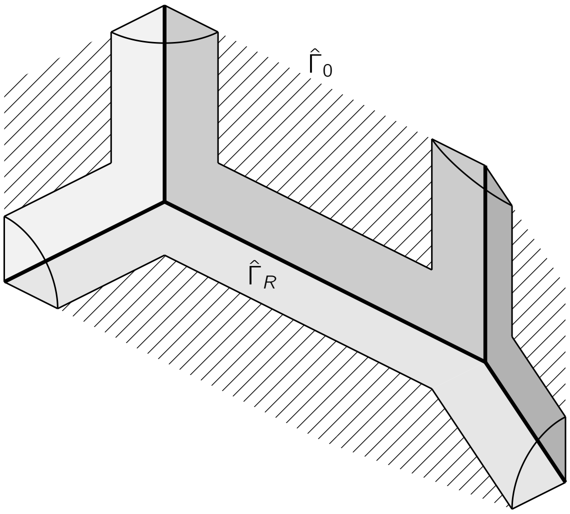


Figure 3.5: Decomposition of the boundary into $\hat{\Gamma}_R$ and $\hat{\Gamma}_0$.

1. *The family of triangulations* $\{\mathcal{T}_h\}_{h>0}$ *is quasi-uniform (i.e.* $\mu = 1$*) and the singular exponents satisfy* $\lambda > 1$.
2. *The family of triangulations* $\{\mathcal{T}_h\}_{h>0}$ *is refined according to* (3.56) *with refinement parameter*
$$\mu < \lambda.$$

In the remainder of this section, the finite element error on the boundary Γ is investigated. The initial step of the convergence proof is an appropriate decomposition of Γ. In order to extract those parts of the domain which are under influence of singularities we define the sets

$$\begin{aligned} \Omega_R &:= \{x\colon 0 < r(x) < R\} \cap \Omega, & \Gamma_R &:= \partial\Omega_R \cap \Gamma, \\ \hat{\Omega}_R &:= \{x\colon 0 < r(x) < R/2\} \cap \Omega, & \hat{\Gamma}_R &:= \partial\hat{\Omega}_R \cap \Gamma, \end{aligned} \tag{3.81}$$

illustrated in Figure 3.5. Remember that $r(\cdot) := \min_{k\in\mathcal{E}} r_k(\cdot)$ stands for the minimum distance to the singular points. The boundary part which is not influenced by singularities is denoted by $\hat{\Gamma}_0 := \Gamma\backslash\hat{\Gamma}_R$. Without loss of generality we will set $R = 1$ in the following, because as the circumstances require the domain Ω has to be rescaled appropriately.

For technical reasons we introduce a decomposition of the domain Ω_R as follows. Let $d_i := 2^{-i}$, $i = 0, \ldots, I$ and let $c_I > 0$ be a constant independent of h such that $d_I = c_I\, h^{1/\mu}$ holds. This implies the property $I \sim |\ln h|$. The constant c_I will be specified at the end of the proof of Theorem 3.4.13, where a kick-back argument is applied. In some steps of our proof, when the constant is immaterial, we will hide it in the generic constant c. As illustrated in Figure 3.6 we introduce the dyadic decomposition

$$\Omega_R = \operatorname{int}\bigcup_{i=0}^{I}\overline{\Omega}_i \quad \text{with} \quad \Omega_i := \begin{cases} \{x \in \Omega_R\colon d_{i+1} < r(x) < d_i\}, & \text{for } \ i = 0, 1, \ldots, I-1, \\ \{x \in \Omega_R\colon 0 < r(x) < d_I\}, & \text{for } \ i = I. \end{cases}$$

A decomposition of the boundary part Γ_R is then given by

$$\Gamma_i := \partial\Omega_i \cap \Gamma, \qquad i = 0, \ldots, I. \tag{3.82}$$

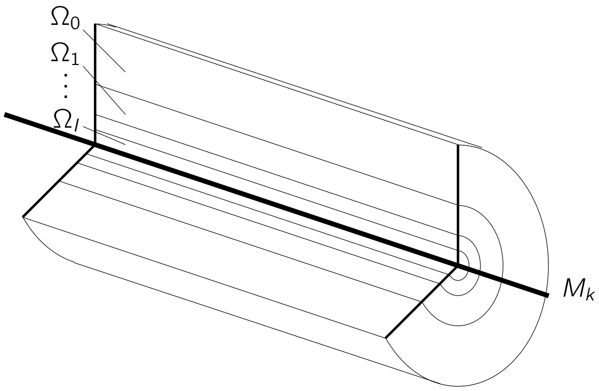
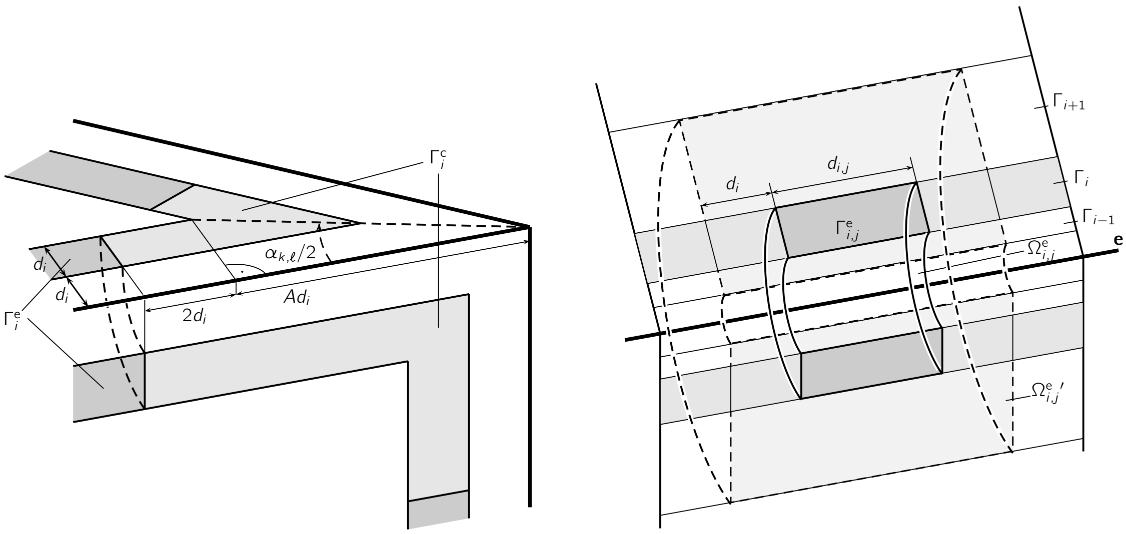


Figure 3.6: Dyadic decomposition of Ω_R along an edge.



(a) Definition of Ω_i^e and Ω_i^c (b) Definition of $\Omega_{i,j}^e$ and its patch

Figure 3.7: Illustration of the domains Ω_i^c and $\Omega_{i,j}^e$.

Note that the elements contained in Ω_i or intersecting Ω_i satisfy

$$h_T \sim \begin{cases} hd_i^{1-\mu}, & \text{for } \ i = 0, 1, \ldots, l-1, \\ h^{1/\mu}, & \text{for } \ i = l. \end{cases}$$

We will further need the patches of Ω_i with its adjacent sets defined by

$$\Omega_i^{(m)} := \operatorname{int}\left(\bar{\Omega}_{\max\{0,i-m\}} \cup \ldots \cup \bar{\Omega}_i \cup \ldots \cup \bar{\Omega}_{\min\{l,i+m\}}\right), \quad m \in \mathbb{N},$$

and write $\Omega_i' := \Omega_i^{(1)}$, $\Omega_i'' := \Omega_i^{(2)}$. In order to separate the parts of Ω_i where only edge singularities and where both corner and edge singularities are present we introduce a further decomposition of Ω_i. Let (x_k, y_k, z_k), $k \in \mathcal{E}$, denote Cartesian coordinate systems having origin in some corner $\boldsymbol{c} = x^{(j)}$, $j \in \mathcal{C}$ with $k \in X_j$, such that the z_k-axes coincide with the edges M_k. Moreover, define $\alpha_{min}^j := \min_{k,\ell \in X_j} \alpha_{k,\ell}$ where $\alpha_{k,\ell}$ is the angle between the edges M_k and M_ℓ, and introduce the set

$$\Omega_i^{c} := \bigcup_{k\in X_j} \{x \in \Omega_i \colon z_k(x) \in (0,(2+A)\,d_i)\}\,, \qquad \Gamma_i^{c} := \partial\Omega_i^{c} \cap \Gamma,$$

with

$$A := 2\min_{j\in\mathcal{C}} \cot\frac{\alpha^j_{min}}{2} \sim 1.$$

This set is illustrated in Figure 3.7a). It is easy to confirm that $|\Gamma_i^{c}| \sim d_i^2$. The remaining parts of Γ_i are away from the singular corners and are defined as follows. We fix an edge $\boldsymbol{e} := M_k$ having length $L_{\boldsymbol{e}}$ and endpoints $x^{(j)}$, $x^{(j')}$, introduce the interval

$$Z_{\boldsymbol{e}} := ((2+A)\,d_i, L_{\boldsymbol{e}} - (2+A)\,d_i)\,,$$

and define the sets

$$\Omega_i^{\boldsymbol{e}} := \{x \in \Omega_i \colon z_k(x) \in Z_{\boldsymbol{e}}\}\,, \qquad \Gamma_i^{\boldsymbol{e}} := \partial\Omega_i^{\boldsymbol{e}} \cap \Gamma.$$

We observe that the boundary part Γ_i is covered completely by the sets defined above, i. e.

$$\Gamma_i = \operatorname{int}\left(\bigcup_{j\in\mathcal{C}} \overline{\Gamma_i^{x^{(j)}}} \cup \bigcup_{k\in\mathcal{M}} \overline{\Gamma_i^{M_k}}\right) \tag{3.83}$$

It remains to define appropriate patches

$$\Omega_i^{c,(m)} := \bigcup_{k\in X_j} \left\{x \in \Omega_i^{(m)} \colon z_k(x) \in (0,(2+m-A)d_i)\right\},$$

$$\Omega_i^{\boldsymbol{e},(m)} := \left\{x \in \Omega_i^{(m)} \colon z_k(x) \in ((2-m+A)d_i, L_{\boldsymbol{e}} - (2-m+A)d_i)\right\},$$

for $m \in \{1,2\}$. We write in the following

$$\Omega_i^{c,(1)} = \Omega_i^{c\prime}, \quad \Omega_i^{c,(2)} = \Omega_i^{c\prime\prime}, \quad \Omega_i^{\boldsymbol{e},(1)} = \Omega_i^{\boldsymbol{e}\prime}, \quad \Omega_i^{\boldsymbol{e},(2)} = \Omega_i^{\boldsymbol{e}\prime\prime}.$$

The essential property that we exploit in the proof of Lemma 3.4.12 is, that

$$\operatorname{dist}\left(\partial\Omega_i^{\boldsymbol{e}\prime} \setminus \Gamma, \partial\Omega_i^{\boldsymbol{e}} \setminus \Gamma\right) \sim d_i, \qquad \operatorname{dist}\left(\partial\Omega_i^{c\prime} \setminus \Gamma, \partial\Omega_i^{c} \setminus \Gamma\right) \sim d_i.$$

We moreover require a dyadic decomposition of $\Omega_i^{\boldsymbol{e}}$ and its patches $\Omega_i^{\boldsymbol{e},(m)}$ in order to carve out the influence of the corner singularity. For $j = 0,\dots,i$ and $m \in \{0,1,2\}$ we define

$$\begin{aligned}
\Omega_{i,j}^{\boldsymbol{e},+,(m)} &:= \left\{x \in \Omega_i^{\boldsymbol{e},(m)} \colon z_k(x) \in ((1+A+2^j-m)d_i, (1+A+2^{j+1}+m)d_i)\right\},\\
\Omega_{i,j}^{\boldsymbol{e},-,(m)} &:= \left\{x \in \Omega_i^{\boldsymbol{e},(m)} \colon z_k(x) \in (L_{\boldsymbol{e}} - (1+A+2^{j+1}+m)d_i, L_{\boldsymbol{e}} - (1+A+2^j-m)d_i)\right\},\\
\tilde\Omega_i^{\boldsymbol{e},(m)} &:= \left\{x \in \Omega_i^{\boldsymbol{e},(m)} \colon z_k(x) \in ((1+A+2^{i+1}-m)d_i, L_{\boldsymbol{e}} - (1+A+2^{i+1}-m)d_i)\right\},
\end{aligned} \tag{3.84}$$

and we observe that

$$\Omega_i^{\boldsymbol{e},(m)} = \bigcup_{j=0}^{i} \Omega_{i,j}^{\boldsymbol{e},\pm,(m)} \cup \tilde\Omega_i^{\boldsymbol{e},(m)}.$$

As usual, the boundary parts are denoted by

$$\Gamma_{i,j}^{e,\pm} := \partial\Omega_{i,j}^{e,\pm} \cap \Gamma, \qquad \tilde{\Gamma}_i^e := \partial\tilde{\Omega}_i^e \cap \Gamma.$$

One easily confirms that the properties

$$\begin{aligned} |\Omega_{i,j}^{e,\pm,(m)}| &\sim d_i^2 d_{i,j}, & |\tilde{\Omega}_i^{e,(m)}| &\sim d_i^2, \\ |\Gamma_{i,j}^{e,\pm,(m)}| &\sim d_i d_{i,j}, & |\tilde{\Gamma}_i^{e,(m)}| &\sim d_i, \end{aligned} \tag{3.85}$$

hold for $i = 0, \ldots, I$ and $j = 0, \ldots, i$, with

$$d_{i,j} := 2^j d_i = 2^{j-i} \leq 1.$$

The first step of the proof is to derive some interpolation error estimates on the subdomains Ω_i^e and Ω_i^c. We will require estimates in the H^ℓ-norm ($\ell = 0, 1$) as well as in the L^∞-norm.

Lemma 3.4.10. *Let some function $u \in H^1(\Omega_i^{(m+1)})$ with $m \in \{0, 1\}$ be given and assume that $D^{\boldsymbol{\alpha}} u \in W^{1,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{(m+1)})$ for all $|\boldsymbol{\alpha}| = 1$. Assume that $p \in [2, \infty]$ and that the weights satisfy*

$$\begin{aligned} &0 \leq \alpha_j < 5/2 - 3/p, && j \in \mathcal{C}, \\ &0 \leq \delta_k < 5/3 - 2/p, && k \in \mathcal{E}. \end{aligned}$$

Let $\boldsymbol{e} := M_k$, $k \in \mathcal{E}$, and $\boldsymbol{c} := x^{(j)}$, $j \in \mathcal{C}$, be an arbitrary edge and corner, respectively. Moreover, define the numbers $\kappa_j := \max\{\alpha_j, \max_{k \in X_j} \delta_k\}$, $\tilde{\alpha}_k := \max\{\alpha_j, \alpha_{j'}\}$ where $j \neq j'$ are the corner indices such that $k \in X_j \cap X_{j'}$, $s_k := 1/2 - 1/p + \delta_k - \tilde{\alpha}_k$, and $\Theta_\ell := (7/2 - \ell - 3/p)(1 - \mu)$.

a) For $i = 0, \ldots, I - 2$ there hold the estimates

$$\begin{aligned} |u - Z_h u|_{H^\ell(\Omega_i^{c,(m)})} &\leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+3/2-3/p-\kappa_j} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{c,(m+1)})}, \\ |u - Z_h u|_{H^\ell(\Omega_i^{e,(m)})} &\leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+1-2/p-\delta_k+[s_k]_-} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{e,(m+1)})}. \end{aligned}$$

b) For $i = I - 1, I$ there hold the estimates

$$\begin{aligned} |u - Z_h u|_{H^\ell(\Omega_i^{c,(m)})} &\leq c c_I^{[\Theta_\ell - \kappa_j]_+ + 3/2 - 3/p} h^{(7/2 - 3/p - \ell - \kappa_j)/\mu} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{c,(m+1)})}, \\ |u - Z_h u|_{H^\ell(\Omega_i^{e,(m)})} &\leq c c_I^{[\Theta_\ell - \delta_k]_+ + 1 - 2/p} h^{(3 - 2/p - \ell - \delta_k + [s_k]_-)/\mu} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{e,(m+1)})}. \end{aligned}$$

Proof. Without loss of generality we prove the assertion for $m = 0$. The same arguments can be applied in case of $m = 1$ either. We begin with the estimate on Ω_i^c. To prove the assertion we merely apply the discrete Hölder inequality

$$|u - Z_h u|^2_{H^\ell(\Omega_i^c)} \leq \left(\sum_{T \cap \Omega_i^c \neq \emptyset} 1 \right)^{1-2/p} \left(\sum_{T \cap \Omega_i^c \neq \emptyset} |u - Z_h u|^p_{H^\ell(T)} \right)^{2/p}, \tag{3.86}$$

and insert the local estimates from Lemma 3.2.6 as well as an estimate for the number of elements intersecting Ω_i^c.

For the case $i = 0, \dots, l-2$, the number of elements contained in Ω_i^c is of order

$$\sum_{T \cap \Omega_i^c \neq \emptyset} 1 \leq c \max_{T \cap \Omega_i^c \neq \emptyset} \frac{|\Omega_i^c|}{|T|} \leq c \max_{T \cap \Omega_i^c \neq \emptyset} \frac{d_i^3}{|T|}. \tag{3.87}$$

For all $T \cap \Omega_i^c \neq \emptyset$ we obtain with Lemma 3.2.6 and the properties $h_T \sim h r_T^{1-\mu}$ and $r_T \sim d_i$ the estimate

$$|u - Z_h u|_{H^\ell(T)} \leq c h_T^{2-\ell} |T|^{1/2-1/p} d_i^{-\kappa_j} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}. \tag{3.88}$$

Inserting this together with (3.87) into (3.86) leads to

$$|u - Z_h u|^2_{H^\ell(\Omega_i^c)} \leq c h^{2(2-\ell)} d_i^{2\left((2-\ell)(1-\mu)+3(1/2-1/p)-\kappa_j\right)} |u|^2_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{c\prime})}. \tag{3.89}$$

Extracting the root yields the desired estimate on Ω_i^c for $i = 0, \dots, l-2$.

In order to derive the estimate on Ω_i^e we can basically use the same technique. Certainly, we have to decompose the domain Ω_i^e into subsets defined in (3.84). For all elements intersecting $\Omega_{i,j}^{e,\pm}$ or $\tilde{\Omega}_i^e$ we get from Lemma 3.2.6 the local estimates

$$\begin{aligned} |u - Z_h u|_{H^\ell(T)} &\leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)-\delta_k} d_{i,j}^{\delta_k - \tilde{\alpha}_k} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}, && \text{if } T \cap \Omega_{i,j}^{e,\pm} \neq \emptyset, \\ |u - Z_h u|_{H^\ell(T)} &\leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)-\delta_k} |T|^{1/2-1/p} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}, && \text{if } T \cap \tilde{\Omega}_i^e \neq \emptyset. \end{aligned} \tag{3.90}$$

The number of elements which intersect $\Omega_{i,j}^{e,\pm}$ and $\tilde{\Omega}_i^e$ is of order

$$\sum_{T \cap \Omega_{i,j}^{e,\pm} \neq \emptyset} 1 \leq c \max_{T \cap \Omega_{i,j}^{e,\pm}} \frac{d_i^2 d_{i,j}}{|T|} \quad \text{and} \quad \sum_{T \cap \tilde{\Omega}_i^e \neq \emptyset} 1 \leq c \max_{T \cap \tilde{\Omega}_i^e} \frac{d_i^2}{|T|},$$

respectively, compare also (3.87). From the Hölder inequality (3.86) we then obtain

$$|u - Z_h u|_{H^\ell(\Omega_i^e)} \leq c h^{2-\ell} d_i^{(2-\ell)(1-\mu)+1-2/p-\delta_k} \left(\sum_{j=0}^{i} d_{i,j}^{(1/2-1/p+\delta_k-\tilde{\alpha}_k)p'} \right)^{1/p'} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{e\prime})}, \tag{3.91}$$

where $p^{-1} + p'^{-1} = 1$. The limit value of the geometric series yields

$$\sum_{j=0}^{i} d_{i,j}^{s_k p'} = d_i^{s_k p'} \sum_{j=0}^{i} 2^{j s_k p'} \leq c d_i^{s_k p'} (2^{(i+1) s_k p'} - 1) \leq c (2^{s_k p'} + d_i^{s_k p'}) \leq c d_i^{[s_k]_- p'}, \tag{3.92}$$

and we conclude from (3.91) the desired estimate on Ω_i^e for $i = 0, \dots, l-2$.

Let us now consider the case $i = l-1, l$. We start with an estimate on Ω_i^c, where $\boldsymbol{c} = x^{(j)}$ for some $j \in \mathcal{C}$. Taking $d_i \sim d_l = c_l h^{1/\mu}$ into account, the number of elements can be estimated by

$$\sum_{T \cap \Omega_i^c \neq \emptyset} 1 \leq c d_i^3 |T_{min}|^{-1} \leq c c_l^3 h^{3/\mu} |T_{min}|^{-1}, \tag{3.93}$$

where $|T_{min}| \sim h^{3/\mu}$. Due to the mesh condition we have to distinguish between the cases whether the patch S_T touches the singular points or not. If $r_{S_T} > 0$ the estimate (3.88) can be applied again. Using the mesh condition $h_T \sim hd_i^{1-\mu} \le c_l^{1-\mu}h^{1/\mu}$, the property $h^{1/\mu} \le r_T \le d_i \sim c_l h^{1/\mu}$ as well as

$$|T| \le ch^3 d_l^{3(1-\mu)} \le cc_l^{3(1-\mu)}h^{3/\mu} \le cc_l^{3(1-\mu)}|T_{min}|, \tag{3.94}$$

we obtain

$$|u - Z_h u|_{H^\ell(T)} \le cc_l^{[\Theta_\ell - \kappa_j]_+} h^{(2-\ell-\kappa_j)/\mu}|T_{min}|^{1/2-1/p}|u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}. \tag{3.95}$$

From Lemma 3.2.6 we directly conclude that the same estimate holds also for $r_{S_T} = 0$ even without the factor $c_l^{[\Theta_\ell - \kappa_j]_+}$. Next, we apply the Hölder inequality (3.86) together with (3.93) and obtain

$$|u - Z_h u|_{H^\ell(\Omega_i^c)} \le cc_l^{[\Theta_\ell - \kappa_j]_+ + 3/2 - 3/p} h^{(7/2-\ell-3/p-\kappa_j)/\mu}|u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i')}. \tag{3.96}$$

With a similar technique we can show an estimate on $\Omega_{i,j}^{e,\pm}$ for $i = I-1, I$ and $j = 0, \dots, i$. For all $T \cap \Omega_{i,j}^{e,\pm}$ with $r_{S_T} > 0$ we conclude from (3.90) using the properties (3.94) and $d_i \sim c_l h^{1/\mu}$ the estimate

$$|u - Z_h u|_{H^\ell(T)} \le cc_l^{[\Theta_\ell - \delta_k]_+} h^{(2-\ell-\delta_k)/\mu}|T_{min}|^{1/2-1/p} d_{i,j}^{\delta_k - \tilde{\alpha}_k}|u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)}.$$

One easily confirms that this estimate holds also in case of $r_{S_T} = 0$ when taking Lemma 3.2.6 into account. The number of elements which intersect $\Omega_{i,j}^{e,\pm}$ is of order

$$\sum_{T \cap \Omega_{i,j}^e \ne \emptyset} 1 \le cd_i^2 d_{i,j}|T_{min}|^{-1} \le c_l^2 h^{2/\mu} d_{i,j}|T_{min}|^{-1}.$$

Consequently, we get using the local estimates and the Hölder inequality (3.86)

$$|u - Z_h u|_{H^\ell(\Omega_{i,j}^{e,\pm})} \le cc_l^{[\Theta_\ell - \delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k)/\mu} d_{i,j}^{1/2-1/p+\delta_k-\tilde{\alpha}_k} \left(\sum_{T \cap \Omega_{i,j}^{e,\pm} \ne \emptyset} |u|^p_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(S_T)} \right)^{1/p}.$$

Summing up over all $\Omega_{i,j}^{e,\pm}$ for $j = 0, \dots, i$ yields

$$\begin{aligned} \left(\sum_{j=0}^{i} |u - Z_h u|^2_{H^\ell(\Omega_{i,j}^{e,\pm})} \right)^{1/2} &\le cc_l^{[\Theta_\ell - \delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k)/\mu} \left(\sum_{j=0}^{i} d_{i,j}^{s_k p'} \right)^{1/p'} |u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i')} \\ &\le cc_l^{[\Theta_\ell - \delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k+[s_k]_-)/\mu}|u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i')}, \end{aligned} \tag{3.97}$$

where we used the estimate (3.92) and the fact that $c_l^{[s_k]_-} \le 1$ in the last step.

For all $T \cap \tilde{\Omega}_i^e \ne \emptyset$ there holds $\rho_{j,S_T} \sim 1$ and as the number of these elements is of order

$$\sum_{T \cap \tilde{\Omega}_i^e \ne \emptyset} 1 \le cd_i^2|T_{min}|^{-1} \le c_l^2 h^{2/\mu}|T_{min}|^{-1},$$

we get

$$|u - Z_h u|_{H^\ell(\tilde{\Omega}_i^e)} \le cc_l^{[\Theta_\ell - \delta_k]_+ + 1 - 2/p} h^{(3-2/p-\ell-\delta_k)/\mu}|u|_{W^{2,p}_{\vec{\alpha},\vec{\delta}}(\Omega_i^{e\prime})}. \tag{3.98}$$

Finally, from the decomposition (3.84) and the estimates (3.97) and (3.98) we conclude the estimate on Ω_i^e for $i = I-1, I$. □

Lemma 3.4.11. *Let some function $u \in L^\infty(\Omega_i^{(m+1)})$, $m \in \{0,1\}$, be given and assume that $D^{\boldsymbol{\alpha}} u \in W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i^{(m+1)})$ for all $|\boldsymbol{\alpha}| = 1$, and $u \equiv 0$ on $\Omega \setminus \Omega_R$. Moreover, assume that the weight vectors $\vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\varrho} \in \mathbb{R}^d$ satisfy*

$$\begin{aligned} &0 \le \beta_j < 2, && j \in \mathcal{C}, \\ &0 \le \varrho_k < 5/3, && k \in \mathcal{E}. \end{aligned}$$

Define $\kappa_j = \max\{\beta_j, \max_{k \in X_j} \varrho_k\}$, $\tilde{\beta}_k := \max\{\beta_j : j \in \mathcal{C}$ such that $k \in X_j\}$ and $\Theta := 2(1-\mu)$. Then, for all corners $\boldsymbol{c} := x^{(j)}$, $j \in \mathcal{C}$, and edges $\boldsymbol{e} := M_k$, $k \in \mathcal{E}$, the following estimates hold:

a) For $i = 0, 1, \dots, l-2-m$ there hold the estimates

$$\begin{aligned} \|u - I_h u\|_{L^\infty(\Omega_i^{\boldsymbol{c},(m)})} &\le c h^2 d_i^{2(1-\mu)-\kappa_j} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i^{\boldsymbol{c},(m+1)})}, \\ \|u - I_h u\|_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm,(m)})} &\le c h^2 d_i^{2(1-\mu)-\varrho_k} d_{i,j}^{\varrho_k - \tilde{\beta}_k} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_{i,j}^{\boldsymbol{e},\pm,(m+1)})}, \\ \|u - I_h u\|_{L^\infty(\tilde{\Omega}_i^{\boldsymbol{e},(m)})} &\le c h^2 d_i^{2(1-\mu)-\varrho_k} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\tilde{\Omega}_i^{\boldsymbol{e},(m+1)})}. \end{aligned}$$

b) For $i = l-1-m, \dots, l$ there hold the estimates

$$\begin{aligned} \|u - I_h u\|_{L^\infty(\Omega_i^{\boldsymbol{c},(m)})} &\le c_l^{[\Theta-\kappa_j]_+} h^{(2-\kappa_j)/\mu} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i^{\boldsymbol{c},(m+1)})}, \\ \|u - I_h u\|_{L^\infty(\Omega_{i,j}^{\boldsymbol{e},\pm,(m)})} &\le c_l^{[\Theta-\varrho_k]_+} h^{(2-\varrho_k)/\mu} d_{i,j}^{\varrho_k - \tilde{\beta}_k} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_{i,j}^{\boldsymbol{e},\pm,(m+1)})}, \\ \|u - I_h u\|_{L^\infty(\tilde{\Omega}_i^{\boldsymbol{e},(m)})} &\le c_l^{[\Theta-\varrho_k]_+} h^{(2-\varrho_k)/\mu} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\tilde{\Omega}_i^{\boldsymbol{e},(m+1)})}. \end{aligned}$$

Proof. We prove the assertion merely for $m = 0$ since the extension to $m = 1$ is simple. Let $T^* \cap \Omega_i^{\boldsymbol{c}} \neq \emptyset$ be the element where the maximum of $|u(x) - I_h u(x)|$ within $\Omega_i^{\boldsymbol{c}}$ is attained. We first investigate the case $i = 0, \dots, l-2$. It suffices to insert the local estimate from Lemma 3.2.3 which leads to

$$\|u - I_h u\|_{L^\infty(\Omega_i^{\boldsymbol{c}})} \le \|u - I_h u\|_{L^\infty(T^*)} \le c h^2 d_i^{2(1-\mu)-\kappa_j} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i^{\boldsymbol{c}\prime})},$$

where we exploited the mesh criterion $h_T \sim h r_T^{1-\mu}$ as well as the property $r_T \sim d_i$ in the last step.

To obtain the desired estimates for $i = l-1, l$ we distinguish the cases that T^* touches the singular points or not. If $r_{T^*} = 0$ we get from Lemma 3.2.3 and $h_T \sim h^{1/\mu}$ the estimate

$$\|u - I_h u\|_{L^\infty(T^*)} \le c h^{(2-\kappa_j)/\mu} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(T^*)}. \tag{3.99}$$

Otherwise, if $r_{T^*} > 0$, we use $d_l = c_l h^{1/\mu}$ to obtain $h_{T^*} \le h d_l^{1-\mu} = c_l^{1-\mu} h^{1/\mu}$ and $d_l^{-\kappa_j} = c_l^{-\kappa_j} h^{-\kappa_j/\mu}$, and from Lemma 3.2.3 we conclude that

$$\|u - I_h u\|_{L^\infty(T^*)} \le c h_{T^*}^2 d_i^{-\kappa_j} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(T^*)} \le c c_l^{2(1-\mu)-\kappa_j} h^{(2-\kappa_j)/\mu} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(T^*)}. \tag{3.100}$$

The estimates (3.99) and (3.100) with $T^* \subset \Omega_i^{\boldsymbol{c}\prime}$ imply the assertion for the domains $\Omega_i^{\boldsymbol{c}}$.

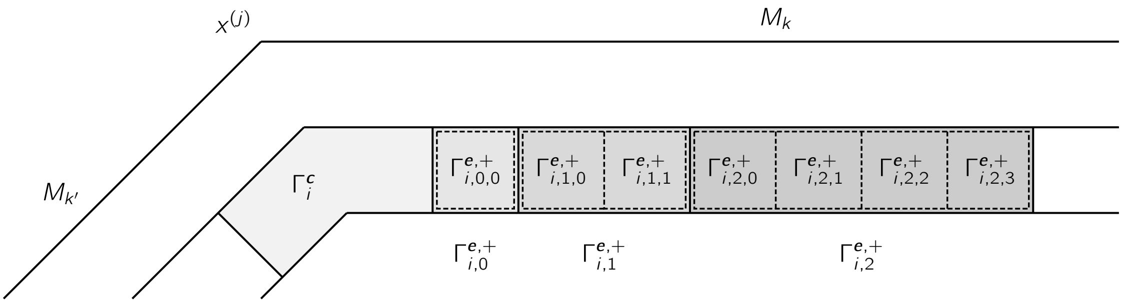


Figure 3.8: Illustration of the sets introduced in (3.109).

Next, we show the estimate on $\Omega_{i,j}^{e,\pm}$ in case of $i = 0, \dots, I-2$. Let $x^{(j_1)}$ and $x^{(j_2)}$, $j_1, j_2 \in \mathcal{C}$, denote the endpoints of the edge $\boldsymbol{e}$. We apply Lemma 3.2.3 and exploit that

$$\begin{aligned} r_{k,T^*} &\sim d_i && \text{if} \quad T^* \cap \Omega_{i,j}^{e,\pm} \neq \emptyset, \\ \rho_{j_1,T^*} &\sim d_{i,j} && \text{if} \quad T^* \cap \Omega_{i,j}^{e,+} \neq \emptyset, \\ \rho_{j_2,T^*} &\sim d_{i,j} && \text{if} \quad T^* \cap \Omega_{i,j}^{e,-} \neq \emptyset. \end{aligned} \tag{3.101}$$

This leads to the local estimate

$$\|u - I_h u\|_{L^\infty(T^*)} \le c h^2 d_i^{2(1-\mu)-\varrho_k} d_{i,j}^{\varrho_k - \tilde{\beta}_k} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(T^*)} \tag{3.102}$$

from which we conclude the assertion for $i = 0, \dots, I-2$. For $i = I-1, I$ we distinguish among the cases $r_{T^*} > 0$ and $r_{T^*} = 0$. To show an estimate for $r_{T^*} > 0$ we insert the property $d_i \sim c_I h^{1/\mu}$ into (3.102). In case of $r_{T^*} = 0$ we insert (3.101) into the local estimate from Lemma 3.2.3. In both cases we then obtain

$$\|u - I_h u\|_{L^\infty(T^*)} \le c c_I^{[\Theta - \varrho_k]_+} h^{(2-\varrho_k)/\mu} d_{i,j}^{\varrho_k - \beta_j} |u|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(T^*)},$$

which yields the assertion as $T^* \subset \Omega_{i,j}^{e,\pm\prime}$. The estimates on $\tilde{\Omega}_i^e$ follow from the same strategy exploiting that $\rho_{j,T^*} \sim 1$ for all $T \cap \tilde{\Omega}_i^e \neq \emptyset$. □

The next step is to show an initial error estimate on a single boundary strip Γ_i. Afterwards we will combine the following result to a global estimate in Theorem 3.4.13.

Lemma 3.4.12. *Let $y \in H^1(\Omega_R) \cap L^\infty(\Omega_R)$ and denote by y_h its Ritz projection, i. e.*

$$\int_{\Omega_R} \left(\nabla(y - y_h)(x) \cdot \nabla v_h(x) + (y - y_h)(x) v_h(x)\right) \mathrm{d}x = 0 \qquad \forall v_h \in V_h.$$

Then, for arbitrary $i \in \{1, \dots, I\}$ the local estimates

$$\|y - y_h\|_{L^2(\Gamma_i^c)} \le c \left(d_i |\ln h| \|y - I_h y\|_{L^\infty(\Omega_i^{c\prime})} + d_i^{-1/2} \|y - y_h\|_{L^2(\Omega_i')} \right), \tag{3.103}$$

$$\begin{aligned} \|y - y_h\|_{L^2(\Gamma_i^e)} \le c \Bigg(&\sum_{j=0}^{i} d_i^{1/2} d_{i,j}^{1/2} |\ln h| \|y - I_h y\|_{L^\infty(\Omega_{i,j}^{e,\pm\prime})} \\ &+ d_i^{1/2} |\ln h| \|y - I_h y\|_{L^\infty(\tilde{\Omega}_i^{e\prime})} + d_i^{-1/2} \|y - y_h\|_{L^2(\Omega_i')} \Bigg), \end{aligned} \tag{3.104}$$

hold for all corners $\boldsymbol{c} := x^{(j)}$, $j \in \mathcal{C}$, and edges $\boldsymbol{e} := M_k$, $k \in \mathcal{E}$.

Proof. To obtain the desired estimate on Γ_i^c we apply the Hölder inequality with $|\Gamma_i^c| \sim d_i^2$, and a trace theorem which leads to

$$\|y - y_h\|_{L^2(\Gamma_i^c)} \le d_i \|y - y_h\|_{L^\infty(\Gamma_i^c)} \le d_i \|y - y_h\|_{L^\infty(\Omega_i^c)} \tag{3.105}$$

Now we can apply the local maximum norm estimate from Theorem 10.1 and Example 10.1 in [90], which reads in our situation

$$\|y - y_h\|_{L^\infty(\Omega_i^c)} \le c \left(|\ln h| \|y - I_h y\|_{L^\infty(\Omega_i^{c\prime})} + d^{-3/2} \|y - y_h\|_{L^2(\Omega_i')} \right), \tag{3.106}$$

with $d := \operatorname{dist}(\partial\Omega_i^{c\prime} \setminus \Gamma,\ \partial\Omega_i^c \setminus \Gamma)$. Due to our construction we find that $d \sim d_i$. Inserting (3.106) into (3.105) yields (3.103) for $i = 1, \dots, I-2$.

To show the estimate on Γ_i^e we cannot apply this technique directly as the measure of Γ_i^e is only of order d_i. We would then obtain a worse estimate. One can apply a coordinate transformation with the aim that the edge $\boldsymbol{e}$ coincides with the z-axis, and that $z = 0$ and $z = L$ correspond to the endpoints of $\boldsymbol{e}$. We introduce a further decomposition, namely

$$\begin{aligned} \Omega_{i,j,k}^{e,+,(m)} &:= \Big\{ x \in \Omega_{i,j}^{e,+,(m)} \colon z(x) \in \big((1 + A + 2^j + k - m) d_i, \\ &\qquad\qquad (2 + A + 2^j + k + m) d_i \big) \Big\}, \\ \Omega_{i,j,k}^{e,-,(m)} &:= \Big\{ x \in \Omega_{i,j}^{e,-,(m)} \colon z(x) \in \big(L - (2 + A + 2^j + k + m) d_i, \\ &\qquad\qquad L - (1 + A + 2^j + k - m) d_i \big) \Big\}, \end{aligned} \tag{3.107}$$

for $k = 0, \dots, 2^j - 1$ and $m \in \{0, 1\}$. To shorten the notation we write

$$\Omega_{i,j,k}^{e,\pm} := \Omega_{i,j,k}^{e,\pm,(0)} \qquad \text{and} \qquad \Omega_{i,j,k}^{e,\pm\,\prime} := \Omega_{i,j,k}^{e,\pm,(1)}.$$

The sets $\{\Omega_{i,j,k}^{e,\pm,(m)}\}_{k=0}^{2j-1}$ form a decomposition of $\Omega_{i,j}^{e,\pm,(m)}$. Analogously we introduce a decomposition of $\tilde\Omega_i^{e,(m)}$, namely

$$\begin{aligned} \tilde\Omega_{i,k}^{e,(m)} := \Big\{ x \in \tilde\Omega_i^{e,(m)} \colon z(x) \in \big(&(1 + A + 2^{i+1} + k - m) d_i, \\ &(2 + A + 2^{i+1} + k + m) d_i \big) \Big\} \end{aligned} \tag{3.108}$$

for $k = 0, \dots, K$ with some $K \sim d_i^{-1}$ and $m \in \{0, 1\}$. Again, we denote the boundary parts by

$$\Gamma_{i,j,k}^{e,\pm} := \partial\Omega_{i,j,k}^{e,\pm} \cap \Gamma, \qquad \tilde\Gamma_{i,k}^{e} := \partial\tilde\Omega_{i,k}^{e} \cap \Gamma, \tag{3.109}$$

which are illustrated in Figure 3.8, and confirm the desired properties

$$|\Gamma_{i,j,k}^{e,\pm}| \sim d_i^2, \qquad |\tilde\Gamma_{i,k}^{e}| \sim d_i^2. \tag{3.110}$$

Due to this construction we moreover have the properties

$$\operatorname{dist}(\partial\Omega_{i,j,k}^{e,\pm\,\prime} \setminus \Gamma,\ \partial\Omega_{i,j,k}^{e,\pm} \setminus \Gamma) \sim d_i \quad \text{and} \quad \operatorname{dist}(\partial\tilde\Omega_{i,k}^{e\,\prime} \setminus \Gamma,\ \partial\tilde\Omega_{i,k}^{e} \setminus \Gamma) \sim d_i, \tag{3.111}$$

which play a role in the local maximum norm estimate (3.106). Exploiting the decompositions (3.107) and (3.108), the Hölder inequality with (3.110) and a trace theorem leads to

$$\begin{aligned}\|y-y_h\|^2_{L^2(\Gamma^e_i)} &= \sum_{j=0}^{i}\sum_{k=0}^{2^j-1}\|y-y_h\|^2_{L^2(\Gamma^{e,\pm}_{i,j,k})}+\sum_{k=0}^{K}\|y-y_h\|^2_{L^2(\tilde\Gamma^e_{i,k})}\\ &\le c d_i^2\left(\sum_{j=0}^{i}\sum_{k=0}^{2^j-1}\|y-y_h\|^2_{L^\infty(\Omega^{e,\pm}_{i,j,k})}+\sum_{k=0}^{K}\|y-y_h\|^2_{L^\infty(\tilde\Omega^e_{i,k})}\right)\end{aligned}$$

Several applications of the local maximum norm estimate (3.106) with the properties (3.111) yields

$$\begin{aligned}\|y-y_h\|^2_{L^2(\Gamma^e_i)} \le c d_i^2 &\Bigg(\sum_{j=0}^{i}\sum_{k=0}^{2^j-1}\left(|\ln h|^2\|y-I_hy\|^2_{L^\infty(\Omega^{e,\pm}_{i,j,k}{}')}+d_i^{-3}\|y-y_h\|^2_{L^2(\Omega^{e,\pm}_{i,j,k}{}')}\right)\\ &+\sum_{k=0}^{K}\left(|\ln h|^2\|y-I_hy\|^2_{L^\infty(\tilde\Omega^e_{i,k}{}')}+d_i^{-3}\|y-y_h\|^2_{L^2(\tilde\Omega^e_{i,k}{}')}\right)\Bigg)\\ \le c &\Bigg(\sum_{j=0}^{i} d_i d_{i,j}|\ln h|^2\|y-I_hy\|^2_{L^\infty(\Omega^{e,\pm}_{i,j}{}')}\\ &+d_i|\ln h|^2\|y-I_hy\|^2_{L^\infty(\tilde\Omega^{e}_{i}{}')}+d_i^{-1}\|y-y_h\|^2_{L^2(\Omega_i')}\Bigg).\end{aligned}$$

In the last step we exploited that $K\sim d_i^{-1}$ and that $d_i 2^j = d_{i,j}$. Extracting the root yields estimate (3.104).

It remains to show the desired estimates also for $i = I-1, I$ which cannot be shown with the same technique, since the local maximum norm estimate (3.106) is not applicable if $\Omega^{c}_i{}'$ and $\Omega^{e}_i{}'$ contain the singular points. Therefore, we insert I_hy as intermediate function and apply the triangle inequality which leads to

$$\|y-y_h\|_{L^2(\Gamma^c_i)} \le c\left(\|y-I_hy\|_{L^2(\Gamma^c_i)}+\|I_hy-y_h\|_{L^2(\Gamma^c_i)}\right). \tag{3.112}$$

Next, we apply the Hölder inequality with $|\Gamma^c_i|\sim d_i^2$, and a trace theorem to get

$$\|y-I_hy\|_{L^2(\Gamma^c_i)} \le c d_i\|y-I_hy\|_{L^\infty(\Omega^c_i)}. \tag{3.113}$$

For the second part of (3.112) we exploit that I_hy-y_h is a function from a finite-dimensional space. On an arbitrary boundary element $E\in\mathcal{E}_h$ and its corresponding tetrahedron $T\in\mathcal{T}_h$ we obtain using a trace theorem on a reference setting as well as norm equivalences in finite-dimensional spaces

$$\|I_hy-y_h\|_{L^2(E)} \le c h_T^{-1/2}\|I_hy-y_h\|_{L^2(T)}. \tag{3.114}$$

Consequently, due to $h_T^{-1}\le h^{-1/\mu}\sim c_I d_i^{-1}$ for all $T\cap\Omega^c_i{}'\neq\emptyset$, as well as $|\Omega^c_i|\sim d_i^3$, we get

$$\|I_hy-y_h\|_{L^2(\Gamma^c_i)} \le d_i^{-1/2}\|I_hy-y_h\|_{L^2(\Omega^c_i{}')} \le c\left(d_i\|y-I_hy\|_{L^\infty(\Omega^c_i{}')}+d_i^{-1/2}\|y-y_h\|_{L^2(\Omega^c_i{}')}\right).$$

The constant c_I is neglected as it is not needed for this term. This estimate together with (3.113) and (3.112) yields (3.103) for $i = I-1, I$.

On Ω_i^e we use again the decomposition (3.84), the triangle inequality, and the Hölder inequality with (3.85) to arrive at

$$\begin{aligned}\|y-y_h\|^2_{L^2(\Gamma_i^e)} &\le \sum_{j=0}^{i}\left(\|y-I_hy\|^2_{L^2(\Gamma_{i,j}^{e,\pm})}+\|I_hy-y_h\|^2_{L^2(\Gamma_{i,j}^{e,\pm})}\right)\\ &\quad+\|y-I_hy\|^2_{L^2(\tilde\Gamma_i^e)}+\|I_hy-y_h\|^2_{L^2(\tilde\Gamma_i^e)}\\ &\le \sum_{j=0}^{i}\left(d_id_{i,j}\|y-I_hy\|^2_{L^\infty(\Omega_{i,j}^{e,\pm\prime})}+\|I_hy-y_h\|^2_{L^2(\Gamma_{i,j}^{e,\pm})}\right)\\ &\quad+d_i\|y-I_hy\|^2_{L^\infty(\tilde\Omega_i^{e\prime})}+\|I_hy-y_h\|^2_{L^2(\tilde\Gamma_i^e)}. \end{aligned} \tag{3.115}$$

From (3.114) and $|\Omega_{i,j}^{e,\pm\prime}|\sim d_i^2d_{i,j}$ we obtain

$$\begin{aligned}\|I_hy-y_h\|_{L^2(\Gamma_{i,j}^{e,\pm})} &\le d_i^{-1/2}\|I_hy-y_h\|_{L^2(\Omega_{i,j}^{e,\pm\prime})}\\ &\le d_i^{1/2}d_{i,j}^{1/2}\|y-I_hy\|_{L^\infty(\Omega_{i,j}^{e,\pm\prime})}+d_i^{-1/2}\|y-y_h\|_{L^2(\Omega_{i,j}^{e,\pm\prime})},\end{aligned}$$

and with the same arguments using $|\tilde\Omega_i^{e\prime}|\sim d_i^2$

$$\|I_hy-y_h\|_{L^2(\tilde\Gamma_i^e)}\le d_i^{1/2}\|y-I_hy\|_{L^\infty(\tilde\Omega_i^{e\prime})}+d_i^{-1/2}\|y-y_h\|_{L^2(\tilde\Omega_i^{e\prime})}.$$

From these estimates and (3.115) we finally conclude (3.104) in case of $i = I-1, I$. □

The next step of the proof is to derive a finite element error estimate on the boundary part $\hat\Gamma_R$ defined in (3.81) which is under influence of corner and edge singularities. Therefore, we localize the solution y with a smooth cut-off function $\omega\in C^\infty(\Omega)$ satisfying

$$\omega|_{\hat\Omega_R}\equiv 1 \qquad\text{and}\qquad \operatorname{supp}\omega\subset\Omega_R, \tag{3.116}$$

and define $\tilde y := \omega y$. For our proof we introduce the Ritz projection of $\tilde y$ as follows. Let

$$V_h(\Omega_R) := \{v_h\in V_h\colon v_h\equiv 0 \text{ in } \Omega\setminus\Omega_R\}$$

denote the space of ansatz functions vanishing outside of Ω_R. The function $\tilde y_h\in V_h$ is the unique solution of

$$a(\tilde y-\tilde y_h, v_h)=0,\quad \text{for all } v_h\in V_h(\Omega_R). \tag{3.117}$$

An error estimate for this Ritz projection is considered in the following theorem:

Theorem 3.4.13. *Let $\tilde y\in H^1(\Omega_R)$ such that $D^{\boldsymbol\alpha}\tilde y\in W^{1,2}_{\vec\alpha,\vec\delta}(\Omega_R)\cap W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega_R)$ for $|\boldsymbol\alpha|=1$ be given. Assume that the weight vectors $\vec\alpha\in[0,1)^{d'}$, $\vec\beta\in[0,2)$, $\vec\delta\in[0,2/3)^d$, $\vec\varrho\in[0,5/3)^d$, and the refinement parameter μ satisfy the inequalities*

$$\begin{aligned} &\alpha_j\le 1-\mu, &&\beta_j\le 3-2\mu, &&\forall j\in\mathcal C\\ &\delta_k\le 1-\mu, &&\varrho_k\le\frac52-2\mu, &&\forall k\in\mathcal E.\end{aligned}\tag{3.118}$$

Then, the estimate

$$\|\tilde{y}-\tilde{y}_h\|_{L^2(\hat{\Gamma}_R)} \le ch^2|\ln h|^{3/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right)$$

holds.

Proof. We consider the decomposition of the boundary $\hat{\Gamma}_R$ into the segments $\Gamma_i := \partial\Omega_i \cap \Gamma$ introduced in (3.82). Taking the two estimates from Lemma 3.4.12 as well as the decomposition (3.83) into account yields

$$\|\tilde{y}-\tilde{y}_h\|^2_{L^2(\Gamma_i)} \le c\Bigg(\sum_{\substack{e:=M_k\\ k\in\mathcal{E}}} |\ln h|^2\left(\sum_{j=0}^{i} d_i d_{i,j}\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\Omega^{e,\pm\prime}_{i,j})} + d_i\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\tilde{\Omega}^{e\prime}_i)}\right) + \sum_{\substack{c:=x^{(j)}\\ j\in\mathcal{C}}} |\ln h|^2 d_i^2\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\Omega^{c\prime}_i)} + d_i^{-1}\|\tilde{y}-\tilde{y}_h\|^2_{L^2(\Omega'_i)}\Bigg) \tag{3.119}$$

for all $i = 1,\dots,I$. Inserting the local estimates from Lemma 3.4.11 yields for $i = 1,\dots,I-3$

$$\begin{aligned}
&\sum_{\substack{e:=M_k\\ k\in\mathcal{E}}}\left(\sum_{j=0}^{i} d_i d_{i,j}\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\Omega^{e,\pm\prime}_{i,j})} + d_i\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\tilde{\Omega}^{e\prime}_i)}\right) + \sum_{\substack{c:=x^{(j)}\\ j\in\mathcal{C}}} d_i^2\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\Omega^{c\prime}_i)}\\
&\le ch^4\Bigg(\sum_{\substack{e:=M_k\\ k\in\mathcal{E}}} d_i^{2(5/2-2\mu-\varrho_k)}\left(\sum_{j=0}^{i} d_{i,j}^{2(1/2+\varrho_k-\tilde{\beta}_k)}|\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega^{e,\pm\prime\prime}_{i,j})} + |\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\tilde{\Omega}^{e\prime\prime}_i)}\right)\\
&\quad + \sum_{\substack{c:=x^{(j)}\\ j\in\mathcal{C}}} d_i^{2(3-2\mu-\kappa_j)}|\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega^{c\prime\prime}_i)}\Bigg) \le ch^4|\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega''_i)},
\end{aligned} \tag{3.120}$$

where we used the refinement condition (3.118) as well as (3.92) in the last step. In case of $i = I-2,\dots,I$ we obtain with Lemma 3.4.11

$$\begin{aligned}
&\sum_{\substack{e:=M_k\\ k\in\mathcal{E}}}\left(\sum_{j=0}^{i} d_i d_{i,j}\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\Omega^{e,\pm\prime}_{i,j})} + d_i\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\tilde{\Omega}^{e\prime}_i)}\right) + \sum_{\substack{c:=x^{(j)}\\ j\in\mathcal{C}}} d_i^2\|\tilde{y}-I_h\tilde{y}\|^2_{L^\infty(\Omega^{c\prime}_i)}\\
&\le c\left(\sum_{\substack{e:=M_k\\ k\in\mathcal{E}}} h^{2(5/2-\varrho_k+[1/2+\varrho_k-\tilde{\beta}_k]_-)/\mu}|\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega^{e\prime\prime}_i)} + \sum_{\substack{c:=x^{(j)}\\ j\in\mathcal{C}}} h^{2(3-\kappa_j)/\mu}|\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega^{c\prime\prime}_i)}\right)\\
&\le ch^4|\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega''_i)}.
\end{aligned} \tag{3.121}$$

Inserting the estimates (3.120) and (3.121) into (3.119) and summing up over all Γ_i for $i = 1,\dots,I$ yields with $I \sim |\ln h|$ the estimate

$$\|\tilde{y}-\tilde{y}_h\|^2_{L^2(\hat{\Gamma}_R)} \le c\left(|\ln h|^3 h^4|\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} + \|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|^2_{L^2(\Omega_R)}\right), \tag{3.122}$$

where $\gamma(x) := d_I + r(x)$. Note, that there holds $\gamma(x) \geq d_i = 2d_{i-1}$ if $x \in \Omega_i$.

In the remainder of the proof we will discuss the second term on the right-hand side of (3.122) which requires an estimate for a weighted $L^2(\Omega_R)$ error. Therefore, we adopt the technique that was applied in the proof of Lemma 6.2 in [81] where a duality argument was used. First we decompose the error into

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_R)} \leq \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_R \setminus (\Omega_0 \cup \Omega_1))} + \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_0 \cup \Omega_1)}. \quad (3.123)$$

On the outermost rings $\Omega_0 \cup \Omega_1$ we exploit that $\gamma \sim 1$ and can directly use the global finite element error estimate from Theorem 3.4.8. As a consequence we get

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_0 \cup \Omega_1)} \leq c\|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_R)} \leq ch^2 |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)}. \quad (3.124)$$

For an error estimate on $\tilde{\Omega}_R := \Omega_R \setminus (\Omega_0 \cup \Omega_1)$ we apply the representation

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} = \sup_{\substack{g \in C_0^\infty(\tilde{\Omega}_R) \\ \|g\|_{L^2(\tilde{\Omega}_R)} = 1}} (\gamma^{-1/2}(\tilde{y} - \tilde{y}_h), g) \quad (3.125)$$

and consider the auxiliary problem

$$-\Delta w + w = \gamma^{-1/2} g \quad \text{in} \quad \Omega_R, \qquad \partial_n w = 0 \quad \text{on} \quad \partial\Omega_R. \quad (3.126)$$

From the weak formulation of (3.126) we can deduce

$$(\gamma^{-1/2}(\tilde{y} - \tilde{y}_h), g) = (\tilde{y} - \tilde{y}_h, \gamma^{-1/2} g) = a(\tilde{y} - \tilde{y}_h, w). \quad (3.127)$$

We introduce a further cut-off function $\eta \in C_0^\infty(\Omega_R)$ such that

$$\eta \equiv 1 \quad \text{on} \quad \tilde{\Omega}_R, \qquad \text{and} \qquad \operatorname{supp} \eta \subset \hat{\Omega}_R,$$

and we make use of the decomposition $w = w_1 + w_2$ with $w_1 := \eta w$ and $w_2 := (1 - \eta)w$. The definition of w_1 implies that $Z_h w_1 \in V_h(\Omega_R)$ which allows us to apply the Galerkin orthogonality (3.117). This yields

$$\begin{aligned} a(\tilde{y} - \tilde{y}_h, w_1) &= a(\tilde{y} - \tilde{y}_h, w_1 - Z_h w_1) \\ &\leq c \sum_{i=0}^{I} \Bigg(\sum_{\substack{\boldsymbol{c} := x^{(j)} \\ j \in \mathcal{C}}} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} \|w_1 - Z_h w_1\|_{H^1(\Omega_i^{\boldsymbol{c}})} \\ &\qquad + \sum_{\substack{\boldsymbol{e} := M_k \\ k \in \mathcal{E}}} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^{\boldsymbol{e}})} \|w_1 - Z_h w_1\|_{H^1(\Omega_i^{\boldsymbol{e}})} \Bigg). \end{aligned} \quad (3.128)$$

First, we insert the local finite element error estimate from Corollary 9.1 in [90], which reads in our situation

$$\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^{\boldsymbol{c}})} \leq c \left(|\tilde{y} - Z_h \tilde{y}|_{H^1(\Omega_i^{\boldsymbol{c}\prime})} + d_i^{-1} \|\tilde{y} - Z_h \tilde{y}\|_{L^2(\Omega_i^{\boldsymbol{c}\prime})} + d_i^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i^{\boldsymbol{c}\prime})} \right). \quad (3.129)$$

The estimate remains true when replacing $\boldsymbol{c}$ by $\boldsymbol{e}$.

In order to derive estimates for the terms on the right-hand side of (3.128) we consider the cases $i = 3, \dots, I-3$ and $i = I-2, \dots, I$ such as $i = 0, 1, 2$ separately.

In case of $i = 3, \dots, I-3$, we obtain with the local estimates from Lemma 3.4.10 and (3.129)

$$\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^c)} \le c \left(h d_i^{5/2-\mu-\kappa_j} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + d_i^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right),$$

$$\|w_1 - Z_h w_1\|_{H^1(\Omega_i^c)} \le c h d_i^{1/2-\mu} |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')},$$

where we also exploited $h d_i^{-\mu} \le h d_I^{-\mu} = c_I^{-\mu} \le 1$ to simplify the interpolation error estimate in $L^2(\Omega_i^c)$. Moreover, we used the property

$$|w_1|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \le |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')},$$

which holds since $\eta \equiv 1$ on $\tilde{\Omega}_R$, and assumed that w possesses the regularity demanded be the right-hand side for arbitrary polyhedra, which we will confirm later.

Combining both estimates yields for $i = 3, \dots, I-3$

$$\begin{aligned} &\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^c)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i^c)} \\ &\le c \left(h^2 d_i^{3-2\mu-\kappa_j} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + h d_i^{-1/2-\mu} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\le c \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_I^{-\mu} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned} \tag{3.130}$$

The last step is a consequence of the assumption upon μ and the definition of the domains Ω_i, more precisely we exploited $d_i^{-\mu} \le d_I^{-\mu} \le c_I^{-\mu} h^{-1}$.

In case of $i = I-2, \dots, I$ we obtain

$$\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^c)} \le c \left(h^{(5/2-\kappa_j)/\mu} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + d_i^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right),$$

$$\|w_1 - Z_h w_1\|_{H^1(\Omega_i^c)} \le c c_I^{\max\{0,1/2-\mu\}} h^{1/(2\mu)} |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')},$$

where we exploited again that $\eta \equiv 1$ on $\tilde{\Omega}_R$. Combining both estimates leads to

$$\begin{aligned} &\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^c)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i^c)} \\ &\le c \left(h^{(3-\kappa_j)/\mu} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_I^{\max\{0,1/2-\mu\}} h^{1/(2\mu)} d_I^{-1} \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\le c \left(h^2 |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_I^{\max\{-1/2,-\mu\}} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')} \right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')}. \end{aligned} \tag{3.131}$$

The last step follows from the assumption upon μ and the fact that $d_I = c_I h^{1/\mu}$. For $i = 0, 1, 2$ we insert the global finite element error estimate from Theorem 3.4.8 and the interpolation error estimate from Lemma 3.4.10, where the factors d_0, d_1 and d_2 are of order one and can thus be neglected. We then obtain

$$\begin{aligned} \|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_1^{c\prime})} \|w_1 - Z_h w_1\|_{H^1(\Omega_1^{c\prime})} &\le c h^2 |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} |w_1|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_1'')} \\ &\le c h^2 |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} \left(|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)} \right). \end{aligned} \tag{3.132}$$

In the last step the Leibniz rule was applied using the fact that $\|D^{\boldsymbol{\alpha}}\eta\|_{L^\infty(\Omega_R)} \le c$ and that the weights are of order one within Ω_1''.

We can repeat the same strategy to show the appropriate estimates on $\Omega_i^{\boldsymbol{e}}$, and apply Lemma 3.4.10 with $s_k = 1/2 + \varrho_k - \tilde{\beta}_k$, as well as (3.129) with $\boldsymbol{c}$ replaced by $\boldsymbol{e}$. Moreover, we have to exploit the refinement condition

$$2\mu \le 5/2 - \varrho_k + [s_k]_- = \begin{cases} 5/2 - \varrho_k, & \text{if } s_k \ge 0, \\ 3 - \tilde{\beta}_k, & \text{if } s_k < 0, \end{cases}$$

which follows from (3.118). Consequently, we arrive at

$$\begin{aligned} &\|\tilde{y} - \tilde{y}_h\|_{H^1(\Omega_i^{\boldsymbol{e}})}\|w_1 - Z_h w_1\|_{H^1(\Omega_i^{\boldsymbol{e}})} \\ &\le c\left(h^2|\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_l^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')}\right)|w|_{W^{2,2}_{1/2,1/2}(\Omega_i')}, \end{aligned} \tag{3.133}$$

for $i = 3,\dots,l$. Finally, we easily confirm that the estimate (3.132) remains true when replacing $\boldsymbol{c}$ by $\boldsymbol{e}$, and we have covered also the cases $i = 0, 1, 2$.

We may now insert the estimates (3.130), (3.131), (3.132) and (3.133) into (3.128) which leads to

$$\begin{aligned} &a(\tilde{y} - \tilde{y}_h, w_1) \\ &\le c\sum_{i=3}^{l}\left(h^2|\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_l^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\Omega_i')}\right)|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \\ &\quad + ch^2\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right)\left(|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)}\right) \\ &\le c\left(h^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right) + c_l^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)}\right) \\ &\quad \times\left(|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)}\right). \end{aligned} \tag{3.134}$$

Next, we show that w possesses the regularity demanded by the right-hand side, which follows from Theorem 2.3.7 and the Lax-Milgram Lemma once we have shown that

$$\gamma^{-1/2}g \in W^{0,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R) \cap [H^1(\Omega_R)]^*. \tag{3.135}$$

To this end we have to find a relation between the weight function $\gamma(\cdot)$ and the weights hidden in the norm of the weighted Sobolev spaces. For some fixed $x \in U_j$ define $\bar{k} \in X_j$ such that $r_{\bar{k}}(x) = r(x)$. The angular distance to the edges M_k with $k \in X_j \setminus \{\bar{k}\}$ is bounded from below, i.e. $r_k/\rho_j \ge c$, compare also Figure 3.2 on page 33. Consequently, we obtain

$$\begin{aligned} \gamma^{-1}(x) \le r(x)^{-1} = r_{\bar{k}}(x)^{-1} = \rho_j(x)^{-1}\left(\frac{r_{\bar{k}}}{\rho_j}(x)\right)^{-1} \\ \le c\rho_j(x)^{-1}\prod_{k\in X_j}\left(\frac{r_k}{\rho_j}(x)\right)^{-1}, \end{aligned} \tag{3.136}$$

and directly conclude

$$\|\gamma^{-1/2}g\|_{W^{0,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} \le c\|g\|_{L^2(\Omega_R)} \le c. \tag{3.137}$$

To show the boundedness in $[H^1(\Omega_R)]^*$ we use the operator norm representation, the Cauchy-Schwarz inequality, as well as the boundedness of g in $L^2(\Omega_R)$, and arrive at

$$\|\gamma^{-1/2}g\|_{[H^1(\Omega_R)]^*} = \sup_{\varphi\in H^1(\Omega_R)} \frac{(g,\gamma^{-1/2}\varphi)_{\Omega_R}}{\|\varphi\|_{H^1(\Omega_R)}} \le c \sup_{\varphi\in H^1(\Omega_R)} \frac{\|\gamma^{-1/2}\varphi\|_{L^2(\Omega_R)}}{\|\varphi\|_{H^1(\Omega_R)}}. \tag{3.138}$$

Taking again (3.136) into account leads to

$$\|\gamma^{-1/2}\varphi\|_{L^2(\Omega_R)} \le c\|\varphi\|_{W^{0,2}_{-\vec{1}/2,-\vec{1}/2}(\Omega_R)} \le c\|\varphi\|_{H^1(\Omega_R)}, \tag{3.139}$$

where the embedding used in the second step is a consequence of Lemma 2.3.4 and the fact that the spaces $W^{1,2}_{\vec{0},\vec{0}}(\Omega_R)$ and $H^1(\Omega_R)$ are equivalent. Inserting (3.139) into (3.138) and taking also (3.137) into account yields (3.135), and consequently

$$|w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)} + \|w\|_{H^1(\Omega_R)} \le c. \tag{3.140}$$

The estimate (3.134) then becomes

$$\begin{aligned} &a(\tilde{y}-\tilde{y}_h, w_1) \\ &\le c\left(h^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right) + c_I^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)}\right). \end{aligned} \tag{3.141}$$

It remains to derive a similar estimate with w_2 instead of w_1. Therefore, we exploit that $w_2 \equiv 0$ on $\tilde{\Omega}_R$, and $\partial_n w_2 \equiv 0$ on $\partial\Omega_R$. Partial integration yields

$$\begin{aligned} a(\tilde{y}-\tilde{y}_h, w_2) &= (\tilde{y}-\tilde{y}_h, -\Delta w_2)_{\Omega_R} + (\tilde{y}-\tilde{y}_h, w_2)_{\Omega_R} + (\tilde{y}-\tilde{y}_h, \partial_n w_2)_{\partial\Omega_R} \\ &\le \|\tilde{y}-\tilde{y}_h\|_{L^2(\Omega_R)}\|w_2\|_{H^2(\Omega_R\setminus\tilde{\Omega}_R)}. \end{aligned} \tag{3.142}$$

We exploit the property $\|D^\alpha\eta\|_{L^\infty(\Omega_R)} \le c$ for all $|\alpha| \le 2$ and the fact that $\Omega_R\setminus\tilde{\Omega}_R$ has positive distance to the singular points, and arrive at

$$\|w_2\|_{H^2(\Omega_R\setminus\tilde{\Omega}_R)} \le c\|w\|_{H^2(\Omega_R\setminus\tilde{\Omega}_R)} \le c\left(\|w\|_{H^1(\Omega_R)} + |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_R)}\right) \le c.$$

The last estimate is another application of (3.140). Moreover, we insert the global estimate from Theorem 3.4.8 into (3.142) and get

$$a(\tilde{y}-\tilde{y}_h, w_2) \le ch^2|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)}. \tag{3.143}$$

Inserting now (3.143) and (3.141) into (3.127) yields together with (3.125)

$$\begin{aligned} \|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \le c\Bigg(& h^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right) \\ &+ c_I^{\max\{-1/2,-\mu\}}\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)}\Bigg). \end{aligned} \tag{3.144}$$

We fix the generic constant c and choose c_I sufficiently large such that

$$c c_I^{\max\{-1/2,-\mu\}} \leq 1/2.$$

This allows us to apply a kick-back argument and we consequently arrive at

$$\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \leq ch^2|\ln h|^{1/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right).$$

Finally, we insert this estimate together with (3.124) into (3.123), insert the resulting estimate into (3.122), and arrive at the assertion. □

Now we are able to prove the main result of this section.

Theorem 3.4.14. *Let y denote the solution of the variational problem (2.23) and y_h its finite element approximation (3.3), with input data satisfying $f \in C^{0,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$, and $g \equiv 0$. Assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of locally refined triangulations according to condition (3.56). Moreover, let be given weights $\vec{\alpha},\vec{\beta} \in \mathbb{R}^{d'}_+$ and $\vec{\delta},\vec{\varrho} \in \mathbb{R}^d_+$ satisfying*

$$\begin{aligned}
&\frac{1}{2}-\lambda_j^c < \alpha_j \leq 1-\mu, && 2-\lambda_j^c < \beta_j \leq 3-2\mu, && \forall j \in \mathcal{C},\\
&1-\lambda_k^e < \delta_k \leq 1-\mu, && 2-\lambda_k^e < \varrho_k \leq \frac{5}{2}-2\mu, && \forall k \in \mathcal{E}.
\end{aligned} \tag{3.145}$$

Then, some $c > 0$ exists such that

$$\|y-y_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2}\left(\sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}y\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|y\|_{L^\infty(\Omega)}\right). \tag{3.146}$$

Proof. For technical reasons we introduce further subsets

$$\breve{\Omega}_R := \operatorname{int}\bigcup_{i=2}^{I}\overline{\Omega_i}, \quad \tilde{\Omega}_R := \operatorname{int}\bigcup_{i=3}^{I}\overline{\Omega_i}, \quad \breve{\Gamma}_R := \partial\breve{\Omega}_R \cap \Gamma, \quad \tilde{\Gamma}_R := \partial\tilde{\Omega}_R \cap \Gamma.$$

Note that we have the relation $\tilde{\Omega}_R \subset \breve{\Omega}_R \subset \hat{\Omega}_R \subset \Omega_R \subset \Omega$. Let ω be the cut-off function defined in (3.116). In order to apply Theorem 3.4.13 we insert the intermediate function $\tilde{y}_h$ and exploit that $\tilde{y} := \omega y$ coincides with y in $\hat{\Omega}_R$. This leads to

$$\|y-y_h\|_{L^2(\breve{\Gamma}_R)} = \|\tilde{y}-\tilde{y}_h\|_{L^2(\hat{\Gamma}_R)} + \|\tilde{y}_h-y_h\|_{L^2(\breve{\Gamma}_R)}. \tag{3.147}$$

For the first part we may now apply the result of Theorem 3.4.13 and obtain

$$\|\tilde{y}-\tilde{y}_h\|_{L^2(\hat{\Gamma}_R)} \leq ch^2|\ln h|^{3/2}\left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)}\right). \tag{3.148}$$

Note that it is possible to construct a cut-off function ω satisfying (3.116) and $\|D^{\boldsymbol{\alpha}}\omega\|_{L^\infty(\Omega_R)} \leq 2^{|\boldsymbol{\alpha}|} \leq c$ for arbitrary $\boldsymbol{\alpha} \in \mathbb{N}_0^3$. Using the Leibniz rule we then get

$$\begin{aligned}
|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} = |\omega y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} &\leq c\left(|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + \|y\|_{W^{1,2}(\Omega\setminus\hat{\Omega}_R)}\right)\\
&\leq c\left(\sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|y\|_{L^2(\Omega)}\right),
\end{aligned} \tag{3.149}$$

and analogously

$$|\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \leq c \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|y\|_{L^\infty(\Omega)} \right). \tag{3.150}$$

Let us discuss the second part of (3.147). The function $\tilde{y}_h - y_h$ is discrete harmonic on $\hat{\Omega}_R$. Hence, the discrete Caccioppoli estimate from Lemma 3.3 in [36] yields

$$\|\tilde{y}_h - y_h\|_{H^1(\breve{\Omega}_R)} \leq c d \|\tilde{y}_h - y_h\|_{L^2(\hat{\Omega}_R)}, \qquad d := \operatorname{dist}(\partial\hat{\Omega}\backslash\Gamma, \partial\breve{\Omega}\backslash\Gamma), \tag{3.151}$$

and with our construction we have $d = 1/4$. With a trace theorem and (3.151) we then obtain

$$\begin{aligned} \|\tilde{y}_h - y_h\|_{L^2(\breve{\Gamma}_R)} &\leq c\|\tilde{y}_h - y_h\|_{H^1(\breve{\Omega}_R)} \leq c\|\tilde{y}_h - y_h\|_{L^2(\hat{\Omega}_R)} \\ &\leq c\left(\|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega_R)} + \|y - y_h\|_{L^2(\Omega)}\right), \end{aligned}$$

where the last step holds due to $y = \tilde{y}$ on $\hat{\Omega}_R$. An application of Theorem 3.4.8 yields

$$\|\tilde{y}_h - y_h\|_{L^2(\breve{\Gamma}_R)} \leq ch^2 \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|y\|_{L^2(\Omega)} \right), \tag{3.152}$$

where we also applied the estimate (3.149). Consequently we get from (3.147) the estimate

$$\|y - y_h\|_{L^2(\breve{\Gamma}_R)} \leq ch^2 |\ln h|^{3/2} \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|y\|_{L^\infty(\Omega)} \right). \tag{3.153}$$

Let us consider the error on the remaining part $\Gamma\backslash\breve{\Gamma}_R$ where we have no influence of the singularities. One can directly apply the trace theorem in the L^∞-norm which yields

$$\|y - y_h\|_{L^2(\Gamma\backslash\breve{\Gamma}_R)} \leq c\|y - y_h\|_{L^\infty(\Gamma\backslash\breve{\Gamma}_R)} \leq c\|y - y_h\|_{L^\infty(\Omega\backslash\breve{\Omega}_R)}. \tag{3.154}$$

With the local maximum norm estimate (3.106) exploiting that $\operatorname{dist}(\partial\tilde{\Omega}_R \setminus \Gamma, \partial\breve{\Omega}_R \setminus \Gamma) \sim c$, we arrive at

$$\|y - y_h\|_{L^\infty(\Omega\backslash\breve{\Omega}_R)} \leq c\left(|\ln h|\|y - I_h y\|_{L^\infty(\Omega\backslash\tilde{\Omega}_R)} + \|y - y_h\|_{L^2(\Omega\backslash\tilde{\Omega}_R)}\right). \tag{3.155}$$

Denote by T^* the element where the maximum of $|y(x) - I_h y(x)|$ within $\Omega \setminus \tilde{\Omega}_R$ is acquired. An application of a standard interpolation error estimate in $L^\infty(T^*)$ implies

$$\|y - I_h y\|_{L^\infty(\Omega\backslash\tilde{\Omega}_R)} \leq \|y - I_h y\|_{L^\infty(T^*)} \leq ch^2 |y|_{W^{2,\infty}(T^*)}$$

and we may insert the weights which are bounded from below by a positive constant within $\Omega\backslash\tilde{\Omega}_R$. For the second term on the right-hand side of (3.155) we insert again the global estimate from Theorem 3.4.8. From (3.155) and (3.154) we hence conclude

$$\|y - y_h\|_{L^2(\Gamma\backslash\tilde{\Gamma}_R)} \leq ch^2 \left(|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |y|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} \right),$$

and together with (3.153) the desired estimate (3.146) follows. □

With some modifications of the proof we can now also show the error estimate on quasi-uniform meshes presented in Theorem 3.3.2.

Proof of Theorem 3.3.2. Before we discuss the modifications required in the proofs of Theorems 3.4.13 and 3.4.14 we show some essential properties we will frequently use.

For all $i = 0, \dots, I-1$ we get with the properties $h \le d_I < d_i \le 1$ for all $k \in \mathcal{E}$ the estimate

$$h^2 d_i^{1/2-\varrho_k} \le h^{\min\{2,5/2-\varrho_k\}} = h^{\min\{2,1/2+\lambda_k^e-\varepsilon\}}, \tag{3.156}$$

when considering the cases $\varrho_k \le 1/2$ and $\varrho_k > 1/2$ separately, and inserting the definition $\varrho_k := \min\{0, 2-\lambda_k^e+\varepsilon\}$. Analogously we can show for $i = I-1, I$ that

$$h^{5/2-\varrho_k} \le h^{\min\{2,1/2+\lambda_k^e-\varepsilon\}}. \tag{3.157}$$

To obtain the estimates in a vicinity of a corner $x^{(j)}$, $j \in \mathcal{C}$, for $i = 0, \dots, I-1$, we moreover require the property

$$h^2 d_i^{1-\beta_j} \le h^{\min\{2,3-\beta_j\}} \le c h^{\min\{2,1+\lambda_j^c-\varepsilon\}}, \tag{3.158}$$

where we distinguished among the cases $\beta_j \le 1$ and $\beta_j > 1$, and inserted the definition $\beta_j := \max\{0, 2-\lambda_j^c+\varepsilon\}$. For $i = I-1, I$ we will use instead

$$h^{3-\beta_j} \le h^{\min\{2,1+\lambda_j^c-\varepsilon\}}. \tag{3.159}$$

Inserting these properties into (3.120) and (3.121) yields together with (3.119) the estimate

$$\|\tilde{y}-\tilde{y}_h\|_{L^2(\Gamma_i)}^2 \le c\left(|\ln h|^2 h^{2\min\{2,1/2+\lambda-\varepsilon\}} |\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i')} + d_i^{-1}\|(\tilde{y}-\tilde{y}_h)\|^2_{L^2(\Omega_i')}\right), \tag{3.160}$$

where we used that

$$\min\{2, 1/2+\lambda-\varepsilon\} = \min\{2, 1+\min_{j\in\mathcal{C}} \lambda_j^c - \varepsilon, 1/2+\min_{k\in\mathcal{E}} \lambda_k^e - \varepsilon\}.$$

Summation over all $i = 1, \dots, I$ leads to

$$\|\tilde{y}-\tilde{y}_h\|_{L^2(\hat{\Gamma}_R)}^2 \le c\left(|\ln h|^3 h^{2\min\{2,1/2+\lambda-\varepsilon\}} |\tilde{y}|^2_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} + \|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|^2_{L^2(\Omega_R)}\right), \tag{3.161}$$

with $\gamma(x) := d_I + r(x)$.

It remains to discuss the weighted finite element error in $L^2(\Omega_R)$ on the right-hand side of (3.161). On the outermost rings we get with the global estimate from Theorem 3.3.1

$$\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\Omega_0\cup\Omega_1)} \le c\|\tilde{y}-\tilde{y}_h\|_{L^2(\Omega_R)} \le c h^{\min\{2,2\lambda-\varepsilon\}} |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)}. \tag{3.162}$$

On the remaining part we proceed as in the steps (3.125)–(3.128). Setting $\mu = 1$ in the estimates (3.130), (3.131) and (3.132) yields with the properties (3.156)–(3.159)

$$\begin{aligned} &\|\tilde{y}-\tilde{y}_h\|_{H^1(\Omega_i^c)} \|w_1 - Z_h w_1\|_{H^1(\Omega_i^c)} \\ &\le c\left(h^{\min\{2,1/2+\lambda-\varepsilon\}} |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_i'')} + c_I^{-1/2}\|\gamma^{-1/2}(\tilde{y}-\tilde{y}_h)\|_{L^2(\Omega_i')}\right) |w|_{W^{2,2}_{\vec{1}/2,\vec{1}/2}(\Omega_i')} \end{aligned} \tag{3.163}$$

for $i = 0, \ldots, I$, where we used also the property $d_i \geq d_I = c_I h^{1/\mu}$. One easily confirms that this estimate holds true when replacing $\boldsymbol{c}$ by $\boldsymbol{e}$ which follows from the same technique. Inserting estimate (3.163) into (3.128) then yields together with (3.140)

$$\begin{aligned} & a(\tilde{y} - \tilde{y}_h, w_1) \\ & \leq c \left(h^{\min\{2,1/2+\lambda-\varepsilon\}} |\ln h|^{1/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right) + c_I^{-1/2} \|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \right). \end{aligned} \tag{3.164}$$

To show an estimate with w_2 instead of w_1 we insert the global estimate from Theorem 3.3.1 into (3.142) which yields

$$a(\tilde{y} - \tilde{y}_h, w_2) \leq c h^{\min\{2,2\lambda-\varepsilon\}} |\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)}. \tag{3.165}$$

The estimates and (3.164) and (3.165) yield together with (3.125) and (3.127)

$$\|\gamma^{-1/2}(\tilde{y} - \tilde{y}_h)\|_{L^2(\tilde{\Omega}_R)} \leq c h^{\min\{2,1/2+\lambda-\varepsilon\}} |\ln h|^{1/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right), \tag{3.166}$$

where we already applied a kick-back argument as at the end of the proof of Theorem 3.4.13.

The proof of Theorem (3.4.14) can be almost repeated. From (3.161) and (3.166) we get instead of (3.148) the estimate

$$\|\tilde{y} - \tilde{y}_h\|_{L^2(\hat{\Gamma}_R)} \leq c h^{\min\{2,1/2+\lambda-\varepsilon\}} |\ln h|^{3/2} \left(|\tilde{y}|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |\tilde{y}|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega_R)} \right). \tag{3.167}$$

Moreover, instead of the estimates (3.152) and (3.155) we get

$$\|\tilde{y}_h - y_h\|_{L^2(\check{\Gamma}_R)} \leq c h^{\min\{2,2\lambda-\varepsilon\}} \left(\sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} y\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \|y\|_{L^2(\Omega)} \right), \tag{3.168}$$

$$\|y - y_h\|_{L^2(\Gamma \setminus \hat{\Gamma}_R)} \leq c h^{\min\{2,2\lambda-\varepsilon\}} \left(|y|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega_R)} + |y|_{W^{2,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} \right), \tag{3.169}$$

if the global estimate in $L^2(\Omega)$ for quasi-uniform meshes from Theorem 3.3.1 is applied. From (3.147), (3.167), (3.168) and (3.169) we finally conclude the assertion of Theorem 3.3.2. □

Looking carefully at the assumption (3.145), we observe that the refinement parameter depends solely on the number λ defined in (3.80). From this we conclude the following simplified version of Theorem 3.4.14.

Corollary 3.4.15. *Assume that $f \in C^{0,\sigma}(\overline{\Omega})$ for arbitrary $\sigma \in (0,1)$, and $g \equiv 0$. The error estimate*

$$\|y - y_h\|_{L^2(\Gamma)} \leq c |\ln h|^{3/2} h^2$$

holds, if one of the following assumptions is satisfied:

1. *The family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform (i. e. $\mu = 1$), and there holds*

$$\lambda > 3/2.$$

2. *The family of triangulations* $\{\mathcal{T}_h\}_{h>0}$ *is refined according to* (3.56) *with parameter*

$$\frac{1}{3} < \mu < \frac{1}{4} + \frac{\lambda}{2}.$$

Here, the constant $c > 0$ *depends also on* f.

Remark 3.4.16. *One observes that the refinement condition necessary for an optimal convergence rate in* $L^2(\Gamma)$*-norm is a different one than for an optimal rate in the* $H^1(\Omega)$*- or* $L^2(\Omega)$*-norm (see Corollary 3.4.9). Due to* $\lambda > 1/2$ *there holds* $1/4 + \lambda/2 < \lambda$. *Thus, the mesh grading condition required for optimal error estimates on the boundary from Corollary 3.4.15 implies the condition required for the estimates in the domain from Corollary 3.4.9.*

CHAPTER 4

Neumann boundary control problems in $L^2(\Gamma)$

The aim of this chapter is to discuss and prove error estimates for the numerical approximation of the Neumann boundary control problem

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \tag{4.1}$$

subject to

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \partial_n y = u & \text{on } \Gamma, \end{cases} \tag{4.2}$$

$$u \in U_{ad} := \{u \in L^2(\Gamma)\colon u_a \le u \le u_b \text{ a. e. on } \Gamma\}. \tag{4.3}$$

Initially, we demand that $f \in [H^1(\Omega)]^*$ and $y_d \in L^2(\Omega)$, but we will require higher regularity of the input data for the error estimates derived in this chapter. The control bounds u_a and u_b are assumed to be constant, and that $u_a < u_b$.

Roughly speaking, the aim is to find a Neumann datum – the control – $u \in L^2(\Gamma)$ such that the corresponding solution of the state equation $y \in H^1(\Omega)$ is as close as possible to the given desired state y_d. The additional regularization term is in many applications modeled as control cost which results in a penalization of high control values. The regularization parameter $\alpha > 0$ can be chosen arbitrarily.

As almost all estimates we derive here have already been proved for polygonal domains we consider in this chapter only polyhedral domains $\Omega \subset \mathbb{R}^3$. In Section 4.1 we will discuss existence of a solution of (4.1)–(4.3), derive necessary optimality conditions and prove regularity results of its solution. These optimality conditions are discretized in order to compute an approximate solution and we will discuss three possible discretization approaches and error estimates in Section 4.2. At the end of this chapter in Section 4.3 we will also confirm the predicted convergence rates in numerical experiments.

4.1 Analysis of the optimal control problem

Before we investigate the numerical solution of the optimization problem (4.1)–(4.3) we discuss the continuous problem in detail. The statements about existence of solutions and optimality conditions can be also found in the text books [52, 88].

Analysis of the state equation

The weak formulation of the state equation (4.2) reads

$$a(y, v) = \langle f, v\rangle_\Omega + (u, v)_\Gamma \qquad \forall v \in H^1(\Omega), \tag{4.4}$$

as already derived in Section 2.1.2. Note, that $u \in L^2(\Gamma)$ allows us to use the inner product in $L^2(\Gamma)$ on the right-hand side. As this equation is linear we can decompose its solution into $y = y_f + y_u$ such that $y_f, y_u \in H^1(\Omega)$ solve

$$a(y_f, v) = \langle f, v\rangle_\Omega \qquad \forall v \in H^1(\Omega), \tag{4.5}$$

$$a(y_u, v) = (u, v)_\Gamma \qquad \forall v \in H^1(\Omega). \tag{4.6}$$

The solution $y_f \in H^1(\Omega)$ of (4.5) does not depend on the control, but only on the input datum f, and is hence not a quantity which has to be optimized. Hence, by a slight abuse of our definitions we will say that y_u is the state corresponding to the control u. The solution operator of (4.6) also being referred to as *control-to-state mapping* is denoted by

$$S : L^2(\Gamma) \to L^2(\Omega), \qquad u \mapsto Su := y_u.$$

The substitution $y = y_f + Su$ allows us to eliminate the state variable in (4.1). As a consequence, (4.1)–(4.3) is equivalent to the optimization problem with reduced target functional

$$j(u) := \frac{1}{2}\|Su + y_f - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \to \min! \quad \text{s. t.} \quad u \in U_{ad}, \tag{4.7}$$

which depends only on the control u.

Existence of solutions and optimality conditions

In order to solve problem (4.7) we derive necessary optimality conditions. In the present case we have a quadratic target functional which is Fréchet differentiable. This directly implies the first-order optimality condition presented in the following Theorem.

Theorem 4.1.1. *The function $\bar u \in U_{ad}$ is the unique solution of (4.7) if and only if it satisfies the variational inequality*

$$(S\bar u + y_f - y_d, S(u - \bar u)) + \alpha(\bar u, u - \bar u)_\Gamma \geq 0 \qquad \forall u \in U_{ad}. \tag{4.8}$$

Proof. The unique solubility is proven in [88, Section 2.5.3] and the first-order optimality condition in [88, Theorem 2.22]. □

To avoid the presence of the control-to-state mapping in the optimality condition (4.8) the *adjoint state* $p \in H^1(\Omega)$ is introduced which is defined as the solution of the *adjoint equation*

$$a(v, p) = (y_u + y_f - y_d, v) \qquad \forall v \in H^1(\Omega). \tag{4.9}$$

As a consequence we obtain an optimality system consisting of two coupled partial differential equations and a variational inequality.

Theorem 4.1.2. *The function $\bar{u} \in U_{ad}$ solves problem (4.7), if and only if a related state $\bar{y}_u \in H^1(\Omega)$ and a related adjoint state $\bar{p} \in H^1(\Omega)$ exist such that $(\bar{y}_u, \bar{u}, \bar{p})$ fulfills the system*

$$\begin{cases} a(\bar{y}_u, v) = (\bar{u}, v)_\Gamma & \forall v \in H^1(\Omega), \\ a(v, \bar{p}) = (\bar{y}_u + y_f - y_d, v) & \forall v \in H^1(\Omega), \\ (\bar{p} + \alpha\bar{u}, u - \bar{u})_\Gamma \geq 0 & \forall u \in U_{ad}. \end{cases} \tag{4.10}$$

Moreover, the variational inequality is equivalent to the projection formula

$$\bar{u} = \Pi_{ad}\left(-\frac{1}{\alpha}\bar{p}|_\Gamma\right), \tag{4.11}$$

where $\Pi_{ad}\colon L^2(\Gamma) \to U_{ad}$ is the $L^2(\Gamma)$-projection onto U_{ad} which possesses the pointwise representation

$$[\Pi_{ad} v](x) := \max\{u_a, \min\{u_b, v(x)\}\}.$$

Proof. This result follows from the substitution $\bar{y} = y_f + S\bar{u}$ as well as the equations (4.4) and (4.9). The first term in (4.8) is then simplified to

$$(\bar{y}_u + y_f - y_d, S(u - \bar{u})) = a(S(u - \bar{u}), \bar{p}) = (\bar{p}, u - \bar{u})_\Gamma .$$

The equivalence of the variational inequality and the projection formula is proved e. g. in [52, Corollary 1.2]. □

Let us now introduce the solution operator of the adjoint equation $P : L^2(\Omega) \to H^1(\Omega)$. In the following we write $p = P(y_u + y_f - y_d)$ if p solves (4.9). It is well-known that the adjoint operator of the control–to–state mapping can be expressed by $S^* := \tau \circ P : L^2(\Omega) \to L^2(\Gamma)$, where $\tau\colon H^1(\Omega) \to L^2(\Gamma)$ is the trace operator onto Γ. Hence, the gradient of the target functional can be represented by means of

$$\nabla j(u) = S^*(Su + y_f - y_d) + \alpha u.$$

In the following we denote the active and inactive sets by

$$\mathcal{A}^+ := \{x \in \Gamma\colon \bar{u}(x) = u_b\}, \quad \mathcal{A}^- := \{x \in \Gamma\colon \bar{u}(x) = u_a\}, \quad \mathcal{I} := \Gamma \setminus (\mathcal{A}^+ \cup \mathcal{A}^-),$$

and write $\mathcal{A}^\pm := \mathcal{A}^+ \cup \mathcal{A}^-$. A pointwise discussion of the variational inequality in (4.10) leads to

$$\nabla j(u) \begin{cases} = 0, & \text{a.e. on } \mathcal{I}, \\ \leq 0, & \text{a.e. on } \mathcal{A}^+, \\ \geq 0, & \text{a.e. on } \mathcal{A}^-, \end{cases} \tag{4.12}$$

compare also the technique applied in [52, Lemma 1.12].

Regularity

Before discussing discretization error estimates we investigate the regularity of the solution of the optimal control problem (4.1)–(4.3). In Lemma 4.1.3 we will prove a regularity result in classical Sobolev spaces. More accurate results in weighted Sobolev spaces are summarized in Theorem 4.1.4.

Lemma 4.1.3. *Assume that the input data satisfy $f, y_d \in L^2(\Omega)$, and denote by $\lambda := \min_{j\in\mathcal{C}} \lambda_j$ the smallest singular exponent. Let $s_0 \in (3/2, 1+\lambda)$ and $\sigma \in (0, \min\{1/2, \lambda - 1/2\})$. Then, the solution $(\bar y_u, \bar u, \bar p)$ of the optimality system (4.10) and y_f from (4.5) possess the regularity*

$$\begin{aligned} \bar y_u, y_f, \bar p &\in H^s(\Omega) \cap C^{0,\sigma}(\overline{\Omega}), \\ \bar u &\in H^1(\Gamma), \end{aligned}$$

for arbitrary $s \le \min\{2, s_0\}$.

Proof. From Theorem 4.1.1 we know that a unique solution $\bar u \in L^2(\Gamma)$ exists and hence $\bar y_u \in H^1(\Omega)$. The assumption $f \in L^2(\Omega)$ implies $y_f \in H^s(\Omega)$ which is proved e. g. in [45, Corollary 2.6.7] or [33, Corollary 23.5], and in the same way $y_f + \bar y_u - y_d \in L^2(\Omega)$ implies $\bar p \in H^s(\Omega)$. From the trace theorem we get that $\bar p|_\Gamma \in H^1(\Gamma)$ as $s_0 > 3/2$. Due to the projection formula (4.11) this regularity is transferred to $\bar u$. In particular, we have $\bar u \in H^{1/2}(\Gamma)$. This implies $\bar y_u \in H^s(\Omega)$ which is also proved in the references mentioned above. The Hölder-continuity is a consequence of the embedding $H^s(\Omega) \hookrightarrow C^{0,\sigma}(\overline{\Omega})$ stated in Lemma 2.1.1. □

In order to outline the regularity of the optimal control more accurately we introduce the following definitions. For some triangulation $\mathcal{T}_h$ of Ω, and its corresponding boundary triangulation $\mathcal{E}_h$ we define the sets

$$\mathcal{K}_1 = \mathrm{cl} \cup \left(\{E \in \mathcal{E}_h \colon E \cap \mathcal{A}^\pm \ne \emptyset \text{ and } E \cap \mathcal{I} \ne \emptyset\}\right), \quad \mathcal{K}_2 := \Gamma \setminus \mathcal{K}_1.$$

On $\mathcal{K}_1$ the control $\bar u$ is switching from the active to the inactive region, and can have a kink. Consequently, there holds $\bar u \notin H^2(\mathcal{K}_1)$. As a remedy one can use regularity results in $W^{1,\infty}(\mathcal{K}_1)$ instead. In the following theorem more accurate regularity results in weighted Sobolev spaces are proven. These results are important for the discretization error estimates proved in Section 4.2.

Theorem 4.1.4. *Assume that the input data satisfy $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Let $\varepsilon > 0$ be a sufficiently small real number, and let $\vec\alpha, \vec\beta, \vec\gamma \in \mathbb{R}^{d'}$ and $\vec\delta, \vec\varrho, \vec\tau \in \mathbb{R}^d$ be weight vectors defined by*

$$\begin{aligned} \alpha_j &:= \max\{0, \tfrac{1}{2} - \lambda_j^c + \varepsilon\}, & \delta_k &:= \max\{0, 1 - \lambda_k^e + \varepsilon\}, \\ \beta_j &:= \max\{0, 2 - \lambda_j^c + \varepsilon\}, & \varrho_k &:= \max\{0, 2 - \lambda_k^e + \varepsilon\}, \\ \gamma_j &:= \max\{0, 1 - \lambda_j^c + \varepsilon\}, & \tau_k &:= \max\{0, \tfrac{3}{2} - \lambda_k^e + \varepsilon\}, \end{aligned}$$

for all $j \in \mathcal{C}$ and $k \in \mathcal{E}$. Then, the solution $(\bar{y}_u, \bar{u}, \bar{p})$ of the optimality system (4.10) and the solution y_f of (4.5) satisfy

$$\begin{aligned} D^{\boldsymbol{\alpha}}\bar{y}_u, D^{\boldsymbol{\alpha}}y_f &\in W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega), \\ D^{\boldsymbol{\alpha}}\bar{p} &\in W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega) \cap W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega) \cap W^{1,2}_{\vec{\gamma},\vec{\tau}}(\Gamma), \\ D^{\boldsymbol{\alpha}}\bar{u} &\in W^{0,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1) \cap W^{1,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2), \end{aligned}$$

for all $|\boldsymbol{\alpha}| = 1$.

Proof. From Theorem 2.3.7 and Theorem 2.3.8 we directly conclude with Lemma 4.1.3

$$D^{\boldsymbol{\alpha}}\bar{y}_u, D^{\boldsymbol{\alpha}}y_f \in W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega), \quad D^{\boldsymbol{\alpha}}\bar{p} \in W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega) \cap W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega), \qquad \forall |\boldsymbol{\alpha}| = 1.$$

A trace theorem and the embeddings from Lemma 2.3.4 imply

$$D^{\boldsymbol{\alpha}}\bar{p} \in W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Gamma) \hookrightarrow W^{1,2}_{\vec{\gamma},\vec{\tau}}(\Gamma) \cap W^{0,\infty}_{\vec{\gamma},\vec{\delta}}(\Gamma).$$

Note, that in order to get the validity of the embeddings one has to take into account that $\varepsilon > 0$ can be chosen arbitrarily but small. Due to (4.12) we moreover have

$$\bar{u} = \begin{cases} -\alpha^{-1}\bar{p}, & \text{on } \mathcal{I}, \\ u_a, & \text{on } \mathcal{A}^-, \\ u_b, & \text{on } \mathcal{A}^+, \end{cases}$$

Consequently, the control $\bar{u}$ inherits the regularity of the adjoint state $\bar{p}$ and the control bounds u_a and u_b. □

4.2 Discretization error estimates

There exist a couple of discretization approaches for the optimal control problem (4.1)–(4.3) that we will discuss in detail now. An overview of related contributions has already been given in Chapter 1, and we are now in the position to improve these results for polyhedral domains when taking into account the accurate regularity results from Chapter 2 and the sharp finite element error estimates from Chapter 3.

4.2.1 Full discretization

A possible approach for the discretization of the optimal control problem (4.1)–(4.3) is, to approximate the state and adjoint state variable with piecewise linear and continuous finite elements, and the control with piecewise constant functions. More precisely, we search

$$\begin{aligned} y_{f,h}, y_{u,h}, p_h \in V_h &:= \left\{ v_h \in C(\overline{\Omega}) \colon v_h \text{ is affine linear on all } T \in \mathcal{T}_h \right\}, \\ u_h \in U_{h,ad} &:= \{ u_h \in L^\infty(\Gamma) \colon u_h \text{ is constant on all } E \in \mathcal{E}_h \} \cap U_{ad}, \end{aligned} \tag{4.13}$$

where $\mathcal{T}_h$ is a conforming triangulation of Ω and $\mathcal{E}_h$ the induced boundary mesh, i. e. each $E \in \mathcal{E}_h$ is also a face (if $n = 3$) or edge (if $n = 2$) of some $T \in \mathcal{T}_h$.

The discrete form of the optimality system (4.10) then reads

Find $y_{u,h}, p_h \in V_h$ and $u_h \in U_h$:

$$\begin{cases} a(y_{u,h}, v_h) = (u_h, v_h)_\Gamma & \forall v_h \in V_h, \\ a(v_h, p_h) = (y_{u,h} + y_{f,h} - y_d, v_h) & \forall v_h \in V_h, \\ (\alpha u_h + p_h, w_h - u_h)_\Gamma \geq 0 & \forall w_h \in U_{h,ad}, \end{cases} \tag{4.14}$$

where the function $y_{f,h} \in V_h$ can be computed in advance from

$$a(y_{f,h}, v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

We introduce the discrete versions of the operators S, S^* and P, defined by

$$\begin{aligned} S_h &\colon L^2(\Gamma) \to V_h \hookrightarrow L^2(\Omega), & w_h = S_h u \quad &:\Longleftrightarrow \quad a(w_h, v_h) = (u, v_h)_\Gamma & \forall v_h \in V_h, \\ P_h &\colon L^2(\Omega) \to V_h \hookrightarrow H^1(\Omega), & w_h = P_h y \quad &:\Longleftrightarrow \quad a(v_h, w_h) = (y, v_h) & \forall v_h \in V_h, \end{aligned}$$

and

$$S_h^* := \tau \circ P_h \colon L^2(\Omega) \to V_h^\partial := \{w_h \in C(\Gamma) \colon w_h = v_h|_\Gamma \text{ for some } v_h \in V_h\} \hookrightarrow L^2(\Gamma).$$

That S_h^* is indeed the adjoint operator to S_h becomes clear by

$$(S_h^* v, w)_\Gamma = (P_h v, w)_\Gamma = a(S_h w, P_h v) = (v, S_h w) \qquad \forall v \in L^2(\Omega), w \in L^2(\Gamma). \tag{4.15}$$

Analogous to the continuous case one can show that the system (4.14) possesses a unique solution $(\bar y_{u,h}, \bar u_h, \bar p_h) \in V_h \times U_{h,ad} \times V_h$, and, that $(\bar y_{u,h}, \bar u_h)$ is also the unique solution of the related discretized optimal control problem

$$\min_{(y_{u,h}, u_h) \in V_h \times U_{h,ad}} \frac{1}{2}\|y_{u,h} + y_{f,h} - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_h\|^2_{L^2(\Gamma)}$$

subject to

$$a(y_{u,h}, v_h) = (u_h, v_h)_\Gamma \qquad \forall v_h \in V_h.$$

Recall that the optimal continuous and discrete state variable can be decomposed by means of

$$\bar y = \bar y_u + y_f, \qquad \bar y_h = \bar y_{u,h} + y_{f,h}.$$

We will derive an *a priori* error estimate for the discrete solution $(\bar y_h, \bar u_h)$ in the next theorem. The proof is similar to the one in [26], but we can improve the results using the sharp error estimates on the boundary from Theorem 3.3.2 and Theorem 3.4.14.

Theorem 4.2.1. *Assume that the input data satisfy $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Then, the error estimates*

$$\sqrt{\alpha}\|\bar u - \bar u_h\|_{L^2(\Gamma)} \leq c h \eta \tag{4.16}$$

and

$$\|\bar y - \bar y_h\|_{H^1(\Omega)} + \|\bar p - \bar p_h\|_{H^1(\Omega)} \leq c \begin{cases} h^{\min\{1, \lambda - \varepsilon\}}\eta, & \text{if } \mu = 1, \\ h\eta, & \text{if } \mu < \lambda, \end{cases}$$

hold, where

$$\eta := \|f\|_{L^2(\Omega)} + \|\bar u\|_{H^1(\Gamma)} + |\bar y|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} \bar p\|_{W^{1,2}_{\vec\alpha,\vec\delta}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} \bar p\|_{W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega)} + \|\bar p\|_{L^\infty(\Omega)},$$

with the weight vectors defined in Theorem 4.1.4 and sufficiently small $\varepsilon > 0$.

Proof. As the error estimate for the control cannot be improved with mesh refinement, we prove the stated estimate on quasi-uniform meshes only. Testing the optimality condition of the continuous problem (4.10) with the discrete solution, and the optimality condition of the discrete problem (4.14) with the $L^2(\Gamma)$-projection $P_h^\partial \bar{u}$ of the continuous solution onto $U_{h,ad}$ yields

$$\begin{aligned} (\alpha \bar{u} + \bar{p}, \bar{u}_h - \bar{u})_\Gamma &\geq 0, \\ \left(\alpha \bar{u}_h + \bar{p}_h, P_h^\partial \bar{u} - \bar{u} + \bar{u} - \bar{u}_h\right)_\Gamma &\geq 0. \end{aligned}$$

Summing up both inequalities implies

$$-\alpha\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} + (\bar{p} - \bar{p}_h, \bar{u}_h - \bar{u})_\Gamma + (\alpha \bar{u}_h + \bar{p}_h, P_h^\partial \bar{u} - \bar{u})_\Gamma \geq 0.$$

Reordering this inequality and exploiting that $(\bar{u}_h, P_h^\partial \bar{u} - \bar{u})_\Gamma = 0$ yields

$$\alpha\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (\bar{p} - \bar{p}_h, \bar{u}_h - \bar{u})_\Gamma + (\bar{p}_h, P_h^\partial \bar{u} - \bar{u})_\Gamma. \tag{4.17}$$

In the following the two terms on the right-hand side are discussed separately. For the first term we insert the representations

$$\bar{p}|_\Gamma = S^*(S\bar{u} + y_f - y_d) \quad \text{and} \quad \bar{p}_h|_\Gamma = S_h^*(S_h\bar{u}_h + y_{f,h} - y_d), \tag{4.18}$$

introduce several intermediate functions, apply the Cauchy-Schwarz inequality, exploit the boundedness of S_h^* as operator from $L^2(\Omega)$ to $L^2(\Gamma)$, and get

$$\begin{aligned} &(\bar{p} - \bar{p}_h, \bar{u}_h - \bar{u})_\Gamma \\ &= (S^*(S\bar{u} + y_f - y_d) - S_h^*(S_h\bar{u}_h + y_{f,h} - y_d), \bar{u}_h - \bar{u})_\Gamma \\ &= ((S^* - S_h^*)(S\bar{u} + y_f - y_d) + S_h^*(S - S_h)\bar{u} + S_h^*(y_f - y_{f,h}) + S_h^* S_h(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u})_\Gamma \\ &\leq c\left(\|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} + \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|y_f - y_{f,h}\|_{L^2(\Omega)}\right)\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}, \end{aligned} \tag{4.19}$$

where we exploited in the last step that

$$(S_h^* S_h(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u})_\Gamma = -\|S_h(\bar{u} - \bar{u}_h)\|^2_{L^2(\Omega)} \leq 0.$$

Inserting the already known error estimates for elliptic problems from Theorem 3.3.1 and Theorem 3.3.2 into (4.19) and applying Young's inequality yields

$$(\bar{p} - \bar{p}_h, \bar{u}_h - \bar{u})_\Gamma \leq c\left(h^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta\right)^2 + \frac{\alpha}{3}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)}. \tag{4.20}$$

For the second part on the right-hand side of (4.17) we take into account the error orthogonality of the $L^2(\Gamma)$-projection P_h^∂, apply the Cauchy-Schwarz and the Young inequality with arbitrary $\nu > 0$, and get

$$\begin{aligned} (\bar{p}_h, P_h^\partial \bar{u} - \bar{u})_\Gamma &= (\bar{p}_h - \bar{p}, P_h^\partial \bar{u} - \bar{u})_\Gamma + (\bar{p} - P_h^\partial \bar{p}, P_h^\partial \bar{u} - \bar{u})_\Gamma \\ &\leq c\left(\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} h\|\bar{u}\|_{H^1(\Gamma)} + h^2\|\bar{p}\|_{H^1(\Gamma)}\|\bar{u}\|_{H^1(\Gamma)}\right) \\ &\leq \nu\|\bar{p} - \bar{p}_h\|^2_{L^2(\Gamma)} + ch^2\left(\|\bar{u}\|_{H^1(\Gamma)} + \|\bar{p}\|_{H^1(\Gamma)}\right)^2. \end{aligned} \tag{4.21}$$

Moreover we can derive an estimate for the adjoint state using the representation from (4.18) which implies after insertion of several intermediate functions

$$\bar{p}|_\Gamma - \bar{p}_h|_\Gamma = (S^* - S_h^*)(\bar{y}_u + y_f - y_d) + S_h^*(S - S_h)\bar{u} + S_h^*(y_f - y_{f,h}) + S_h^* S_h(\bar{u} - \bar{u}_h).$$

Then we obtain using the triangle inequality and error estimates for elliptic problems from Theorem 3.3.1 and Theorem 3.3.2

$$\begin{aligned}\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} &\le c\Big(\|(S^* - S_h^*)(\bar{y}_u + y_f - y_d)\|_{L^2(\Gamma)} + \|(S - S_h)\bar{u}\|_{L^2(\Omega)} \\ &\qquad + \|y_f - y_{f,h}\|_{L^2(\Omega)} + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\Big) \\ &\le c\left(h^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\right). \end{aligned} \tag{4.22}$$

Inserting (4.22) into (4.21) and choosing $\nu = \alpha/3c$ leads to

$$(\bar{p}_h, P_h^\partial \bar{u} - \bar{u})_\Gamma \le c\left(h^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta + h\|\bar{p}\|_{H^1(\Gamma)} + h\|\bar{u}\|_{H^1(\Gamma)}\right)^2 + \frac{\alpha}{3}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)}. \tag{4.23}$$

Now we insert (4.20) and (4.23) into (4.17), apply a kick-back argument to the terms $\alpha/3\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)}$, extract the root, and obtain

$$\sqrt{\alpha}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch\eta,$$

where we exploited that $1/2 + \lambda - \varepsilon > 1$ for arbitrary polyhedral domains, when $\varepsilon > 0$ is sufficiently small.

The error estimate for the state follows from the triangle inequality

$$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \le \|(S - S_h)\bar{u}\|_{H^1(\Omega)} + \|S_h(\bar{u} - \bar{u}_h)\|_{H^1(\Omega)} + \|y_f - y_{f,h}\|_{H^1(\Omega)}, \tag{4.24}$$

the error estimates from Theorem 3.3.1 if $\mu = 1$ or Theorem 3.4.8 if $\mu < \lambda$, the boundedness of S_h as operator from $L^2(\Gamma)$ to $H^1(\Omega)$ and the already shown estimate (4.16). In the same way we get an estimate for the adjoint state when writing $\bar{p} = P(\bar{y} - y_d)$ and $\bar{p}_h = P_h(\bar{y}_h - y_d)$. With the triangle inequality this leads to

$$\begin{aligned}\|\bar{p} - \bar{p}_h\|_{H^1(\Omega)} &\le c\left(\|(P - P_h)(\bar{y} - y_d)\|_{H^1(\Omega)} + \|P_h(\bar{y} - \bar{y}_h)\|_{H^1(\Omega)}\right) \\ &\le c\left(\|(P - P_h)(\bar{y} - y_d)\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}\right), \end{aligned} \tag{4.25}$$

where we exploited the stability of P_h as operator from $H^1(\Omega)$ to $H^1(\Omega)$. Inserting the finite element error estimates from Theorem 3.3.1 in case of $\mu = 1$ and Theorem 3.4.8 if $\mu < \lambda$, as well as the estimate already derived for the state variable yields the assertion. □

Estimates for the state in $L^2(\Omega)$

In the remainder of this section we derive an error estimate for the state variable in the $L^2(\Omega)$-norm. While estimates for the state variables in $H^1(\Omega)$ are very easy to show (compare (4.24) and (4.25)), estimates in weaker norms require advanced techniques. The basic strategy we will use is not new. A proof for distributed control problems can be found in [67] and an extension to Neumann control problems in [62]. Improved error estimates using weighted Sobolev spaces are presented in [9] for two-dimensional polygonal domains and in the following, we will show how these results can be extended to polyhedral domains.

Let $R_h^\partial\colon L^\infty(\Gamma) \to U_h$ denote the midpoint interpolant defined in Section 3.2.4. Using the triangle inequality we obtain an estimate for the state by

$$\|\bar y_u - \bar y_{u,h}\|_{L^2(\Omega)} \le \|(S - S_h)\bar u\|_{L^2(\Omega)} + \|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)} + \|S_h(R_h^\partial \bar u - \bar u_h)\|_{L^2(\Omega)}. \quad (4.26)$$

For the first term we only have to insert the finite-element error estimates from Theorem 3.3.1 for quasi-uniform meshes or Theorem 3.4.8 for locally refined meshes. In the following lemmata we discuss the two other terms on the right-hand side of (4.26).

For technical reasons we require an assumption upon the active set.

Assumption 2. *The set* $g := \overline{\mathcal{A}^\pm} \cap \overline{\mathcal{I}}$ *consists of a finite number of curves having finite length.*

From this assumption one could also conclude the assumption Mateos and Rösch used in [62]. They demanded that $|\cup\{E \in \mathcal{K}_1\}| \le ch$. This is not sufficient for our purposes when g goes through some locally refined region. However, what we apply in Lemma 4.2.2 is some local version of this assumption. More precisely, we will benefit again from the decomposition of the boundary $\{\Gamma_i\}_{i=0}^{I}$ already introduced in (3.82) that we used to show a finite element error estimate in $L^2(\Gamma)$.

For the second term in (4.26) one can exploit that the operator S_h realizes a smoothing of the interpolation error $\bar u - R_h^\partial \bar u$ and it is possible to show convergence with a rate higher than one.

Lemma 4.2.2. *Assume that* $f \in L^2(\Omega)$ *and* $y_d \in C^{0,\sigma}(\overline{\Omega})$ *with some* $\sigma \in (0,1)$*, and let Assumption 2 be satisfied.*

a) If $\mu = 1$*, there holds the estimate*

$$\|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)} \le ch^s \eta, \quad (4.27)$$

with $s = \min\{2, 1/2 + \lambda - \varepsilon\}$*,*

b) and if the refinement parameter satisfies $\mu < \frac{1}{4} + \frac{\lambda}{2}$*, there holds*

$$\|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)} \le ch^2 |\ln h| \eta, \quad (4.28)$$

where

$$\eta := |\bar u|_{H^1(\Gamma)} + |\bar u|_{W^{2,2}_{\vec\gamma,\vec\tau}(\mathcal{K}_2)} + |\bar u|_{W^{1,\infty}_{\vec\gamma,\vec\delta}(\mathcal{K}_1)}$$

with the weight vectors defined in Theorem 4.1.4 and $\varepsilon > 0$ *chosen sufficiently small.*

Proof. We will first prove the estimate (4.28) and mention at the end of the proof where modifications are necessary to show also (4.27). Let $v_h := S_h^* S_h(\bar u - R_h^\partial \bar u) \in V_h^\partial$. This allows us to write

$$\|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)}^2 = (\bar u - R_h^\partial \bar u, v_h)_\Gamma = (\bar u - P_h^\partial \bar u, v_h)_\Gamma + (P_h^\partial \bar u - R_h^\partial \bar u, v_h)_\Gamma, \quad (4.29)$$

where P_h^∂ denotes the $L^2(\Gamma)$-projection onto U_h as already defined in Section 3.2.3. With orthogonality properties of P_h^∂ and standard estimates we get

$$\begin{aligned}(\bar u - P_h^\partial \bar u, v_h)_\Gamma = (\bar u - P_h^\partial \bar u, v_h - P_h^\partial v_h)_\Gamma &\le ch^2 |\bar u|_{H^1(\Gamma)} |\bar v_h|_{H^1(\Gamma)} \\ &\le ch^2 |\bar u|_{H^1(\Gamma)} \|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)}. \end{aligned} \tag{4.30}$$

For the second term in (4.29) we distinguish between boundary elements $E \subset \mathcal{K}_1$ and $E \subset \mathcal{K}_2$. On $\mathcal{K}_2$ the solution possesses the regularity $D^{\boldsymbol{\alpha}} \bar u \in W^{1,2}_{\vec\gamma,\vec\tau}(\mathcal{K}_2)$ for all $|\boldsymbol{\alpha}| = 1$, as stated in Theorem 4.1.4, where the largest weight is defined by

$$\kappa := \max_{j\in\mathcal{C},k\in\mathcal{E}} \{\gamma_j, \tau_k\} = \max_{j\in\mathcal{C},k\in\mathcal{E}} \{0, 1 - \lambda_j^c + \varepsilon, 3/2 - \lambda_k^e + \varepsilon\} = \max\{0, 3/2 - \lambda + \varepsilon\}.$$

Using the element-wise definition of the $L^2(\Gamma)$-projection and the fact that $R_h^\partial \bar u$ is constant on each element we get

$$\begin{aligned}\|P_h^\partial \bar u - R_h^\partial \bar u\|^2_{L^2(\mathcal{K}_2)} &= \sum_{E\subset\mathcal{K}_2} \int_E \left(|E|^{-1} \int_E \bar u(y)\,\mathrm{d}s_y - R_h^\partial \bar u\right)^2 \mathrm{d}s_x \\ &= \sum_{E\subset\mathcal{K}_2} |E|^{-1} \left(\int_E (\bar u(y) - R_h^\partial \bar u)\,\mathrm{d}s_y\right)^2. \end{aligned} \tag{4.31}$$

Now the local estimates from Lemma 3.2.10 can be inserted. In case of $r_E > 0$ we get from (3.43) using the mesh condition $h_E \sim h r_E^{1-\mu}$

$$|E|^{-1} \left(\int_E (\bar u(y) - R_h^\partial \bar u)\,\mathrm{d}s_y\right)^2 \le c \left(h^2 r_E^{2(1-\mu)-\kappa} |\bar u|_{W^{2,2}_{\vec\gamma,\vec\tau}(E)}\right)^2, \tag{4.32}$$

and in case of $r_E = 0$ we get with $h_E = h^{1/\mu}$

$$|E|^{-1} \left(\int_E (\bar u(y) - R_h^\partial \bar u)\,\mathrm{d}s_y\right)^2 \le c \left(h^{(2-\kappa)/\mu} |\bar u|_{W^{2,2}_{\vec\gamma,\vec\tau}(E)}\right)^2. \tag{4.33}$$

Moreover, the assumption $\mu < 1/4 + \lambda/2$ implies $\mu \le 1 - \kappa/2$, since

$$1 - \kappa/2 = 1 - \frac{1}{2}\max\{0, 3/2 - \lambda + \varepsilon\} = \min\{1, 1/4 + \lambda/2 - \varepsilon\} \ge \mu, \tag{4.34}$$

where the last step is valid when ε is chosen sufficiently small. Hence (4.32) and (4.33) become

$$|E|^{-1} \left(\int_E (\bar u(y) - R_h^\partial \bar u)\,\mathrm{d}s_y\right)^2 \le c \left(h^2 |\bar u|_{W^{2,2}_{\vec\gamma,\vec\tau}(E)}\right)^2 \tag{4.35}$$

for arbitrary $E \in \mathcal{E}_h$, $E \subset \mathcal{K}_2$. Inserting this into (4.31) yields

$$\|P_h^\partial \bar u - R_h^\partial \bar u\|_{L^2(\mathcal{K}_2)} \le ch^2 |\bar u|_{W^{2,2}_{\vec\gamma,\vec\tau}(\mathcal{K}_2)}.$$

With the Cauchy-Schwarz inequality and the stability estimate $\|v_h\|_{L^2(\Gamma)} \le \|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)}$ we thus arrive at

$$(P_h^\partial \bar u - R_h^\partial \bar u, v_h)_{L^2(\mathcal{K}_2)} \le ch^2 |\bar u|_{W^{2,2}_{\vec\gamma,\vec\tau}(\mathcal{K}_2)} \|S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Omega)}. \tag{4.36}$$

On the set $\mathcal{K}_1$ the solution satisfies only $D^{\boldsymbol{\alpha}}\bar{u} \in W^{0,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)$ for all $|\boldsymbol{\alpha}| = 1$. We denote the largest weight by

$$\kappa_\infty := \max_{j\in\mathcal{C},k\in\mathcal{E}}\{\gamma_j, \delta_k\} = \max_{j\in\mathcal{C},k\in\mathcal{E}}\{0, 1-\lambda_j^c+\varepsilon, 1-\lambda_k^e+\varepsilon\}. \tag{4.37}$$

Exploiting the definition of P_h^∂ yields the estimate

$$\begin{aligned}(P_h^\partial\bar{u} - R_h^\partial\bar{u}, v_h)_{L^2(\mathcal{K}_1)} &= \sum_{E\subset\mathcal{K}_1}\int_E (P_h^\partial\bar{u} - R_h^\partial\bar{u})v_h(x)\,\mathrm{d}s_x \\ &\le \|v_h\|_{L^\infty(\Gamma)}\sum_{E\subset\mathcal{K}_1}\int_E\left||E|^{-1}\int_E\bar{u}(y)\,\mathrm{d}s_y - R_h^\partial\bar{u}\right|\,\mathrm{d}s_x \\ &\le \|v_h\|_{L^\infty(\Gamma)}\sum_{E\subset\mathcal{K}_1}\left|\int_E\left(\bar{u}(y) - R_h^\partial\bar{u}\right)\,\mathrm{d}s_y\right| \\ &\le \|v_h\|_{L^\infty(\Gamma)}\sum_{E\subset\mathcal{K}_1}\|\bar{u} - R_h^\partial\bar{u}\|_{L^\infty(E)}|E|. \end{aligned} \tag{4.38}$$

To obtain a sharp error estimate, we recall the decomposition of the boundary already used in Section 3.4, namely

$$\Gamma_{R/n} := \{x\in\Gamma\colon r(x) < R/n\}, \qquad \tilde{\Gamma}_{R/n} := \Gamma\setminus\Gamma_{R/n},$$

with sufficiently small $R > 0$ that we set without loss of generality equal to one, and use the dyadic decomposition

$$\Gamma_i := \begin{cases}\{x\in\Gamma\colon d_{i+1} < r(x) < d_i\}, & \text{for } i = 0,\dots,I-1,\\ \{x\in\Gamma\colon \quad 0 < r(x) < d_I\}, & \text{for } i = I,\end{cases} \quad \text{with} \quad d_i = 2^{-i}. \tag{4.39}$$

The inner-most domain has radius $d_I = c_I h^{1/\mu}$ with some $c_I > 1$ independent of h, and hence, $I \sim |\ln h|$. The patch with the neighboring sets is denoted by

$$\Gamma_i' := \operatorname{int}\left(\overline{\Gamma_{\max\{0,i-1\}}}\cup\overline{\Gamma_i}\cup\overline{\Gamma_{\min\{I,i+1\}}}\right).$$

Within the set Γ_i, $i = 0,\dots,I$, all elements E have diameter $h_E \sim hd_i^{1-\mu}$. Assumption 2 then implies that

$$\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\tilde{\Gamma}_R\neq\emptyset}} 1 \le ch^{-1}, \qquad \sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_i\neq\emptyset}} 1 \le ch^{-1}d_i^{-(1-\mu)}, \tag{4.40}$$

for all $i = 0,\dots,I$.

With the decomposition (4.39) we obtain

$$\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_{R/2}\neq\emptyset}}\|\bar{u} - R_h^\partial\bar{u}\|_{L^\infty(E)}|E| \le \sum_{i=1}^{I}\sum_{\substack{E\subset\mathcal{K}_1\\ E\cap\Gamma_i\neq\emptyset}}\|\bar{u} - R_h^\partial\bar{u}\|_{L^\infty(E)}|E|. \tag{4.41}$$

From Lemma 3.2.10 we conclude the local estimate

$$\|\bar{u} - R_h^\partial\bar{u}\|_{L^\infty(E)}|E| \le ch^3 d_i^{3(1-\mu)-\kappa_\infty}|\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(E)} \qquad \forall E\subset\mathcal{K}_1, E\cap\Gamma_i\neq\emptyset, \tag{4.42}$$

for all $i = 1, \ldots, I$, where we used the properties $h_E \sim hd_i^{1-\mu}$, $|E| \sim h_E^2$, and in particular if $r_E = 0$

$$h_E^{3-\kappa_\infty} = h^{3+(3-3\mu-\kappa_\infty)/\mu} \leq ch^3 d_I^{3(1-\mu)-\kappa_\infty}.$$

Inserting (4.40) and (4.42) into (4.41) yields

$$\sum_{\substack{E\subset\mathcal{K}_1 \\ E\cap\Gamma_{R/2}\neq\emptyset}} \|\bar{u} - R_h^\partial \bar{u}\|_{L^\infty(E)}|E| \leq ch^2 \sum_{i=1}^{I} d_i^{2(1-\mu)-\kappa_\infty} |\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\Gamma_i'\cap\mathcal{K}_1)}. \tag{4.43}$$

Next, we observe that the condition $\mu \leq 1 - \kappa_\infty/2$ holds. Taking (4.37) and the assumption upon μ into account yields

$$1 - \frac{\kappa_\infty}{2} = 1 - \frac{1}{2} \max_{j\in\mathcal{C},k\in\mathcal{E}} \{0, 1 - \lambda_k^e + \varepsilon, 1 - \lambda_j^c + \varepsilon\} \geq \min\{1, 1/4 + \lambda/2 - \varepsilon\} \geq \mu.$$

As a consequence, (4.43) leads together with $I \sim |\ln h|$ to

$$\sum_{\substack{E\subset\mathcal{K}_1 \\ E\cap\Gamma_{R/2}\neq\emptyset}} \|\bar{u} - R_h^\partial \bar{u}\|_{L^\infty(E)}|E| \leq ch^2|\ln h||\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}. \tag{4.44}$$

The extension to elements contained in or intersecting $\tilde{\Gamma}_{R/2}$ is easy as $r_E \sim c$ and $h_E \sim h$. Exploiting also (4.40) yields

$$\sum_{\substack{E\subset\mathcal{K}_1 \\ E\cap\tilde{\Gamma}_{R/2}\neq\emptyset}} \|\bar{u} - R_h^\partial \bar{u}\|_{L^\infty(E)}|E| \leq ch|\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)} \sum_{\substack{E\subset\mathcal{K}_1 \\ E\cap\tilde{\Gamma}_{R/2}\neq\emptyset}} |E| \leq ch^2|\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}. \tag{4.45}$$

Consequently, we deduce from (4.44) and (4.45) that

$$\sum_{E\subset\mathcal{K}_1} \|\bar{u} - R_h^\partial \bar{u}\|_{L^\infty(E)}|E| \leq ch^2|\ln h||\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}. \tag{4.46}$$

Inserting (4.46) into (4.38) yields together with the stability estimate $\|S_h^* v\|_{L^\infty(\Gamma)} \leq \|v\|_{L^2(\Omega)}$

$$(P_h^\partial \bar{u} - R_h^\partial \bar{u}, v_h)_{L^2(\mathcal{K}_1)} \leq ch^2|\ln h||u|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}\|S_h(\bar{u} - R_h^\partial \bar{u})\|_{L^2(\Omega)}. \tag{4.47}$$

Together with (4.36), (4.30) and (4.29) we arrive at the desired estimate (4.28).

To show the estimate (4.27) without mesh refinement, only a few modifications of the proof are necessary. First, note that (4.30) remains valid with $\mu = 1$. Moreover, instead of (4.32) and (4.33), we obtain

$$|E|^{-1}\left(\int_E (\bar{u}(y) - R_h^\partial \bar{u})\,\mathrm{d}s_y\right)^2 \leq c\left(h^{s_1}|\bar{u}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)}\right)^2,$$

with

$$s_1 := \min\{2, 1/2 + \lambda - \varepsilon\}.$$

As a consequence we get

$$(P_h^\partial \bar{u} - R_h^\partial \bar{u}, v_h)_{L^2(\mathcal{K}_2)} \leq ch^{s_1}|\bar{u}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)}\|S_h(\bar{u} - R_h^\partial \bar{u})\|_{L^2(\Omega)}. \tag{4.48}$$

The convergence rate for the estimate on $\mathcal{K}_1$ is also reduced and the proof is even simpler. We do not need the dyadic decomposition (4.39) as the mesh is globally quasi-uniform and hence, Assumption 2 implies $|\mathcal{K}_1| \leq ch$. As a consequence, we get with the local estimate from Lemma 3.2.10 and $\mu = 1$

$$\sum_{E\subset\mathcal{K}_1} \|\bar{u} - R_h^\partial \bar{u}\|_{L^\infty(E)}|E| \leq ch^{1-\kappa_\infty}|\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)} \sum_{E\subset\mathcal{K}_1} |E| \leq ch^{s_2}|\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)},$$

where

$$s_2 := 2 - \kappa_\infty = \min_{j\in\mathcal{C},k\in\mathcal{E}}\{2, 1+\lambda_j^c - \varepsilon, 1 + \lambda_k^e - \varepsilon\}.$$

Inserting this into (4.38) leads to

$$(P_h^\partial \bar{u} - R_h^\partial \bar{u}, v_h)_{L^2(\mathcal{K}_1)} \leq ch^{s_2}|\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)}\|S_h(\bar{u} - R_h^\partial \bar{u})\|_{L^2(\Omega)}. \tag{4.49}$$

Inserting the estimates (4.30), (4.48) and (4.49) into (4.29) yields the estimate (4.27) taking into account that $s := s_1 \leq s_2$. □

It remains to derive an estimate for the third term on the right-hand side of (4.26), and we exploit a principle which is called *supercloseness* in the literature. This principle relies on the fact that the interpolant of the continuous solution $\bar{u}$ is closer to the discrete solution $\bar{u}_h$ than $\bar{u}$ itself.

Lemma 4.2.3. *Assume that $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$, and let Assumption 2 be satisfied.*

a) If $\mu = 1$ there holds the estimate

$$\|S_h(R_h^\partial \bar{u} - \bar{u}_h)\|_{L^2(\Omega)} \leq ch^s|\ln h|^{3/2}\eta, \tag{4.50}$$

with $s := \min\{2, 1/2 + \lambda - \varepsilon\}$,

b) and if $\mu < \frac{1}{4} + \frac{\lambda}{2}$, there holds

$$\|S_h(R_h^\partial \bar{u} - \bar{u}_h)\|_{L^2(\Omega)} \leq ch^2|\ln h|^{3/2}\eta, \tag{4.51}$$

where

$$\eta := \|f\|_{L^2(\Omega)} + |\bar{u}|_{H^1(\Gamma)} + |\bar{u}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} + |\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)} + |\bar{y}_u|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)}$$
$$+ |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} + \sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}\bar{p}\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}\bar{p}\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}$$

with the weight vectors defined in Theorem 4.1.4 and $\varepsilon > 0$ chosen sufficiently small.

Proof. We show the error estimate (4.51) and discuss at the end of this proof at which point the convergence rate is reduced for quasi-uniform meshes. Firstly, one confirms that the variational inequality in (4.10) holds also pointwise and hence

$$(\alpha R_h^\partial \bar{u} + R_h^\partial \bar{p}, \bar{u}_h - R_h^\partial \bar{u})_\Gamma \geq 0,$$

where we used $\bar{u}_h$ as test function. Secondly, if we test the discrete variational inequality (4.14) with $R_h^\partial \bar{u}$ we get

$$(\alpha \bar{u}_h + \bar{p}_h, R_h^\partial \bar{u} - \bar{u}_h)_\Gamma \geq 0.$$

Summing up both inequalities yields

$$\alpha \|\bar{u}_h - R_h^\partial \bar{u}\|_{L^2(\Gamma)}^2 \leq (R_h^\partial \bar{p} - \bar{p}_h, \bar{u}_h - R_h^\partial \bar{u})_\Gamma.$$

Once we have shown an estimate for the right-hand side the assertion follows as S_h is bounded in the sense that $\|S_h v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Gamma)}$ for all $v \in L^2(\Gamma)$. Introducing the intermediate functions $\bar{p}$ and $S_h^*(S_h R_h^\partial \bar{u} - y_d)$ leads to

$$\begin{aligned}\alpha \|\bar{u}_h - R_h^\partial \bar{u}\|_{L^2(\Gamma)}^2 \leq\ & (R_h^\partial \bar{p} - \bar{p}, \bar{u}_h - R_h^\partial \bar{u})_\Gamma \\ & + (\bar{p} - S_h^*(S_h R_h^\partial \bar{u} + y_{f,h} - y_d), \bar{u}_h - R_h^\partial \bar{u})_\Gamma \\ & + (S_h^*(S_h R_h^\partial \bar{u} + y_{f,h} - y_d) - \bar{p}_h, \bar{u}_h - R_h^\partial \bar{u})_\Gamma, \end{aligned} \tag{4.52}$$

and it remains to discuss the three terms on the right-hand side. Up to here, the proof coincides with the proof of [62, Proposition 4.5].

Taking into account the decomposition $\mathcal{E}_h$ of Γ and exploiting that $\bar{u}_h$ and $R_h^\partial \bar{u}$ are constant on each boundary element $E \in \mathcal{E}_h$ leads to

$$\begin{aligned}(R_h^\partial \bar{p} - \bar{p}, \bar{u}_h - R_h^\partial \bar{u})_\Gamma &= \sum_{E \in \mathcal{E}_h} \int_E (R_h^\partial \bar{p} - \bar{p}(x))(\bar{u}_h - R_h^\partial \bar{u}) \,\mathrm{d}s_x \\ &= \sum_{E \in \mathcal{E}_h} (\bar{u}_h - R_h^\partial \bar{u})|_E \int_E (R_h^\partial \bar{p} - \bar{p}(x)) \,\mathrm{d}s_x. \end{aligned} \tag{4.53}$$

For the adjoint state we know from Theorem 4.1.4 that $D^{\boldsymbol{\alpha}} \bar{p} \in W^{1,2}_{\vec{\gamma},\vec{\tau}}(\Gamma)$ for all $|\boldsymbol{\alpha}| = 1$. We define the number

$$\kappa := \max_{j \in \mathcal{C}, k \in \mathcal{E}} \{\gamma_j, \tau_k\} = \max_{j \in \mathcal{C}, k \in \mathcal{E}} \{0, 1 - \lambda_j^c + \varepsilon, \frac{3}{2} - \lambda_k^e + \varepsilon\} = \max\{0, 3/2 - \lambda + \varepsilon\},$$

and insert the local estimate (3.43) from Lemma 3.2.10 to arrive at

$$\int_E (R_h^\partial \bar{p} - \bar{p}(x)) \,\mathrm{d}s_x \leq c|E|^{1/2} |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \begin{cases} h^2 r_E^{2(1-\mu)-\kappa}, & \text{if } r_E > 0, \\ h^{(2-\kappa)/\mu}, & \text{if } r_E = 0. \end{cases} \tag{4.54}$$

Inserting the assumption $\mu \leq 1 - \kappa/2$ which follows analogous to (4.34) from $\mu < 1/4 + \lambda/2$, yields

$$\int_E (R_h^\partial \bar{p} - \bar{p}(x)) \,\mathrm{d}s_x \leq c h^2 |E|^{1/2} |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \qquad \forall E \in \mathcal{E}_h.$$

The estimate (4.53) then becomes

$$\begin{aligned}(R_h^\partial \bar{p} - \bar{p}, \bar{u}_h - R_h^\partial \bar{u})_\Gamma &\leq c \sum_{E \in \mathcal{E}_h} \left| (\bar{u}_h - R_h^\partial \bar{u})|_E \right| h^2 |E|^{1/2} |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \\ &\leq c \sum_{E \in \mathcal{E}_h} h^2 |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(E)} \|\bar{u}_h - R_h^\partial \bar{u}\|_{L^2(E)} \\ &\leq c h^2 |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\Gamma)} \|\bar{u}_h - R_h^\partial \bar{u}\|_{L^2(\Gamma)}. \end{aligned} \tag{4.55}$$

For the second term in (4.52) we insert the representation $\bar p|_\Gamma = S^*(S\bar u + y_f - y_d)$ and with appropriate intermediate functions we get

$$\begin{aligned}\|\bar p - S_h^*(S_h R_h^\partial \bar u + y_{f,h} - y_d)\|_{L^2(\Gamma)} &= \|(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)\bar u\|_{L^2(\Gamma)} \\ &\quad + \|S_h^*(y_f - y_{f,h})\|_{L^2(\Gamma)} + \|S_h^* S_h(\bar u - R_h^\partial \bar u)\|_{L^2(\Gamma)} \\ &\le ch^2|\ln h|^{3/2}\eta.\end{aligned}$$

In the last step we inserted the finite element error estimate from Theorem 3.4.14 for the first term, the stability of S_h^* as operator from $L^2(\Omega)$ to $L^2(\Gamma)$ and the estimate of Theorem 3.4.8 for the second and third term, and the result of Lemma 4.2.2 for the fourth term. With an application of the Cauchy-Schwarz inequality we then obtain

$$(\bar p - S_h^*(S_h R_h^\partial \bar u + y_{f,h} - y_d), \bar u_h - R_h^\partial \bar u)_\Gamma \le ch^2|\ln h|^{3/2}\eta\|\bar u_h - R_h^\partial \bar u\|_{L^2(\Gamma)}. \tag{4.56}$$

For the third term in (4.52) we insert the representation of the discrete adjoint state, namely $\bar p_h|_\Gamma = S_h^*(S_h \bar u_h + y_{f,h} - y_d)$, and observe that it is non-positive by

$$(S_h^*(S_h R_h^\partial \bar u + y_{f,h} - y_d) - \bar p_h, \bar u_h - R_h^\partial \bar u)_\Gamma = (S_h(R_h^\partial \bar u - \bar u_h), S_h(\bar u_h - R_h^\partial \bar u)) \le 0,$$

and hence, we can neglect this term. From the estimates (4.52), (4.55) and (4.56) we conclude the estimate (4.51) for locally refined meshes.

The estimate (4.50) can be shown with the following modifications. From (4.54) we get in case of $\mu = 1$ the local estimate

$$\int_E (R_h^\partial \bar p - \bar p(x))\,\mathrm{d}s_x \le ch^s|E|^{1/2}|\bar p|_{W^{2,2}_{\vec\gamma,\vec\tau}(E)} \qquad \forall E \in \mathcal{E}_h,$$

and hence, analogous to (4.55) we deduce

$$(R_h^\partial \bar p - \bar p, \bar u_h - R_h^\partial \bar u)_\Gamma \le ch^s|\bar p|_{W^{2,2}_{\vec\gamma,\vec\tau}(\Gamma)}\|\bar u_h - R_h^\partial \bar u\|_{L^2(\Gamma)}. \tag{4.57}$$

Instead of (4.56) we get with the error estimates for elliptic problems on quasi-uniform meshes from Theorem 3.3.1 and Theorem 3.3.2 the estimate

$$(\bar p - S_h^*(S_h R_h^\partial \bar u + y_{f,h} - y_d), \bar u_h - R_h^\partial \bar u)_\Gamma \le ch^s|\ln h|^{3/2}\eta\|\bar u_h - R_h^\partial \bar u\|_{L^2(\Gamma)}. \tag{4.58}$$

Inserting (4.57) and (4.58) into (4.52) yields the estimate (4.50) when exploiting the a-priori estimate

$$\|S_h v\|_{L^2(\Omega)} \le c\|v\|_{L^2(\Gamma)} \qquad \forall v \in L^2(\Gamma). \tag{4.59}$$

□

With the two lemmata above we can conclude an estimate for the state and adjoint state in the $L^2(\Omega)$- and $L^2(\Gamma)$-norm, respectively.

Theorem 4.2.4. *Let Assumption 2 be satisfied, and assume that $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$.*

a) If $\mu = 1$ there holds the estimate

$$\|\bar y - \bar y_h\|_{L^2(\Omega)} + \|\bar p - \bar p_h\|_{L^2(\Gamma)} \le ch^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta, \tag{4.60}$$

b) and if $\mu < \frac{1}{4} + \frac{\lambda}{2}$ there holds

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq ch^2 |\ln h|^{3/2} \eta, \tag{4.61}$$

where

$$\begin{aligned} \eta := \|f\|_{L^2(\Omega)} + |\bar{u}|_{H^1(\Gamma)} + |\bar{u}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} + |\bar{u}|_{W^{1,\infty}_{\vec{\gamma},\vec{\delta}}(\mathcal{K}_1)} + |\bar{y}_u|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} \\ + |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\Gamma)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} \bar{p}\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}} \bar{p}\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}, \end{aligned}$$

with the weight vectors defined in Theorem 4.1.4 and $\varepsilon > 0$ chosen sufficiently small.

Proof. The estimates for the state variable follow from the decomposition

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq \|\bar{y}_u - \bar{y}_{u,h}\|_{L^2(\Omega)} + \|y_f - y_{f,h}\|_{L^2(\Gamma)},$$

the further decomposition (4.26), Theorem 3.3.1 for $\mu = 1$ or Theorem 3.4.8 for $\mu < 1/4 + \lambda/2 < \lambda$, and the Lemmata 4.2.2 and 4.2.3.

From the representations $\bar{p}|_\Gamma = S^*(\bar{y} - y_d)$ and $\bar{p}_h|_\Gamma = S_h^*(\bar{y}_h - y_d)$, as well as the triangle inequality we get an estimate for the adjoint state

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \leq \|(S^* - S_h^*)(\bar{y} - \bar{y}_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Gamma)}.$$

It remains to insert the error estimate on the boundary from Theorem 3.3.2 for $\mu = 1$ or Theorem 3.4.14 for $\mu < 1/4 + \lambda/2$, the stability of S_h^* from $L^2(\Omega)$ to $L^2(\Gamma)$, and the estimate already derived for the state. □

4.2.2 Variational discretization

Another possibility to discretize the optimal control problem (4.1)–(4.3) is the variational approach first introduced by Hinze in [50] and extended to Neumann control in [51]. In contrast to the choice in (4.13) we do not discretize the control and search the triple

$$(\bar{y}_{u,h}, \bar{u}_h, \bar{p}_h) \in V_h \times U_{ad} \times V_h$$

as the solution of the optimality system

$$\begin{aligned} a(y_{u,h}, v_h) - (u_h, v_h)_\Gamma &= 0 && \forall v_h \in V_h, \\ a(v_h, p_h) - (y_{u,h}, v_h) &= (y_{f,h} - y_d, v_h) && \forall v_h \in V_h, \\ (\alpha u_h + p_h, u - u_h)_\Gamma &\geq 0 && \forall u \in U_{ad}. \end{aligned} \tag{4.62}$$

The variational inequality is equivalent to the projection formula

$$\bar{u}_h = \Pi_{ad} \left(-\frac{1}{\alpha} \bar{p}_h|_\Gamma \right)$$

and hence, $\bar{u}_h$ is piecewise linear, but it is not a finite-element function from V_h^∂. However, we have an implicit discretization of the control by means of the discrete adjoint state and the control bounds. According to [50] this finite-dimensional system forms a necessary and sufficient optimality condition of the infinite-dimensional optimization problem

$$\min_{u\in U_{ad}} \frac{1}{2}\|S_h u + y_{f,h} - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)},$$

where only the control-to-state operator S was replaced by the corresponding finite-element solution operator, but the control remains continuous.

In what follows we show how the convergence rates are improved using this approach. The basic idea of the proof for the following theorem has first been presented in [50], but we repeat it in order to outline the similarity to the proof of Theorem 4.2.1. Moreover, we are in the position to improve the results in [51] due to the sharp finite-element error estimates derived in Chapter 3.

Theorem 4.2.5. *Assume that $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$.*

a) If $\mu = 1$, the error estimates

$$\sqrt{\alpha}\|\bar u - \bar u_h\|_{L^2(\Gamma)} + \|\bar y - \bar y_h\|_{L^2(\Omega)} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le ch^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta,$$
$$\|\bar y - \bar y_h\|_{H^1(\Omega)} + \|\bar p - \bar p_h\|_{H^1(\Omega)} \le ch^{\min\{1,\lambda-\varepsilon\}}\eta,$$

b) and if $\mu < \frac{1}{4} + \frac{\lambda}{2}$, the estimates

$$\sqrt{\alpha}\|\bar u - \bar u_h\|_{L^2(\Gamma)} + \|\bar y - \bar y_h\|_{L^2(\Omega)} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}\eta,$$
$$\|\bar y - \bar y_h\|_{H^1(\Omega)} + \|\bar p - \bar p_h\|_{H^1(\Omega)} \le ch\eta,$$

hold, where

$$\eta := \|f\|_{L^2(\Omega)} + |\bar y_u|_{W^{2,2}_{\vec\alpha,\vec\delta}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}\bar p\|_{W^{1,2}_{\vec\alpha,\vec\delta}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1}\|D^{\boldsymbol{\alpha}}\bar p\|_{W^{1,\infty}_{\vec\beta,\vec\varrho}(\Omega)} + \|\bar p\|_{L^\infty(\Omega)},$$

with the weight vectors defined in Theorem 4.1.4 and $\varepsilon > 0$ chosen sufficiently small.

Proof. Testing the variational inequality from the system (4.10) with the solution $\bar u_h$ of the discretized problem, and the variational inequality from (4.62) with the continuous solution $\bar u$ leads to

$$(\alpha\bar u + \bar p, \bar u_h - \bar u)_\Gamma \ge 0,$$
$$(\alpha\bar u_h + \bar p_h, \bar u - \bar u_h)_\Gamma \ge 0.$$

This is possible, since $\bar u$ is also feasible for the discretized problem. Summing up both inequalities leads to

$$\alpha\|\bar u - \bar u_h\|^2_{L^2(\Gamma)} \le (\bar p - \bar p_h, \bar u_h - \bar u)_\Gamma.$$

We have already derived an estimate for this term in the proof of Theorem 4.2.1. Recall estimate (4.19), which yields

$$\sqrt{\alpha}\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le c\left(\|(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Gamma)} + \|(S - S_h)\bar u\|_{L^2(\Omega)} + \|y_f - y_{f,h}\|_{L^2(\Omega)}\right). \tag{4.63}$$

Inserting the finite element error estimates from Theorem 3.3.1 and Theorem 3.3.2 leads to

$$\sqrt{\alpha}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq ch^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta.$$

To obtain the estimates for the state we consider the estimate (4.24), exploit the stability of S_h as operator from $L^2(\Gamma)$ to $H^1(\Omega)$, insert the finite element error estimate from Theorem 3.3.1 as well as the estimate derived for the control, and get

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} &\leq c\left(\|(S - S_h)\bar{u}\|_{H^1(\Omega)} + \|y_f - y_{f,h}\|_{H^1(\Omega)} + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\right) \\ &\leq c\left(h^{\min\{1,\lambda-\varepsilon\}} + h^{\min\{2,1/2+\lambda\}-\varepsilon}\right)\eta,\end{aligned} \tag{4.64}$$

and since $\min\{1,\lambda\} < \min\{2, 1/2+\lambda\}$ we arrive at the desired estimate. To obtain an estimate in the $L^2(\Omega)$-norm we apply the same technique and can improve the estimate using

$$\|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|y_f - y_{f,h}\|_{L^2(\Omega)} \leq ch^{\min\{2,2\lambda-\varepsilon\}}\eta.$$

The convergence rate is now dominated by $\min\{2, 1/2 + \lambda - \varepsilon\}$.

The error estimates for the adjoint state follow in a similar way from (4.25), the estimates for elliptic problems from Theorem 3.3.1 and the estimate already derived for the state variable. From this we conclude

$$\|\bar{p} - \bar{p}_h\|_{H^1(\Omega)} \leq c\left(\|(P - P_h)(\bar{y} - y_d)\|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}\right) \leq ch^{\min\{1,\lambda-\varepsilon\}}\eta. \tag{4.65}$$

To obtain an estimate in $L^2(\Omega)$ we replace the $H^1(\Omega)$-norm by the $L^2(\Omega)$-norm, and with

$$\|(P - P_h)(\bar{y} - y_d)\|_{L^2(\Omega)} \leq ch^{\min\{2,2\lambda-\varepsilon\}}\eta,$$

which follows from Theorem 3.3.1, and the estimate derived for the state in $L^2(\Omega)$, we conclude the estimate for the adjoint state in $L^2(\Omega)$.

Deriving the error estimates when local mesh refinement is used, is easy. The arguments of the proof can be widely repeated, and we only have to insert the error estimates for elliptic problems on refined meshes from Theorems 3.4.8 and 3.4.14 into (4.63), (4.64) and (4.65). □

4.2.3 Postprocessing approach

Another approach which also exploits the higher regularity of the control variable is the *post-processing approach* which was first considered by Meyer and Rösch in [67] and extended to Neumann control problems in [62]. The idea is to compute a fully discrete solution of the system (4.14) and to get an improved control by means of the projection formula (4.11), more precisely

$$\tilde{u}_h := \Pi_{ad}\left(-\frac{1}{\alpha}\bar{p}_h|_\Gamma\right).$$

The function $\tilde{u}_h$ is finite-dimensional and feasible, but not an element of the discrete control space $U_{h,ad}$.

Our aim is to show that $\tilde{u}_h$ converges to $\bar{u}$ with a higher rate than $\bar{u}_h$.

Theorem 4.2.6. *Assume that $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$ and let Assumption 2 be satisfied.*

a) If $\mu = 1$ *there holds*

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \le ch^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}\eta,$$

b) and if $\mu < \frac{1}{4} + \frac{\lambda}{2}$ *there holds*

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\eta,$$

where

$$\eta := \|f\|_{L^2(\Omega)} + |\bar{u}|_{H^1(\Gamma)} + |\bar{u}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\mathcal{K}_2)} + |\bar{u}|_{W^{1,\infty}_{\vec{\alpha},\vec{\delta}}(\mathcal{K}_1)} + |\bar{y}_u|_{W^{2,2}_{\vec{\alpha},\vec{\delta}}(\Omega)}$$
$$+ |\bar{p}|_{W^{2,2}_{\vec{\gamma},\vec{\tau}}(\Gamma)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}}\bar{p}\|_{W^{1,2}_{\vec{\alpha},\vec{\delta}}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=1} \|D^{\boldsymbol{\alpha}}\bar{p}\|_{W^{1,\infty}_{\vec{\beta},\vec{\varrho}}(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)},$$

with the weight vectors defined in Theorem 4.1.4 and $\varepsilon > 0$ *chosen sufficiently small.*

Proof. Inserting the projection formula (4.11) and exploiting the non-expansivity of the projection operator Π_{ad} onto the convex set U_{ad}, see e. g. [96, Proposition 46.5], leads to

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} = \|\Pi_{ad}\left(-\frac{1}{\alpha}\bar{p}\right) - \Pi_{ad}\left(-\frac{1}{\alpha}\bar{p}_h\right)\|_{L^2(\Gamma)} \le c\alpha^{-1}\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)}.$$

The assertion then directly follows from the error estimate for the adjoint state derived in Theorem 4.2.4. □

4.3 Numerical experiments

In order to confirm the results proven in this chapter we consider a slightly modified problem which allows us to construct an analytic solution. The difficulty is to find a solution which possesses the regularity stated in Theorem 4.1.4 and which satisfies the Neumann boundary conditions of state and adjoint equation. As a remedy, we follow the construction used already in [62]. In this paper functions $g_1, g_2 \in L^2(\Gamma)$ were introduced and the optimization problem

$$J(y, u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} + \int_\Gamma y(x)g_2(x)\,\mathrm{d}s_x \to \min!$$

subject to

$$-\Delta y + y = f \text{ in } \Omega, \qquad \partial_n y = u + g_1 \text{ on } \Gamma,$$
$$u \in U_{ad} := \left\{u \in L^2(\Gamma) \colon u_a \le u \le u_b \text{ a. e. on } \Gamma\right\},$$

was considered. The corresponding first-order optimality system reads

$$\begin{aligned} -\Delta y + y &= f & -\Delta p + p &= y - y_d & &\text{in } \Omega, \\ \partial_n y &= u + g_1 & \partial_n p &= g_2 & &\text{on } \Gamma, \end{aligned}$$

$$u = \Pi_{ad}\left(-\frac{1}{\alpha}p|_\Gamma\right).$$

The optimal state $\bar{y}$ and adjoint state $\bar{p}$ can be chosen arbitrarily, and the input data f, y_d, g_1, g_2 as well as the optimal control $\bar{u}$ can be calculated by means of the optimality system.

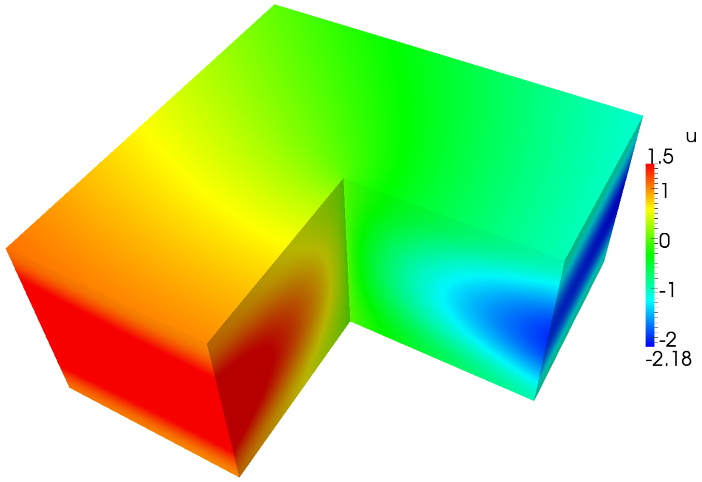


Figure 4.1: The optimal control $\bar{u}$ of the problem in Example 4.3.2.

Remark 4.3.1. *In the experiments the* primal dual active set strategy *was used to compute a solution of the discretized problem. This algorithm has been discussed in [49], [88, Section 2.12.4]. An extensive description of implementation details can be also found in [83]. Some further remarks regarding the implementation using the variational discretization approach can be found in [50]. To compute a solution of the unconstrained problems a CG or GMRES method was applied to the reduced form of the discretized optimality condition. The linear equation systems coming from the finite element formulation of state and adjoint equation were solved with the direct solver MUMPS and the computed Cholesky factorization was reused as all equation systems involve only the system matrix corresponding to* $-\Delta + I$.

Example 4.3.2. *The computational domain is the three-dimensional L-shaped domain* $\Omega := (-1,1)^2 \setminus [0,1]^2 \times (0,1)$. *We introduce cylindrical coordinates* (r,φ,z) *around the* x_3*-axis such that* $\varphi = 0$ *and* $\varphi = 3\pi/2$ *coincide with the two faces which intersect each other in the singular edge. The optimal state and adjoint state are chosen as*

$$\bar{y}(r,\varphi,z) = \bar{p}(r,\varphi,z) := r^{\lambda}\cos(\lambda\varphi),$$

and by some calculations the input data and the optimal control, which is illustrated in Figure 4.1, are obtained. In this example only the upper control bound $u \leq 1.5$ *is considered. Moreover, the regularization parameter* $\alpha = 1$ *has been chosen. It is important that the solution we construct contains the same singularities one would expect for general problems. This is in our example indeed the case with* $\lambda = 2/3$. *However, we neglected the corner singularities since a construction of an exact solution would be very difficult on the one hand, and on the other hand it is known that the corner singularities are mild in comparison to the edge singularities in the L-shaped domain [87].*

For a decreasing sequence of mesh parameters $\{h_k\}_{k\in\mathbb{N}}$ *we computed the corresponding sequence of discrete solutions* $\{(u_k, y_k, p_k)\}_{k\in\mathbb{N}}$ *with the full discretization approach discussed in*

h	# DOF	# DOF BD	$\|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)}$	$\|\bar{y}-\bar{y}_h\|_{H^1(\Omega)}$	$\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}$
1/20	26901	11200	0.023972 (1.00)	0.066934 (0.66)	0.002004 (1.39)
1/30	87451	25200	0.015964 (1.00)	0.05130 (0.66)	0.001147 (1.38)
1/40	203401	44800	0.011964 (1.00)	0.042456 (0.66)	0.000773 (1.37)
1/50	392751	70000	0.009565 (1.00)	0.036652 (0.66)	0.000570 (1.37)
1/60	673501	100800	0.007967 (1.00)	0.032498 (0.66)	0.000444 (1.37)
1/70	1063651	137200	0.006825 (1.00)	0.029353 (0.66)	0.000360 (1.36)
1/80	1581201	179200	0.005969 (1.00)	0.026874 (0.66)	0.000300 (1.37)
1/90	2244151	226800	0.005304 (1.00)	0.024860 (0.66)	0.000256 (1.35)

Table 4.1: Numerical experiments using the full discretization approach on quasi-uniform meshes. The numbers in parenthesis indicate the corresponding experimental convergence rate.

Section 4.2.1, and determined the experimental convergence rate by the well-known formula

$$\mathrm{eoc}(\|e_k\|) := \frac{\log\left(\frac{\|e_k\|}{\|e_{k-1}\|}\right)}{\log\left(\frac{h_k}{h_{k-1}}\right)}, \qquad k \geq 2.$$

The measured error and corresponding convergence rate on a family of quasi-uniform meshes is presented in Table 4.1. The results confirm the convergence rate from Theorem 4.2.1 predicted for the control in $L^2(\Gamma)$ and the state in $H^1(\Omega)$. However, we observe a better convergence rate than predicted for the state in $L^2(\Omega)$, for that we have proven the rate $1/2+\lambda = 7/6$ in Theorem 4.2.4, but obviously we obtain the rate $2\lambda = 4/3$. Indeed, it is easy to confirm that the estimate for the state in $L^2(\Omega)$ is not even sharp as we applied suboptimal a-priori estimates in the proofs of Lemmata 4.2.2 and 4.2.3, more precisely in the steps (4.30), (4.36), (4.47) and (4.59). Similar results were obtained in computations on the two-dimensional L-shaped domain in [74, Table 4.4].

In this example the quite large regularization parameter $\alpha = 1$ has been used which is an unusual choice in applications. However, when computing this experiment with smaller regularization parameters one would observe a much better convergence rate for the state. This is a consequence of the fact that the control is scaled according to α^{-1} and has large values. In order to illustrate this recall the estimate

$$\|\bar{y}-\bar{y}_h\|_{H^1(\Omega)} \leq \|(S-S_h)\bar{u}\|_{H^1(\Omega)} + \|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)} + \|y_f - y_{f,h}\|_{H^1(\Omega)} \tag{4.66}$$

from (4.24). For small α the second term is, for the mesh parameters we chose, the dominating one and converges with rate one. Thus, it would require computations on much finer grids to observe the desired convergence rate for the state as well.

Let us now check whether the results derived in Section 4.2.3 for the postprocessing approach are sharp. We have proven that the sequence of functions $\tilde{u}_k := \Pi_{ad}(-\alpha^{-1}p_k)$ possesses better convergence properties and converges with the best-possible rate under the refinement condition

$$\mu < \frac{1}{4} + \frac{\lambda}{2} = \frac{7}{12}.$$

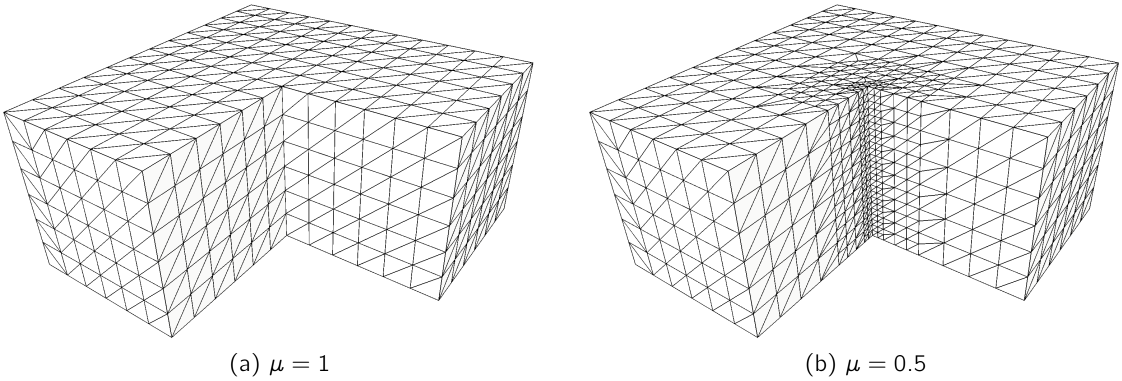


Figure 4.2: Local refinement of a very coarse grid.

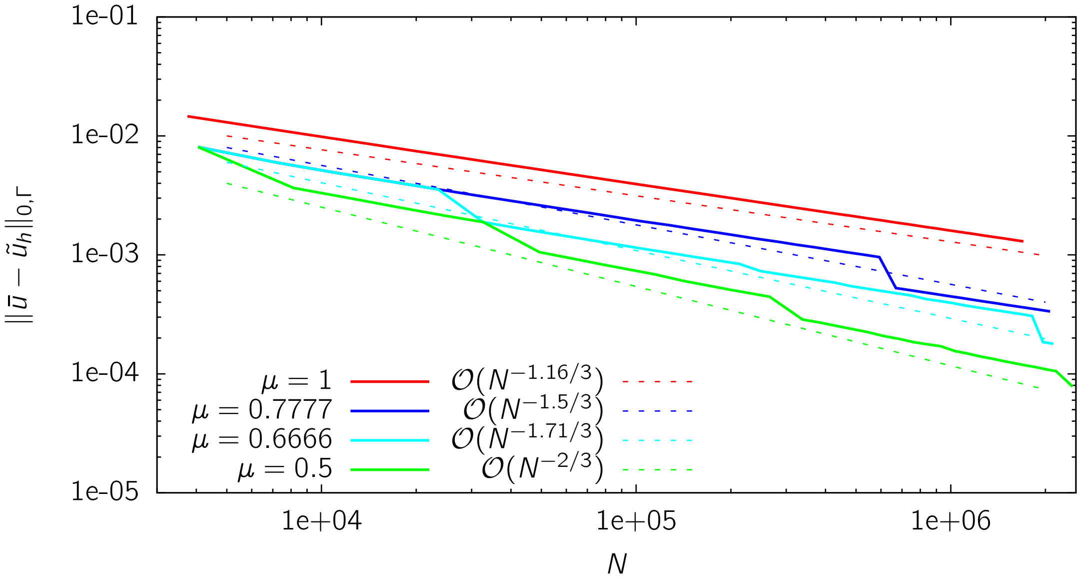


Figure 4.3: Error $\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)}$, solid lines: measured error; dotted lines: expected behavior.

To confirm that this criterion is sharp we computed the solution of our model problem on quasi-uniform meshes ($\mu = 1$), on slightly refined meshes ($\mu = 0.7777$), on meshes which guarantee optimal convergence of the finite element error in $L^2(\Omega)$-norm but not in $L^2(\Gamma)$-norm ($\mu = 0.6666$), and on meshes satisfying the assumption $\mu < 1/4 + \lambda/2$ which is required in Theorem 4.2.6 ($\mu = 0.5$). In this experiment the regular refinement strategy from [21] already described in Section 3.4.2 was used. The resulting mesh from this technique with refinement parameter $\mu = 0.5$ and $R = 0.2$ is illustrated in Figure 4.2.

Figure 4.3 confirms that the measured error coincides with the theoretically predicted behavior which is illustrated by the dotted lines. The results widely coincide with experiments for two-dimensional problems in [9].

CHAPTER 5

Neumann boundary control problems in $H^{-1/2}(\Gamma)$

In this chapter we consider the energy regularization approach that was already used in the book of Lions [60]. Similar investigations for Dirichlet control problems have been presented in [70, 71]. The motivation of this approach is to demand merely as much regularity for the control as it is needed to guarantee that the corresponding state variable is in $H^1(\Omega)$. This means, that we search the solution of a Neumann control problem in the space $H^{-1/2}(\Gamma)$. The problem we are going to investigate reads

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{H^{-1/2}(\Gamma)} \to \min! \tag{5.1}$$

subject to

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \partial_n y = u & \text{on } \Gamma, \end{cases} \tag{5.2}$$

$$u \in U_{ad} := \left\{ u \in H^{-1/2}(\Gamma) \colon u_a \le u \le u_b \right\}. \tag{5.3}$$

Initially, we demand that $f \in [H^1(\Omega)]^*$ and $y_d \in L^2(\Omega)$, but will assume higher regularity later in order to derive our error estimates. For simplification we assume that the control bounds are constant, i. e. $u_a, u_b \in \mathbb{R}$ with $u_a < u_b$. The regularization parameter $\alpha > 0$ can be chosen arbitrarily.

For the analysis of the optimal control problem discussed in Section 5.1 one can widely reuse the techniques applied in case of $L^2(\Gamma)$-regularization in Chapter 4. The fundamental difference regarding the analysis in relation to the $L^2(\Gamma)$-regularization is that the regularization term appears in a negative norm which we replace by an equivalent quadratic functional involving an inverse Steklov-Poincaré operator. This representation allows us to derive an optimality system which has the structure of a Signorini problem. In Section 5.2 we investigate a finite element approximation of the optimality condition and observe that an additional stability condition for the discrete control and state spaces is required to get unique solubility of the discretized problem as well. For simplification purposes we thus consider only two-dimensional domains

$\Omega \in \mathbb{R}^2$ having polygonal boundary. The first main result in this chapter is the error estimates for unconstrained problems that are sharp as we will see in the numerical experiments in Section 5.4. We moreover observe that the control exhibits a similar behavior like the optimal control of a Dirichlet control problem with $L^2(\Gamma)$-regularization, meaning that as a general rule, the control is drawn to zero at convex corners and tends to infinity at reentrant corners. This motivates to consider problems involving control-constraints where the control is automatically active in a vicinity of reentrant corners, and is hence regular. Thus, as a second main result improved error estimates are derived in Section 5.3.

5.1 Analysis of the optimal control problem

In this section we discuss the continuous optimal control problem (5.1)–(5.3) in detail, investigate a possible realization of the regularization functional, and derive optimality conditions. Note that feasible controls are not necessarily functions and this is the reason why the control constraints (5.3) in the space $H^{-1/2}(\Gamma)$ are defined by duality, more precisely we say that the relation $u_a \le u \le u_b$ holds if and only if

$$\begin{cases} \langle u - u_b, v\rangle_\Gamma \le 0 \\ \langle u - u_a, v\rangle_\Gamma \ge 0 \end{cases} \qquad \forall v \in H^{1/2}(\Gamma) \text{ with } v \ge 0 \text{ a. e. on } \Gamma.$$

Analysis of the state equation

As the state equation (5.2) is linear it is possible to decompose its solution according to $y = y_u + y_f$ where y_u and y_f satisfy the equations

$$\begin{aligned} -\Delta y_u + y_u &= 0 & \qquad -\Delta y_f + y_f &= f & \qquad &\text{in } \Omega, \\ \partial_n y_u &= u & \partial_n y_f &= 0 & &\text{on } \Gamma. \end{aligned} \tag{5.4}$$

We consider the weak formulations of these equations using the bilinear form

$$a(y, v) = \int_\Omega (\nabla y(x) \cdot \nabla v(x) + y(x)v(x))\, \mathrm{d}x$$

corresponding to the operator $-\Delta + I$, which read

$$a(y_u, v) = \langle u, v\rangle_\Gamma \qquad \forall v \in H^1(\Omega), \tag{5.5}$$

$$a(y_f, v) = \langle f, v\rangle_\Omega \qquad \forall v \in H^1(\Omega). \tag{5.6}$$

Moreover, we define the *control-to-state* mapping by

$$S\colon H^{-1/2}(\Gamma) \to L^2(\Omega), \qquad u \mapsto Su := y_u.$$

With the Lax-Milgram Theorem we easily verify that the images of S belong even to $H^1(\Omega)$, and that S is bounded in the sense $\|Su\|_{H^1(\Omega)} \le c\|u\|_{H^{-1/2}(\Gamma)}$.

Optimality conditions

The first question which arises is, how the regularization term can be realized. Therefore, the inverse Steklov-Poincaré operator defined by

$$\mathcal{N}\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad u \mapsto \mathcal{N}u := y_u|_\Gamma \tag{5.7}$$

is taken into account. Using this operator we obtain several equivalent formulations for a norm in $H^{-1/2}(\Gamma)$ by

$$\begin{aligned}\|y_u\|^2_{H^1(\Omega)} &= \int_\Omega \nabla y_u(x) \cdot \nabla y_u(x)\,\mathrm{d}x + \int_\Omega y_u(x)^2\,\mathrm{d}x \\ &= \int_\Gamma u(x)\,y_u(x)\,\mathrm{d}s_x = \langle u, \mathcal{N}u\rangle_\Gamma =: \|\!|u|\!\|^2_{H^{-1/2}(\Gamma)}.\end{aligned} \tag{5.8}$$

One can show that this definition of the $H^{-1/2}(\Gamma)$-norm is equivalent to the definition we used in (2.5). From the Lax-Milgram Lemma we immediately get

$$\|\!|u|\!\|_{H^{-1/2}(\Gamma)} = \|y_u\|_{H^1(\Omega)} \le c\|u\|_{H^{-1/2}(\Gamma)},$$

and from the definition (2.5) as well as the harmonic extension $B\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ which is known to be continuous, i. e. $\|Bv\|_{H^1(\Omega)} \le c\|v\|_{H^{1/2}(\Gamma)}$ for all $v \in H^{1/2}(\Gamma)$, we get

$$\|u\|_{H^{-1/2}(\Gamma)} = \sup_{\substack{v\in H^{1/2}(\Gamma)\\ v\not\equiv 0}} \frac{\langle u, v\rangle_\Gamma}{\|v\|_{H^{1/2}(\Gamma)}} = \sup_{\substack{v\in H^{1/2}(\Gamma)\\ v\not\equiv 0}} \frac{a(y_u, Bv)}{\|v\|_{H^{1/2}(\Gamma)}} \le c\|y_u\|_{H^1(\Omega)} = c\|\!|u|\!\|_{H^{-1/2}(\Gamma)},$$

where c is the norm of the extension operator. In the following we will not distinguish between the two equivalent definitions of norms in $H^{-1/2}(\Gamma)$ and will use $\|\cdot\|_{H^{-1/2}(\Gamma)}$ only.

Taking the norm representation (5.8) and the substitution $y = Su + y_f$ into account leads to an equivalent formulation of the optimal control problem (5.1)–(5.3) in the reduced form, namely

$$\min_{u\in U_{ad}} j(u) := \frac{1}{2}\|Su + y_f - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\,\langle u, \mathcal{N}u\rangle_\Gamma\,. \tag{5.9}$$

In order to derive optimality conditions, we discuss the differentiability properties of the regularization term.

Lemma 5.1.1. *The operator $\mathcal{N} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is linear, continuous and self-adjoint. Moreover, the functional*

$$R\colon H^{-1/2}(\Gamma) \to \mathbb{R}, \qquad R(u) := \frac{1}{2}\,\langle u, \mathcal{N}u\rangle_\Gamma$$

is Fréchet differentiable with derivative

$$\left[R'(u)\right] h = \langle h, \mathcal{N}u\rangle_\Gamma \qquad \text{for all } h \in H^{-1/2}(\Gamma).$$

Proof. The linearity and continuity follow from the structure of the underlying boundary value problem. In order to show the self-adjointness we introduce functions $y_1, y_2 \in H^1(\Omega)$ which solve in the weak sense

$$\begin{aligned} -\Delta y_1 + y_1 &= 0 & \qquad -\Delta y_2 + y_2 &= 0 & \qquad &\text{in } \Omega,\\ \partial_n y_1 &= u_1 & \partial_n y_2 &= u_2 & &\text{on } \Gamma,\end{aligned}$$

with arbitrary $u_1, u_2 \in H^{-1/2}(\Gamma)$. Exploiting the weak formulations of both boundary value problems leads to

$$\langle u_1, \mathcal{N} u_2\rangle_\Gamma = \langle u_1, y_2|_\Gamma\rangle_\Gamma = a(y_1, y_2) = a(y_2, y_1) = \langle u_2, y_1|_\Gamma\rangle_\Gamma = \langle u_2, \mathcal{N} u_1\rangle_\Gamma .$$

The properties of $\mathcal{N}$ then imply the differentiability of R which is a consequence of

$$R(u + v) = R(u) + \langle v, \mathcal{N} u\rangle + \frac{1}{2} \langle v, \mathcal{N} v\rangle_\Gamma$$

and the norm definition (5.8) which yields $\langle v, \mathcal{N} v\rangle_\Gamma \sim \|v\|^2_{H^{-1/2}(\Gamma)}$. □

From this lemma we deduce that the target functional of problem (5.9) is Fréchet-differentiable and with standard techniques we can derive the necessary optimality condition

$$\left\langle u - \bar{u}, j'(\bar{u})\right\rangle_\Gamma = (S\bar{u} + y_f - y_d, S(u - \bar{u})) + \alpha \langle u - \bar{u}, \mathcal{N}\bar{u}\rangle_\Gamma \geq 0 \quad \forall u \in U_{ad}. \tag{5.10}$$

Let us summarize the linear and constant part of the optimality condition by introducing the operator $T^\alpha : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and the element $g \in H^{1/2}(\Gamma)$ defined by

$$T^\alpha := S^*S + \alpha\mathcal{N}, \qquad g := S^*(y_f - y_d). \tag{5.11}$$

Here, $S^*\colon L^2(\Omega) \to H^{1/2}(\Gamma)$ is the operator defined by $S^*v := [Pv]|_\Gamma$ where $P\colon L^2(\Omega) \to H^1(\Omega)$ is the solution operator of the adjoint equation, i. e.

$$w = Pv \quad :\Longleftrightarrow \quad a(z, w) = (v, z) \qquad \forall z \in H^1(\Omega).$$

That S^* is the adjoint operator of S can be confirmed by

$$(Su, v) = a(Su, Pv) = \langle u, Pv\rangle_\Gamma = \langle u, S^*v\rangle_\Gamma \qquad \forall u \in H^{-1/2}(\Gamma),\ v \in L^2(\Omega). \tag{5.12}$$

Using the definition of T^α and g the optimality condition can be written in a compact form as

$$\text{Find } \bar{u} \in U_{ad}\colon \qquad \langle u - \bar{u}, T^\alpha\bar{u}\rangle_\Gamma \geq \langle u - \bar{u}, g\rangle_\Gamma \qquad \forall u \in U_{ad}. \tag{5.13}$$

The operator T^α possesses the following properties:

Lemma 5.1.2. *The bilinear form defined by $\langle \cdot, T^\alpha \cdot\rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ is continuous and $H^{-1/2}(\Gamma)$-elliptic, i. e. some constants $M, \gamma > 0$ exist such that*

$$\langle u, T^\alpha v\rangle_\Gamma \leq M\|u\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)},$$
$$\langle u, T^\alpha u\rangle_\Gamma \geq \gamma\|u\|^2_{H^{-1/2}(\Gamma)}$$

hold for all $u, v \in H^{-1/2}(\Gamma)$.

Proof. The continuity follows directly from the definition of T^α, the continuity of $\mathcal{N}$ and S, and the norm equivalence $\|Sv\|_{H^1(\Omega)} \sim \|v\|_{H^{-1/2}(\Gamma)}$ stated in (5.8). This implies

$$\begin{aligned}\langle u, T^\alpha v\rangle_\Gamma &= (Su, Sv) + \alpha \langle u, \mathcal{N} v\rangle_\Gamma \\ &\leq c\left(\|Su\|_{H^1(\Omega)}\|Sv\|_{H^1(\Omega)} + \|u\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)}\right) \\ &\leq c\|u\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

To show the $H^{-1/2}(\Gamma)$-ellipticity we express the $H^{-1/2}(\Gamma)$-norm by the representation (5.8) which leads to

$$\begin{aligned}\langle u, T^\alpha u\rangle_\Gamma &= (Su, Su) + \alpha\, \langle u, \mathcal{N} u\rangle_\Gamma \\ &= \|y_u\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{H^{-1/2}(\Gamma)} \geq \alpha \|u\|^2_{H^{-1/2}(\Gamma)}.\end{aligned}$$

□

Corollary 5.1.3. *The variational inequality (5.13) possesses a unique solution $\bar u \in H^{-1/2}(\Gamma)$.*

Proof. The assertion follows directly from Lemma 5.1.2 and a standard theorem on existence and uniqueness of a solution for variational inequalities, e. g. from [96, Proposition 54.1]. □

In order to find a representation of the optimality condition (5.10) which does not involve the operators S and S^* explicitly, we introduce the adjoint state $p := P(Su + y_f - y_d) \in H^1(\Omega)$. Due to the representation $p|_\Gamma = S^*(y - y_d)$ the optimality condition (5.10) can be written in the following form.

Theorem 5.1.4. *The tuple $(\bar y, \bar u) \in H^1(\Omega) \times U_{ad}$ solves the problem (5.1)–(5.3), if and only if some $\bar p \in H^1(\Omega)$ exists with*

$$\begin{cases} a(\bar y_u, v) - \langle \bar u, v\rangle_\Gamma = 0 & \forall v \in H^1(\Omega), \\ a(v, \bar p) - (\bar y_u, v) = (y_f - y_d, v) & \forall v \in H^1(\Omega), \\ \langle u - \bar u, \alpha \bar y_u + \bar p\rangle_\Gamma \geq 0 & \forall u \in U_{ad}. \end{cases} \tag{5.14}$$

Note, that we already used the decomposition $\bar y = \bar y_u + y_f$ and $\mathcal{N}\bar u = \bar y_u|_\Gamma$.

5.2 Error estimates for the unconstrained problem

In this section we neglect the control constraints and set $U_{ad} := H^{-1/2}(\Gamma)$. This simplifies the proof of error estimates significantly. Estimates with additional control constraints are considered in Section 5.3.

As the variational inequality in the optimality system (5.14) becomes a variational problem when no constraints are present we conclude that

$$\bar y_u(x) = -\frac{1}{\alpha} \bar p(x) \qquad \text{for a. a. } x \in \Gamma.$$

Consequently, there exists some $y_0 \in H^1_0(\Omega)$ such that the state can be decomposed into

$$\bar y_u(x) = y_0(x) - \frac{1}{\alpha}\bar p(x) \qquad \text{for a. a. } x \in \Omega. \tag{5.15}$$

Inserting this decomposition into the state equation (5.4) then yields the differential equation

$$\begin{aligned} -\Delta y_0 + y_0 &= \frac{1}{\alpha}(-\Delta \bar p + \bar p) = \frac{1}{\alpha}(\bar y_u + y_f - y_d) && \text{in } \Omega, \\ y_0 &= 0 && \text{on } \Gamma. \end{aligned} \tag{5.16}$$

Regularity of the solution

Before we are in the position to derive error estimates for the numerical approximation of the optimality system (5.14) we collect some regularity results of its solution. In the following lemma we use results in classical Sobolev spaces.

Lemma 5.2.1. *Let $f \in L^q(\Omega)$, $y_d \in L^{\max\{2,q\}}(\Omega)$ for some q satisfying*

$$q \in \begin{cases} [1, 2/(2-\lambda)), & \text{if } \lambda < 2, \\ [1, \infty), & \text{otherwise,} \end{cases} \tag{5.17}$$

and

$$2 - 2/q \neq \lambda_{j,m} := m\pi/\omega_j \qquad \forall j \in \mathcal{C}, m \in \mathbb{N}. \tag{5.18}$$

Then, the solution of the optimality system (5.14) possesses the regularity

$$\bar{u} \in W^{1-1/q,q}(\Gamma^{(j)}), \ \forall j \in \mathcal{C}, \quad \bar{y}_u, y_f \in W^{2,q}(\Omega), \quad \bar{p} \in W^{2,q}(\Omega).$$

Proof. From Theorem 5.1.3 it is known that a unique optimal control $\bar{u} \in H^{-1/2}(\Gamma)$ exists. Then, the Lax-Milgram lemma guarantees the existence of a unique state $\bar{y} \in H^1(\Omega)$ and hence $\bar{y} \in L^q(\Omega)$ for arbitrary $q \in [1, \infty)$. Under the assumption that q satisfies (5.17) and (5.18) Theorem 2.2.2 implies $\bar{p} \in W^{2,q}(\Omega)$. With the decomposition (5.15) this regularity is transferred to the state variable $\bar{y}_u$, as standard results imply $y_0 \in W^{2,q}(\Omega)$, because the right-hand side of (5.16) belongs to $L^q(\Omega)$. Moreover, $y_f \in W^{2,q}(\Omega)$ follows in case of $f \in L^q(\Omega)$ and we thus have $\bar{y} = \bar{y}_u + y_f \in W^{2,q}(\Omega)$. With a standard trace theorem we obtain that $\bar{u} \in W^{1-1/q,q}(\Gamma^{(j)})$ for all $j \in \mathcal{C}$. □

In the numerical experiments we observe that the control exhibits a similar behavior like the optimal control of a Dirichlet control problem with $L^2(\Gamma)$-regularization (see e. g. [63, 71]). More precisely, the control is drawn down to zero at convex corners and tends to ∞ or $-\infty$ at concave corners. In the following we will study this behavior in detail. Let (r, φ) denote polar coordinates centered at some corner $x^{(j)}$ and let B be a neighborhood of $x^{(j)}$ containing no other corners. Since $\bar{p}$ is the solution of a Neumann problem it admits a decomposition as in Theorem 2.2.2, namely

$$\bar{p}(x) = p_R(x) + cr^\lambda \cos(\lambda\varphi), \qquad \text{for } x \in B, \qquad \lambda = \frac{\pi}{\omega_j},$$

with a regular part p_R in $H^2(B)$. Note that we omitted the cut-off function η as introduced in (2.12) which is possible due to local considerations. Further singular terms with exponents $\lambda_k := k\pi/\omega_j$ for $k \geq 2$ are neglected since the corresponding singular functions belong to $H^2(B)$. Due to the homogeneous Neumann conditions we have

$$0 = \partial_n \bar{p} = \partial_n p_R \pm c\lambda r^{\lambda-1} \sin(\lambda\varphi). \tag{5.19}$$

Since $\sin(\lambda\varphi) = 0$ for $\varphi = 0$ and $\varphi = \omega_j$ we have $\partial_n p_R = 0$. As the auxiliary function y_0 introduced in (5.16) is the solution of a Dirichlet problem it can be decomposed into a regular part $y_{0,R} \in W^{2,q}(B)$ with $q \in (2, (1-\lambda)^{-1})$ if $\lambda < 1$ or $q \in (2, \infty)$ if $\lambda \geq 1$, and a singular part

$$y_0(x) = y_{0,R}(x) + cr^\lambda \sin(\lambda\varphi), \qquad x \in B.$$

Exploiting this decomposition and $\bar{u} = \partial_n \bar{y}_u = \partial_n y_0$ we obtain by some calculations

$$\bar{u}|_{\varphi=0} = \partial_n y_{0,R} - c\lambda\, r^{\lambda-1}\cos(0),$$
$$\bar{u}|_{\varphi=\omega_j} = \partial_n y_{0,R} + c\lambda\, r^{\lambda-1}\cos(\pi),$$

and consequently,

$$\bar{u}(x) = \partial_n y_{0,R}(x) - c\lambda r^{\lambda-1}, \qquad x \in \partial B \cap \Gamma. \tag{5.20}$$

As we are able to choose $q > 2$ the regular part of the state is differentiable due to the embedding $y_{0,R} \in W^{2,q} \hookrightarrow C^1(B)$, and hence, the normal derivative is piecewise continuous, i. e. $\partial_n y_{0,R} \in C(\Gamma^{(j)} \cap B)$ for $j \in \mathcal{C}$. Due to $y_{0,R} \equiv 0$ on Γ the tangential derivatives on the boundary vanish and since the normal vector in a corner can be represented as linear combination of the tangential vectors, this implies that

$$\lim_{r\to 0} \partial_n y_{0,R}(r,\varphi) = 0 \qquad \text{for } \varphi \in \{0, \omega_j\}.$$

However, the term $\lambda r^{\lambda-1}$ in (5.20) could either grow unboundedly or could tend to zero, which depends on λ. If $x^{(j)}$ is a reentrant corner we have $\lambda < 1$ and in case of a convex corner $\lambda > 1$. Consequently, there holds

$$\lim_{r\to 0} \bar{u}(r,\varphi) \to \begin{cases} 0, & \text{if } \omega_j < \pi, \\ \pm\infty, & \text{if } \omega_j > \pi, \end{cases} \qquad \text{for } \varphi \in \{0, \omega_j\}.$$

Note that in case of $\omega_j > \pi$ the control tends either to $+\infty$ on both legs, or to $-\infty$, but the case that it tends to $+\infty$ on the one leg and to $-\infty$ on the other one can never occur.

Let us collect some regularity results in weighted Sobolev spaces.

Theorem 5.2.2. *Let $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Moreover, denote by $\varepsilon > 0$ a sufficiently small real number, and let $\vec{\alpha}, \vec{\beta}, \vec{\gamma} \in \mathbb{R}^d$ be weight vectors defined by*

$$\alpha_j := \max\{0, 1 - \lambda_j + \varepsilon\},$$
$$\beta_j := \max\{0, 2 - \lambda_j + \varepsilon\},$$
$$\gamma_j := \max\{0, 3/2 - \lambda_j + \varepsilon\},$$

for all $j \in \mathcal{C}$. Then, the solution $(\bar{y}_u, \bar{u}, \bar{p})$ of (5.14) with $U_{ad} = H^{-1/2}(\Gamma)$ and y_f from (5.6) satisfy

$$\bar{y}_u, \bar{p} \in W^{2,2}_{\vec{\alpha}}(\Omega) \cap W^{2,\infty}_{\vec{\beta}}(\Omega) \cap W^{2,2}_{\vec{\gamma}}(\Gamma),$$
$$y_f \in W^{2,2}_{\vec{\alpha}}(\Omega),$$
$$\bar{u} \in W^{1,2}_{\vec{\gamma}}(\Gamma).$$

Proof. Due $\bar{y}, y_d \in L^2(\Omega)$ Theorem 2.3.5 implies that the adjoint state belongs to $W^{2,2}_{\vec{\alpha}}(\Omega)$ as $\vec{\alpha}$ satisfies its assumptions by construction. The $W^{2,\infty}_{\vec{\beta}}(\Omega)$-regularity of $\bar{p}$ follows from Theorem 2.3.6 and Lemma 5.2.1 using the fact that $\bar{y} \in W^{2,q}(\Omega) \hookrightarrow C^{0,\sigma'}(\bar{\Omega})$ which holds for some $\sigma' \in (0, 1/2)$ as $q \in (4/3, 2/(2-\lambda)) \neq \emptyset$.

The regularity of $\bar{p}$ can be transferred to $\bar{y}_u$ with similar arguments used already in the proof of Lemma 5.2.1. We may apply the regularity results from [54] and [11, Theorem 2.2] and obtain

$$y_0 \in V^{2,2}_{\vec{\alpha}}(\Omega) \cap V^{2,\infty}_{\vec{\beta}}(\Omega),$$

where

$$V^{2,q}_{\vec{\delta}}(\Omega) := \left\{ v \in \mathcal{D}(\Omega) \colon \sum_{|\boldsymbol{\alpha}| \le 2} \| r^{\delta_j - 2 + |\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} v \|_{L^q(\Omega)} < \infty \right\}.$$

However, as $2 - |\boldsymbol{\alpha}| \ge 0$, there holds the embedding

$$V^{2,q}_{\vec{\delta}}(\Omega) \hookrightarrow W^{2,q}_{\vec{\delta}}(\Omega), \qquad \forall q \in [1, \infty],\ \vec{\delta} \in \mathbb{R}^d,$$

and taking the decomposition (5.15) into account implies

$$\bar{y}_u \in W^{2,2}_{\vec{\alpha}}(\Omega) \cap W^{2,\infty}_{\vec{\beta}}(\Omega).$$

Moreover, the assumption $f \in L^2(\Omega)$ and Theorem 2.3.5 imply the desired regularity of y_f.

Furthermore, we have $\nabla \bar{y}_u \in \left(W^{1,\infty}_{\vec{\beta}}(\Omega) \right)^2$ and thus $\bar{u} \in W^{1,\infty}_{\vec{\beta}}(\Gamma)$. Finally, from the embedding stated in Lemma 2.3.3 we conclude

$$\bar{y}_u, \bar{p} \in W^{2,\infty}_{\vec{\beta}}(\Omega) \hookrightarrow W^{2,2}_{\vec{\gamma}}(\Gamma), \qquad \bar{u} \in W^{1,\infty}_{\vec{\beta}}(\Gamma) \hookrightarrow W^{1,2}_{\vec{\gamma}}(\Gamma).$$

Note that $\varepsilon > 0$ can be chosen arbitrarily but sufficiently small, which is required to get the validity of the embeddings. □

Discretization and general convergence results

In the following we construct a conforming finite element discretization of the optimality system (5.14) from Theorem 5.1.4. Let us introduce some notation. A family of admissible and quasi-uniform finite element triangulations $\mathcal{T}_h$ with mesh size $h > 0$ is considered. We approximate the state and adjoint state variable with continuous and piecewise linear functions, i. e. we seek y_h and p_h in the finite-dimensional subspace

$$V_h := \left\{ v_h \in C(\bar{\Omega}) \colon v_h \text{ is affine linear on all } T \in \mathcal{T}_h \right\} \tag{5.21}$$

with $\dim(V_h) = N_\Omega$. We further seek an approximation of the control u_h in the finite-dimensional space $U_h \subset H^{-1/2}(\Gamma)$ with $\dim(U_h) = N_\Gamma$. Since multiple choices of U_h are possible, we want to keep the analysis here as general as possible and investigate certain choices later in detail. The discretized optimality system of Theorem 5.1.4 reads now

$$\left\{ \begin{aligned} a(y_{u,h}, v_h) - \langle u_h, v_h \rangle_\Gamma &= 0 && \forall v_h \in V_h, \\ a(v_h, p_h) - (y_{u,h}, v_h) &= (y_{f,h} - y_d, v_h) && \forall v_h \in V_h, \\ \langle w_h, \alpha y_{u,h} + p_h \rangle_\Gamma &= 0 && \forall w_h \in U_h, \end{aligned} \right. \tag{5.22}$$

where $y_{f,h}$ can be computed in advance from

$$a(y_{f,h}, v_h) = \langle f, v_h \rangle_\Omega \qquad \forall v_h \in V_h.$$

Solving the system (5.22) leads to the approximate solution $(\bar{y}_{u,h}, \bar{u}_h, \bar{p}_h)$ and we obtain the discrete state variable from

$$\bar{y}_h := \bar{y}_{u,h} + y_{f,h}.$$

Existence and uniqueness of a discrete solution is discussed later as additional assumptions upon the choice of U_h are necessary.

We introduce the finite-element solution operators S_h and P_h, defined by

$$\begin{aligned} S_h : H^{-1/2}(\Gamma) \to V_h \hookrightarrow L^2(\Omega), \quad & y_h = S_h u \quad :\Longleftrightarrow \quad a(y_h, v_h) = \langle u, v_h \rangle_\Gamma \quad \forall v_h \in V_h, \\ P_h : L^2(\Omega) \to V_h \hookrightarrow H^1(\Omega), \quad & p_h = P_h y \quad :\Longleftrightarrow \quad a(v_h, p_h) = (y, v_h) \quad \forall v_h \in V_h. \end{aligned}$$

The adjoint operator to S_h possesses the representation $S_h^* y := [P_h y]|_\Gamma$ which follows from the arguments used in (5.12). The discrete *Steklov-Poincaré* operator can be written in terms of $\mathcal{N}_h u = [S_h u]|_\Gamma$. Similarly to (5.10) we may write the system (5.22) in the compact form

$$0 = \langle w_h, T_h^\alpha u_h + g_h \rangle_\Gamma \qquad \forall w_h \in U_h \tag{5.23}$$

with

$$T_h^\alpha := S_h^* S_h + \alpha \mathcal{N}_h \quad \text{and} \quad g_h := S_h^*(y_{f,h} - y_d).$$

The operator T_h^α is an approximation of the operator T^α defined in (5.11). As the properties of T^α presented in Lemma 5.1.2 cannot be directly transferred to T_h^α we require an additional assumption.

Assumption 3. *The spaces U_h and V_h satisfy the* Ladyshenskaya-Babuška-Brezzi *condition, i. e. some $c > 0$ exists such that*

$$\|u_h\|_{H^{-1/2}(\Gamma)} \le c \sup_{v_h \in V_h} \frac{\langle u_h, v_h \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \quad \text{for all } u_h \in U_h.$$

This is a natural assumption for mixed finite element discretizations. As a consequence the discrete counterpart to Lemma 5.1.2 follows:

Lemma 5.2.3. *Let Assumption 3 be satisfied. Then, the bilinear form $\langle \cdot, T_h^\alpha \cdot \rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ is continuous and U_h-elliptic.*

Proof. The continuity can be proven in analogy to Lemma 5.1.2 since the stability properties of S and S^* also hold for their discrete versions. In order to show the U_h-ellipticity we take into account Assumption 3 which leads to

$$\|u_h\|_{H^{-1/2}(\Gamma)} \le c \sup_{v_h \in V_h} \frac{\langle u_h, v_h \rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} = c \sup_{v_h \in V_h} \frac{a(S_h u_h, v_h)}{\|v_h\|_{H^1(\Omega)}} \le c \|S_h u_h\|_{H^1(\Omega)}.$$

The definitions of S_h and $\mathcal{N}_h$ then imply

$$\|u_h\|^2_{H^{-1/2}(\Gamma)} \le c \|S_h u_h\|^2_{H^1(\Omega)} \le c a(S_h u_h, S_h u_h) = c \langle u_h, \mathcal{N}_h u_h \rangle_\Gamma \le c \langle u_h, T_h^\alpha u_h \rangle_\Gamma .$$

The last step follows from

$$\langle u_h, S_h^* S_h u_h \rangle_\Gamma = (S_h u_h, S_h u_h) \ge 0.$$

□

Theorem 5.2.4. *Let Assumption 3 be satisfied. Then, the discrete optimality system (5.22) possesses a unique solution $(\bar{y}_{u,h}, \bar{u}_h, \bar{p}_h) \in V_h \times U_h \times V_h$, and the tuple $(\bar{y}_{u,h}, \bar{u}_h)$ is the unique solution of the problem*

$$J_h(y_{u,h}, u_h) := \frac{1}{2}\|y_{u,h} + y_{f,h} - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\langle u_h, \mathcal{N}_h u_h\rangle \to \min! \tag{5.24}$$

subject to $(y_{u,h}, u_h) \in V_h \times U_h$ and

$$a(y_{u,h}, v_h) = \langle u_h, v_h\rangle_\Gamma \qquad \forall v_h \in V_h. \tag{5.25}$$

Proof. The existence of a unique optimal control $\bar{u}_h$ follows from the Lax-Milgram Lemma and the properties of T_h^α stated in Lemma 5.2.3. The corresponding state $\bar{y}_{u,h} := S_h\bar{u}_h$ and adjoint state $\bar{p}_h = P_h(\bar{y}_{u,h} + y_{f,h} - y_d)$ are also unique since S_h and P_h are well-defined. Showing that (5.22) forms an optimality system for the optimization problem (5.24)–(5.25) is easy since the arguments we used to derive the continuous optimality system (5.14) can be repeated. □

In the next theorem we derive a general error estimate for the solution of the discrete optimality system (5.14).

Theorem 5.2.5. *Let the Assumption 3 be satisfied. For the solutions $\bar{u} \in H^{-1/2}(\Gamma)$ and $\bar{u}_h \in U_h$ of (5.10) and (5.23), respectively, the error estimate*

$$\begin{aligned}\|\bar{u} - \bar{u}_h\|_{H^{-1/2}(\Gamma)} \le c\Big(&\|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|(S^* - S_h^*)(\bar{y} - y_d)\|_{H^{1/2}(\Gamma)} \\ &+\alpha\|(\mathcal{N} - \mathcal{N}_h)\bar{u}\|_{H^{1/2}(\Gamma)} + \|y_f - y_{f,h}\|_{L^2(\Omega)} + \inf_{\chi\in U_h}\|\bar{u} - \chi\|_{H^{-1/2}(\Gamma)}\Big)\end{aligned} \tag{5.26}$$

holds.

Proof. We introduce the auxiliary function $\tilde{u}_h \in U_h$ as the unique solution of

$$\langle v_h, T^\alpha\tilde{u}_h + g\rangle_\Gamma = 0 \qquad \forall v_h \in U_h. \tag{5.27}$$

Due to $U_h \subset H^{-1/2}(\Gamma)$ we get with (5.13) the orthogonality equation

$$\langle v_h, T^\alpha(\bar{u} - \tilde{u}_h)\rangle_\Gamma = 0 \qquad \forall v_h \in U_h. \tag{5.28}$$

As a consequence of the $H^{-1/2}(\Gamma)$-ellipticity and boundedness of $\langle\cdot, T^\alpha\cdot\rangle_\Gamma$ (see Lemma 5.1.2) and equation (5.28), the Céa-Lemma leads to

$$\|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)} \le c\inf_{\chi\in U_h}\|\bar{u} - \chi\|_{H^{-1/2}(\Gamma)}. \tag{5.29}$$

Next, an estimate for $w_h := \tilde{u}_h - \bar{u}_h$ is derived. We may now apply the U_h-ellipticity of T_h^α as well as (5.23), (5.27) and (5.13) which leads to

$$\begin{aligned}\|w_h\|^2_{H^{-1/2}(\Gamma)} &\le \langle w_h, T_h^\alpha(\tilde{u}_h - \bar{u}_h)\rangle_\Gamma = \langle w_h, T_h^\alpha\tilde{u}_h + g_h\rangle_\Gamma \\ &= \langle w_h, (T_h^\alpha - T^\alpha)\,\tilde{u}_h - g + g_h\rangle_\Gamma \\ &= \langle w_h, (T_h^\alpha - T^\alpha)\,(\tilde{u}_h - \bar{u})\rangle_\Gamma + \langle w_h, (T_h^\alpha - T^\alpha)\,\bar{u} - g + g_h\rangle_\Gamma\,.\end{aligned} \tag{5.30}$$

The boundedness of T^α and T_h^α together with (5.29) imply

$$\langle w_h, (T_h^\alpha - T^\alpha)\,(\tilde{u}_h - \bar{u})\rangle_\Gamma \le c\,\|w_h\|_{H^{-1/2}(\Gamma)} \inf_{\chi\in U_h} \|\bar{u}-\chi\|_{H^{-1/2}(\Gamma)}. \tag{5.31}$$

Exploiting the definition of T^α and T_h^α yields for the second term in (5.30)

$$\begin{aligned}
&\langle w_h, (T_h^\alpha - T^\alpha)\,\bar{u} + g - g_h\rangle_\Gamma \\
&= \langle w_h, S_h^*(S_h - S)\bar{u} + (S_h^* - S^*)S\bar{u} + \alpha(\mathcal{N}_h - \mathcal{N})\bar{u} + (S_h^* - S^*)(y_f - y_d) + S_h^*(y_{f,h} - y_f)\rangle_\Gamma \\
&\le c\|w_h\|_{H^{-1/2}(\Gamma)}\Big(\|S_h^*(S_h - S)\bar{u}\|_{H^{1/2}(\Gamma)} + \|(S_h^* - S^*)(\bar{y} - y_d)\|_{H^{1/2}(\Gamma)} \\
&\qquad + \alpha\|(\mathcal{N}_h - \mathcal{N})\bar{u}\|_{H^{1/2}(\Gamma)} + \|S_h^*(y_{f,h} - y_f)\|_{H^{1/2}(\Gamma)}\Big).
\end{aligned} \tag{5.32}$$

Note, that the operator S_h^* is bounded from $L^2(\Omega)$ to $H^{1/2}(\Gamma)$, and thus

$$\begin{aligned}
&\|S_h^*(S_h - S)\bar{u}\|_{H^{1/2}(\Gamma)} + \|S_h^*(y_{f,h} - y_f)\|_{H^{1/2}(\Gamma)} \\
&\le c\left(\|(S_h - S)\bar{u}\|_{L^2(\Omega)} + \|y_{f,h} - y_f\|_{L^2(\Omega)}\right).
\end{aligned} \tag{5.33}$$

Inserting the estimates (5.31), (5.32) and (5.33) into (5.30) and dividing by $\|w_h\|_{H^{-1/2}(\Gamma)}$ yields

$$\begin{aligned}
\|w_h\|_{H^{-1/2}(\Gamma)} \le c\Big(&\|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|(S^* - S_h^*)(\bar{y} - y_d)\|_{H^{1/2}(\Gamma)} \\
&+ \alpha\|(\mathcal{N} - \mathcal{N}_h)\bar{u}\|_{H^{1/2}(\Gamma)} + \inf_{\chi\in U_h}\|\bar{u} - \chi\|_{H^{-1/2}(\Gamma)} + \|y_f - y_{f,h}\|_{L^2(\Omega)}\Big).
\end{aligned}$$

This estimate, as well as (5.29), and the triangle inequality

$$\|\bar{u} - \bar{u}_h\|_{H^{-1/2}(\Gamma)} \le \|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)} + \|\tilde{u}_h - \bar{u}_h\|_{H^{-1/2}(\Gamma)}$$

imply the assertion. □

Actually we have already derived estimates for almost all terms on the right-hand side in Chapter 3 as these terms depend only on the finite element error of the state and adjoint equation. Only the last term on the right-hand side of (5.26) has to be treated. Note that this is the only term which depends upon the choice of the discrete control space U_h.

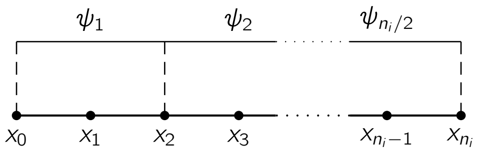
Approximation and error estimates for the control variable

We can now construct some finite-dimensional spaces for the control which satisfy Assumption 3. This is mandatory because otherwise, no unique solution of the variational problem (5.23) exists. The choice of approximating the control with piecewise constant functions on the boundary mesh $\mathcal{E}_h$ of $\mathcal{T}_h$ is known to be not inf-sup stable, and as a consequence, numerically computed solutions exhibit oscillations. An overview over possible pairs V_h and U_h which satisfy Assumption 3 can e.g. be found in [94, Section 1.2]. In the following we discuss three possible choices in detail.

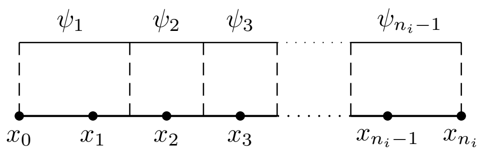
First, we consider an approximation by piecewise constant functions, which is indeed possible but on another boundary mesh $\mathcal{E}_H$ having maximal element diameter $H > 0$. More precisely, we set $U_h = U_h^0$ defined by

$$U_h^0 := \{v_h \in L^\infty(\Gamma)\colon v_{h|E} \in \mathcal{P}_0 \quad \forall E \in \mathcal{E}_H\}.$$

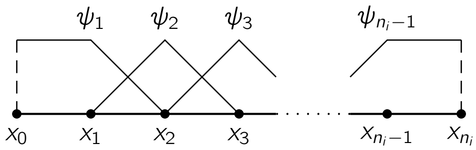
However, further assumptions are necessary to obtain the validity of Assumption 3. Our analysis covers the following two choices:



(a) Piecewise constant on a coarse mesh



(b) Piecewise constant on the dual mesh



(c) Piecewise linear and constant at corners

Figure 5.1: Possible choices for the discretization of the control.

a) The boundary mesh $\mathcal{E}_H$ is assumed to be coarser than the boundary mesh of $\mathcal{T}_h$, i. e. there holds $H/h \geq \gamma$ with some sufficiently large $\gamma > 0$. A proof of Assumption 3 for this choice can be found in [85, Section 11.3]. Certainly, it is not known how large γ has to be, but in the numerical experiments we observed, that $\gamma = 2$ is sufficient in our case. This setting occurs for instance when we refine a given initial mesh $k-1$ times globally to obtain $\mathcal{E}_H$, and refine uniformly once more to obtain $\mathcal{T}_h$ (see Figure 5.1a).

b) We can also choose $\mathcal{E}_H$ as the dual mesh of the boundary mesh induced by $\mathcal{T}_h$. Therefore, assume that the boundary edge $\Gamma^{(i)}$, $i \in \mathcal{C}$, coincides with the x-axis and that the $n_i + 1$, $n_i \in \mathbb{N}$, boundary nodes of $\mathcal{T}_h$ on $\Gamma^{(i)}$ are numerated such that

$$x^{(i)} = x_0 < x_1 < \ldots < x_{n_i} = x^{(i+1)}.$$

Then, the elements of the dual mesh $\{E_k\}_{k=1}^{n_i-1}$ lying on $\Gamma^{(i)}$ are defined by

$$E_k := \operatorname{conv}\left\{\frac{1}{2}(x_{k-1}+x_k), \frac{1}{2}(x_k+x_{k+1})\right\}, \quad \text{for } k = 2, \ldots, n_i-2,$$
$$E_1 := \operatorname{conv}\left\{x_0, \frac{1}{2}(x_1+x_2)\right\}, \quad E_{n_i-1} := \operatorname{conv}\left\{\frac{1}{2}(x_{n_i-2}+x_{n_i-1}), x_{n_i}\right\},$$

which is also illustrated in Figure 5.1b. A proof of Assumption 3 for this choice can be found in [94, Section 1.2]. Due to $H \sim h$ we do not distinguish between h and H in the following.

We prove a best-approximation property of these spaces in order to get an estimate for the last term in (5.26).

Lemma 5.2.6. *Let some function $u \in W^{1,2}_{\vec{\gamma}}(\Gamma)$ be given, where $\vec{\gamma}$ is defined as in Theorem 5.2.2, and denote by $P_h^\partial \colon L^2(\Gamma) \to U_h^0$ the $L^2(\Gamma)$-projection onto the space U_h^0. Then, the error estimate*

$$\|u - P_h^\partial u\|_{H^{-s}(\Gamma)} \leq c h^{\min\{1, \lambda - 1/2 - \varepsilon\} + s} |u|_{W^{1,2}_{\vec{\gamma}}(\Gamma)}$$

holds for $s \in \{0, 1/2\}$ and sufficiently small $\varepsilon > 0$.

Proof. With the definition of negative norms by duality and the standard estimate

$$\|\varphi - P_h^\partial \varphi\|_{L^2(\Gamma)} \le ch^{1/2}\|\varphi\|_{H^{1/2}(\Gamma)}$$

we obtain

$$\begin{aligned}\|u - P_h^\partial u\|_{H^{-1/2}(\Gamma)} &= c \sup_{\varphi\in H^{1/2}(\Gamma)} \left\langle u - P_h^\partial u, \varphi\right\rangle_\Gamma / \|\varphi\|_{H^{1/2}(\Gamma)} \\ &= c \sup_{\varphi\in H^{1/2}(\Gamma)} (u - P_h^\partial u, \varphi - P_h^\partial \varphi)/\|\varphi\|_{H^{1/2}(\Gamma)} \\ &\le ch^{1/2}\, \|u - P_h^\partial u\|_{L^2(\Gamma)}. \end{aligned} \tag{5.34}$$

Note, that the global projection coincides with the local projection on each boundary element $E \in \mathcal{E}_h$ and it remains to insert the local estimates from Lemma 3.2.8. For all $E \in \mathcal{E}_h$ with $r_E = 0$ we get the estimate

$$\|u - P_h^\partial u\|_{L^2(E)} \le ch^{1-\gamma_j}|u|_{W^{1,2}_{\gamma_j}(E)}, \qquad \text{for } E \subset U_j.$$

In case of $r_E > 0$ we arrive at the same estimate exploiting $r_E^{-\gamma_j} \le ch^{-\gamma_j}$ which follows from $\gamma_j \ge 0$. Summation over all $E \in \mathcal{E}_h$ yields

$$\|u - P_h^\partial u\|_{L^2(\Gamma)} \le ch^{1-\max_j \gamma_j}|u|_{W^{1,2}_{\vec{\gamma}}(\Gamma)}.$$

Inserting now the definition of $\vec{\gamma}$ yields the assertion for $s = 0$, and together with (5.34) we conclude the assertion for $s = 1/2$. □

Another possibility is to approximate the control with specific piecewise linear functions which are continuous on each boundary edge $\Gamma^{(i)}$, $i \in \mathcal{C}$. We denote by $\mathcal{E}_h$ the boundary mesh induced by $\mathcal{T}_h$. Hence, the boundary edge $\Gamma^{(i)}$ is decomposed into boundary elements

$$E_k := \operatorname{conv}\{x_{k-1}, x_k\} \in \mathcal{E}_h, \qquad k = 1, \ldots, n_i,$$

where

$$x^{(i)} = x_0 < x_1 < \ldots < x_{n_i} = x^{(i+1)}$$

are again the boundary nodes on $\Gamma^{(i)}$ for some $i \in \mathcal{C}$. We introduce the spaces

$$U_h^1(\Gamma^{(i)}) := \{v_h \in C(\Gamma^{(i)}) \colon v_h|_{E_k} \in \mathcal{P}_1,\ k = 2, \ldots, n_i - 1,\ \text{and } v_h|_{E_1}, v_h|_{E_{n_i}} \in \mathcal{P}_0\},$$

for all $i \in \mathcal{C}$, and define the discrete control space by

$$U_h^1 := \{v_h \in L^\infty(\Gamma) \colon v_h|_{\Gamma^{(i)}} \in U_h^1(\Gamma^{(i)}) \text{ for all } i \in \mathcal{C}\}. \tag{5.35}$$

Note, that we allow discontinuities at corner points, and demand that the slope of functions in U_h^1 is zero on edges touching a corner. This property is necessary to ensure the stability condition in Assumption 3. A proof can be found in [19], we refer also to [75]. In the following $\{\psi_j \colon j = 1, \ldots, n_i - 1\}$ is the nodal basis of $U_h^1(\Gamma^{(i)})$, $i \in \mathcal{C}$, i. e.

$$\psi_j(x_k) = \delta_{j,k} \qquad \forall j, k = 1, \ldots, n_i - 1,$$

which is illustrated in Figure 5.1c.

We can show a best-approximation property of U_h^1 similar to the one from Lemma 5.2.6.

Lemma 5.2.7. *Let some $u \in W^{1,2}_{\vec{\gamma}}(\Gamma)$ be given with $\vec{\gamma}$ defined as in Theorem 5.2.2, and denote by $P_h^\partial \colon L^2(\Gamma) \to U_h^1$ the $L^2(\Gamma)$-projection onto the space U_h^1. Then, the error estimate*

$$\|u - P_h^\partial u\|_{H^{-s}(\Gamma)} \leq c h^{\min\{1, \lambda - 1/2 - \varepsilon\} + s} |u|_{W^{1,2}_{\vec{\gamma}}(\Gamma)}$$

holds for $s \in \{0, 1/2\}$ and sufficiently small $\varepsilon > 0$.

Proof. Analogous to the proof of Lemma 5.2.6 we can show that

$$\|u - P_h^\partial u\|_{H^{-1/2}(\Gamma)} \leq c \sup_{\varphi \in H^{1/2}(\Gamma)} \|u - P_h^\partial u\|_{L^2(\Gamma)} \|\varphi - P_h^\partial \varphi\|_{L^2(\Gamma)} / \|\varphi\|_{H^{1/2}(\Gamma)}. \tag{5.36}$$

Since P_h^∂ is the best-approximation in $L^2(\Gamma)$ we replace P_h^∂ by an appropriate local interpolation operator onto $U_h^1(\Gamma^{(i)})$, $i \in \mathcal{C}$. Therefore, we introduce the operator $C_h^\partial \colon L^1(\Gamma^{(i)}) \to U_h^1(\Gamma^{(i)})$ defined by

$$[C_h^\partial u](x) := \sum_{k=1}^{n_i - 1} [\Pi_{\sigma_k} v](x_k) \psi_k(x), \quad \sigma_k := E_k \text{ or } E_{k+1}, \quad \forall x \in \Gamma^{(i)},$$

where Π_{σ_k} denotes the L^2-projection onto the constant functions on σ_k. This quasi-interpolation operator is similar to the operator introduced by Clément [28] and has the advantage that the stability property

$$\|C_h^\partial u\|_{L^2(E_k)} \leq c \|u\|_{L^2(S_{E_k})}, \quad \text{with} \quad S_{E_k} := \operatorname{int}(\overline{E}_k \cup \overline{\sigma}_{k-1} \cup \overline{\sigma}_k), \tag{5.37}$$

holds for all $k = 1, \ldots, n_i$, which would not hold for the usual Lagrange interpolation operator. For arbitrary $p \in \mathcal{P}_0(S_{E_k})$ we have the property $p = C_h^\partial p$. Using the triangle inequality and (5.37) we get

$$\|\varphi - C_h^\partial u\|_{L^2(E_k)} \leq c \|\varphi - p\|_{L^2(S_{E_k})} \leq c h^s |\varphi|_{H^s(S_{E_k})}, \qquad s \in (0, 1], \tag{5.38}$$

for all $k = 1, \ldots, n_i$, where the last step follows from Theorem 4.2 (for $s = 1$) and Proposition 6.1 (for $s \in (0, 1)$) in [38]. An estimate in weighted Sobolev spaces can be deduced from (3.40) and we get

$$\|u - C_h^\partial u\|_{L^2(E_k)} \leq c \|u - p\|_{L^2(S_{E_k})} \leq c h^{1 - \gamma_j} |u|_{W^{1,2}_{\gamma_j}(S_{E_k})}. \tag{5.39}$$

From (5.38) for $s = 1/2$ and (5.39) we conclude for all $i \in \mathcal{C}$ the global estimates

$$\|u - C_h^\partial u\|_{L^2(\Gamma^{(i)})} \leq c h^{1 - \max_{j \in \mathcal{C}} \gamma_j} |u|_{W^{1,2}_{\vec{\gamma}}(\Gamma^{(i)})},$$
$$\|\varphi - C_h^\partial \varphi\|_{L^2(\Gamma^{(i)})} \leq c h^{1/2} |\varphi|_{H^{1/2}(\Gamma^{(i)})},$$

where the first estimate yields the assertion for $s = 0$ after summation over all $i \in \mathcal{C}$ together with the definition of $\vec{\gamma}$. The assertion for $s = -1/2$ follows after insertion into (5.36). □

Error estimates for the optimal control problem on quasi-uniform meshes

Now we are in the position to formulate a convergence result for the discrete solution of the optimal control problem. Inserting the finite element error estimates from Section 3.3 and the estimate for the $L^2(\Gamma)$-projection from Lemma 5.2.6 in case of a piecewise constant control approximation, or Lemma 5.2.7 in case of continuous and piecewise linear controls, into the estimate of Theorem 5.2.5 yields an error estimate for the control approximation in the $H^{-1/2}(\Gamma)$-norm. It is also possible to prove a convergence result in other norms as well as for the state variable.

Theorem 5.2.8. *Let $\{\mathcal{T}_h\}_{h>0}$ be a family of quasi-uniform meshes. Assume that the input data satisfy $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Let $(\bar y, \bar u, \bar p) \in L^2(\Omega) \times H^{-1/2}(\Gamma) \times L^2(\Omega)$ and $(\bar y_h, \bar u_h, \bar p_h) \in V_h \times U_h^k \times V_h$, $k = 0, 1$, denote the solutions of (5.14) and (5.22), respectively. Then, the* a priori *error estimates*

$$\|\bar u - \bar u_h\|_{H^{-1/2}(\Gamma)} + h^{1/2}\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le c h^{\min\{3/2,\lambda-\varepsilon\}} |\ln h|^{3/2} \eta, \tag{5.40}$$

$$\|\bar y - \bar y_h\|_{H^1(\Omega)} \le c \max\{h, h^{\lambda-\varepsilon} |\ln h|^{3/2}\}\eta, \tag{5.41}$$

hold, where

$$\begin{aligned}\eta := {} & |\bar u|_{W^{1,2}_{\vec\gamma}(\Gamma)} + |\bar y_u|_{W^{2,2}_{\vec\gamma}(\Gamma)} + \|\bar y_u\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|\bar y_u\|_{W^{2,\infty}_{\vec\beta}(\Omega)} + \|y_f\|_{W^{2,2}_{\vec\alpha}(\Omega)} \\ & + |\bar p|_{W^{2,2}_{\vec\gamma}(\Gamma)} + \|\bar p\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|\bar p\|_{W^{2,\infty}_{\vec\beta}(\Omega)},\end{aligned}$$

with the weight vectors $\vec\alpha, \vec\beta, \vec\gamma \in \mathbb{R}^d$ defined in Theorem 5.2.2 and $\varepsilon > 0$ chosen sufficiently small.

Proof. To obtain the desired estimate for the control in $H^{-1/2}(\Gamma)$ we insert the estimates for the best-approximation in U_h^k from Lemma 5.2.6 if $k = 0$, or Lemma 5.2.7 if $k = 1$, as well as the finite-element error estimates from Theorems 3.3.1 and 3.3.3, into the estimate from Theorem 5.2.5. The regularity which is necessary for these estimates has been discussed in Theorem 5.2.2.

Next, we show the error estimate in $L^2(\Gamma)$. We will need the inverse estimate

$$\|w_h\|_{L^2(\Gamma^{(i)})} \le c h^{-1/2} \|w_h\|_{H^{-1/2}(\Gamma^{(i)})} \qquad \forall w_h \in U_h^k,\ k = 0, 1,\ i \in \mathcal{C} \tag{5.42}$$

which is e.g. proven in [85, Lemma 10.10] for $k = 0$, but the arguments used in the proof can be repeated also for $k = 1$. We introduce the $L^2(\Gamma)$-projection onto U_h^k as intermediate function, apply the triangle inequality and the inverse estimate (5.42), and get

$$\begin{aligned}\|\bar u - \bar u_h\|_{L^2(\Gamma)} &\le \|\bar u - P_h^\partial \bar u\|_{L^2(\Gamma)} + \|P_h^\partial \bar u - \bar u_h\|_{L^2(\Gamma)} \\ &\le c\left(\|\bar u - P_h^\partial \bar u\|_{L^2(\Gamma)} + h^{-1/2}\left(\|\bar u - P_h^\partial \bar u\|_{H^{-1/2}(\Gamma)} + \|\bar u - \bar u_h\|_{H^{-1/2}(\Gamma)}\right)\right).\end{aligned} \tag{5.43}$$

Furthermore, we apply Lemma 5.2.6 (for piecewise constant controls) or Lemma 5.2.7 (for piecewise linear controls that are continuous on each $\Gamma^{(i)}$) which leads to

$$\|\bar u - P_h^\partial \bar u\|_{L^2(\Gamma)} + h^{-1/2}\|\bar u - P_h^\partial \bar u\|_{H^{-1/2}(\Gamma)} \le c h^{\min\{1,\lambda-1/2-\varepsilon\}} |\bar u|_{W^{1,2}_{\vec\gamma}(\Gamma)}. \tag{5.44}$$

Inserting now (5.44) together with (5.40) into (5.43) implies the estimate for the controls in $L^2(\Gamma)$.

The error estimate for the state variable follows from the triangle inequality

$$\begin{aligned}\|\bar y - \bar y_h\|_{H^1(\Omega)} &= \|S\bar u - S_h\bar u_h\|_{H^1(\Omega)} + \|y_f - y_{f,h}\|_{H^1(\Omega)}\\ &\le c\left(\|(S-S_h)\bar u\|_{H^1(\Omega)} + \|S_h(\bar u - \bar u_h)\|_{H^1(\Omega)}\right) + \|y_f - y_{f,h}\|_{H^1(\Omega)}\\ &\le c\left(h^{\min\{1,\lambda-\varepsilon\}}\left(\|\bar y_u\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|y_f\|_{W^{2,2}_{\vec\alpha}(\Omega)}\right) + h^{\min\{3/2,\lambda-\varepsilon\}}|\ln h|^{3/2}\eta\right),\end{aligned} \tag{5.45}$$

where we inserted the finite element error estimate stated in Theorem 3.3.1 for the first and third term, and the boundedness of S_h as operator from $H^{-1/2}(\Gamma)$ to $H^1(\Omega)$ as well as the estimate derived already for the control for the second term. □

Remark 5.2.9. *As already observed for optimal control problems in $L^2(\Gamma)$ discussed in Chapter 4, we have also in this case an optimal convergence rate for the controls provided that $\lambda_j \ge 3/2$ for all $j \in \mathcal{C}$. This is the case when the interior angles of all corner points in the domain are less than $120°$. If one angle is larger than $120°$ we have to apply local mesh refinement to restore the optimal convergence rate.*

Error estimates for the optimal control problem on refined meshes

In the following we will derive improved error estimates taking local mesh refinement into account. The investigations are based again on Theorem 5.2.5. The terms on the right-hand side of (5.26) depending on the finite element error of a single boundary value problem have already been discussed in Section 3.4. Hence, we merely have to show a best-approximation property of the discrete control space.

Throughout this section we prove the results only for the space U_h^0. The proof of the following lemma cannot be directly extended to discrete control space U_h^1 introduced on page 111 as the related $L^2(\Gamma)$-projection is not defined locally. Using the locally defined Clément interpolant instead as we did in the proof of Lemma 5.2.7 would also lead to a suboptimal result because the interpolation error does not provide the orthogonality property which is required to show optimal estimates in negative norms. As a remedy, one could use the Carstensen interpolant which provides the advantages both operators. Corresponding interpolation error estimates in negative norms on quasi-uniform meshes have been proven in the Appendix of [7], and an extension to refined meshes is possible. However, this is not considered in the present thesis and we merely discuss the choice $U_h = U_h^0$.

Lemma 5.2.10. *Let $u \in W^{1,2}_{\vec\gamma}(\Gamma)$ with the weight vector $\vec\gamma \in \mathbb{R}^d$ defined in Theorem 5.2.2, and denote by $P_h^\partial\colon L^2(\Gamma) \to U_h^0$ the $L^2(\Gamma)$-projection onto the space U_h^0. Assume that the mesh criterion (3.57) holds with refinement parameters $\mu_j \le 1 - 2\gamma_j/3$ for all $j \in \mathcal{C}$. Then, the estimate*

$$\|u - P_h^\partial u\|_{H^{-1/2}(\Gamma)} \le c h^{3/2} |u|_{W^{1,2}_{\vec\gamma}(\Gamma)}$$

holds.

Proof. To obtain a global estimate we exploit the definition of $\|\cdot\|_{H^{-1/2}(\Gamma)}$ and the orthogonality property of the $L^2(\Gamma)$-projection P_h^∂, which leads to

$$\begin{aligned}\|u - P_h^\partial u\|_{H^{-1/2}(\Gamma)} &= \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle u - P_h^\partial u, \varphi\rangle_\Gamma}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\ &\le c \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle u - P_h^\partial u, \varphi - P_h^\partial\varphi\rangle_\Gamma}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\ &\le c \sup_{\varphi\in H^{1/2}(\Gamma)} \|\varphi\|_{H^{1/2}(\Gamma)}^{-1} \sum_{E\in\mathcal{E}_h} \|u - P_h^\partial u\|_{L^2(E)} \|\varphi - P_h^\partial\varphi\|_{L^2(E)}. \end{aligned} \tag{5.46}$$

It remains to insert local error estimates. We take the standard estimate

$$\|\varphi - P_h^\partial\varphi\|_{L^2(E)} \le c h_E^{1/2} |\varphi|_{H^{1/2}(E)}$$

from [85, Theorem 10.1] and the local estimates in weighted spaces from Lemma 3.2.8 into account. In case of $r_E > 0$ we get

$$\|u - P_h^\partial u\|_{L^2(E)} \|\varphi - P_h^\partial\varphi\|_{L^2(E)} \le c h_E^{3/2} r_E^{-\gamma_j} |u|_{W^{1,2}_{\gamma_j}(E)} |\varphi|_{H^{1/2}(E)}, \qquad \forall E \subset U_j \cap \Gamma,$$

and using the mesh property $h_E \sim h r_E^{1-\mu_j}$ as well as the assumed condition $\mu_j \le 1 - 2\gamma_j/3$ we obtain

$$h_E^{3/2} r_E^{-\gamma_j} \le c h^{3/2} r_E^{3/2(1-\mu_j)-\gamma_j} \le c h^{3/2}.$$

In case of $r_E = 0$ we use instead the estimate

$$\|u - P_h^\partial u\|_{L^2(E)} \|\varphi - P_h^\partial\varphi\|_{L^2(E)} \le c h_E^{3/2-\gamma_j} |u|_{W^{1,2}_{\gamma_j}(E)} |\varphi|_{H^{1/2}(E)}, \qquad \forall E \subset U_j \cap \Gamma,$$

whereas we can show

$$h_E^{3/2-\gamma_j} \le c h^{(3/2-\gamma_j)/\mu_j} \le c h^{3/2}$$

exploiting $h_E \sim h^{1/\mu_j}$ and the property $\mu_j \le 1 - 2\gamma_j/3$ again. Inserting these local estimates into (5.46) yields together with the discrete Hölder inequality

$$\begin{aligned}\|u - P_h^\partial u\|_{H^{-1/2}(\Gamma)} &\le c h^{3/2} \sup_{\varphi\in H^{1/2}(\Gamma)} \|\varphi\|_{H^{1/2}(\Gamma)}^{-1} \sum_{E\in\mathcal{E}_h} |u|_{W^{1,2}_{\vec\gamma}(E)} |\varphi|_{H^{1/2}(E)} \\ &\le c h^{3/2} \sup_{\varphi\in H^{1/2}(\Gamma)} \|\varphi\|_{H^{1/2}(\Gamma)}^{-1} \left(\sum_{E\in\mathcal{E}_h} |u|^2_{W^{1,2}_{\vec\gamma}(E)}\right)^{1/2} \cdot \left(\sum_{E\in\mathcal{E}_h} |\varphi|^2_{H^{1/2}(E)}\right)^{1/2} \\ &\le c h^{3/2} |u|_{W^{1,2}_{\vec\gamma}(\Gamma)}. \end{aligned}$$

□

As a consequence we obtain an error estimate for the solution of the discrete optimality system (5.22).

Theorem 5.2.11. *Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations that are refined locally according to (3.57) with refinement parameters satisfying*

$$\mu_j < 2\lambda_j/3 \qquad \forall j \in \mathcal{C}.$$

Assume that the input data satisfy $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Then, the solution $\bar u$ of (5.13) with $U_{ad} = H^{-1/2}(\Gamma)$, and $\bar u_h \in U_h^0$ of (5.23) satisfy the estimate

$$\|\bar u - \bar u_h\|_{H^{-1/2}(\Gamma)} \le ch^{3/2}|\ln h|^{3/2}\eta, \tag{5.47}$$

where

$$\eta := |\bar u|_{W^{1,2}_{\vec\gamma}(\Gamma)} + \|f\|_{L^2(\Omega)} + |\bar y_u|_{W^{2,2}_{\vec\gamma}(\Gamma)} + \|\bar y_u\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|\bar y_u\|_{W^{2,\infty}_{\vec\beta}(\Omega)} + |\bar p|_{W^{2,2}_{\vec\gamma}(\Gamma)} + \|\bar p\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|\bar p\|_{W^{2,\infty}_{\vec\beta}(\Omega)},$$

with weights $\vec\alpha$, $\vec\beta$, $\vec\gamma \in \mathbb{R}^d$ as defined in Theorem 5.2.2 and $\varepsilon > 0$ chosen sufficiently small.

Proof. It suffices to estimate the terms on the right-hand side of (5.26) separately. We apply Theorem 3.4.5 and exploit the regularity of $\bar y_u$ and $\bar p$ stated in Theorem 5.2.2 which leads to

$$\|(S^* - S_h^*)(\bar y - y_d)\|_{H^{1/2}(\Gamma)} + \alpha\|(\mathcal{N} - \mathcal{N}_h)\bar u\|_{H^{1/2}(\Gamma)} \le ch^{3/2}|\ln h|^{3/2}\eta$$

One easily confirms that the assumption $\mu_j < 2\lambda_j/3$ together with the definition of $\vec\alpha, \vec\beta, \vec\gamma$ imply the assumptions of Theorem 3.4.5.

The error terms in $L^2(\Omega)$ of (5.26) can be estimated using Theorem 3.4.1 and the refinement condition $\mu_j < 2\lambda_j/3$ which obviously implies $\mu_j < \lambda_j$. Hence, we obtain

$$\|(S - S_h)\bar u\|_{L^2(\Omega)} + \|y_f - y_{f,h}\|_{L^2(\Omega)} \le ch^2\left(\|\bar y_u\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|y_f\|_{W^{2,2}_{\vec\alpha}(\Omega)}\right).$$

For the last term in (5.26) we insert the result of Lemma 5.2.10 and arrive at the assertion. □

Note that the refinement condition $\mu_j < 2\lambda_j/3$, $j \in \mathcal{C}$, guarantees only optimal convergence for the control in the $H^{-1/2}(\Gamma)$-norm, but not in $L^2(\Gamma)$. As we observed in numerical experiments this would require a much stronger refinement condition. However, deriving such a condition requires advanced techniques and will not be investigated in this thesis.

5.3 Error estimates for the constrained problem

Let us now investigate how the results of the foregoing sections change in case of additional control constraints. We consider the model problem (5.1)–(5.3) when control constraints are present. An optimality system has been presented in Theorem 5.1.4 which is equivalent to the variational inequality

$$\langle u - \bar u, T^\alpha \bar u + g\rangle_\Gamma \ge 0 \qquad \forall u \in U_{ad}. \tag{5.48}$$

Apparently, we can expect better regularity for the optimal control. For unconstrained problems we had e. g. $\lim_{x\to x^{(j)}} u = \pm\infty$ for concave corners $x^{(j)}$. This is certainly not possible when control constraints are present. Instead, the control is in general active in a vicinity of reentrant corners and is hence regular which allows us to improve error estimates.

In what follows we abbreviate by $\hat{\mathcal{C}}$ and $\check{\mathcal{C}}$ the index sets of concave and convex corners, respectively. As the error estimates we derive in this section depend on the singular exponent of the largest convex angle we introduce the quantity

$$\check{\lambda} := \min_{j\in\check{\mathcal{C}}} \lambda_j.$$

The active and inactive sets are denoted by

$$\begin{aligned} \mathcal{A}^+ &:= \{x\in\Gamma\colon (T^\alpha u+g)(x)<0\}\,, \\ \mathcal{A}^- &:= \{x\in\Gamma\colon (T^\alpha u+g)(x)>0\}\,, \qquad \mathcal{I} := \Gamma\backslash(\mathcal{A}^+\cup\mathcal{A}^-). \end{aligned} \tag{5.49}$$

Obviously, we cannot define active and inactive sets via the solution $\bar{u}$ as it is not measurable, but via the Lagrange multiplier which is in $H^{1/2}(\Gamma)$. First of all we need a structural assumption upon the active set which is in most cases satisfied.

Assumption 4. *Assume that the control bounds are strictly active in the vicinity of all reentrant corners $x^{(j)}$, $j\in\hat{\mathcal{C}}$, i. e. there exist some constants $R,\tau>0$ such that*

$$|(T^\alpha\bar{u}+g)(x)|>\tau \qquad \textit{for a. a. } x\in\Gamma\cap B_R(x^{(j)}).$$

Moreover, the number of transition points $x_T^{(j)}$, $j\in\mathcal{T}:=\{1,\ldots,d_T\}$, between active and inactive set is finite, and transition points can only occur in the interior of a boundary edge $\Gamma^{(i)}$, $i\in\mathcal{C}$.

For technical reasons we introduce balls around each corner or transition point, namely

$$\Omega_R^j := \{x\in\Omega\colon |x-x^{(j)}|<R\} \qquad j\in\mathcal{C}, \tag{5.50}$$

$$\tilde{\Omega}_R^k := \{x\in\Omega\colon |x-x_T^{(k)}|<R\} \qquad k\in\mathcal{T}, \tag{5.51}$$

$$\Omega_R^{reg} := \Omega\setminus\left(\bigcup_{j\in\mathcal{C}}\Omega_R^j\cup\bigcup_{k\in\mathcal{T}}\tilde{\Omega}_R^k\right), \tag{5.52}$$

and assume that $R>0$ is chosen sufficiently small such that Ω_{2R}^j contains only the corner $x^{(j)}$ and $\tilde{\Omega}_{2R}^k$ only the transition point $x_T^{(k)}$, but no other corners or transition points. The outer boundaries are denoted by

$$\Gamma_R^j := \partial\Omega_R^j\cap\Gamma, \qquad \tilde{\Gamma}_R^k := \partial\tilde{\Omega}_R^k\cap\Gamma, \qquad \Gamma_R^{reg} := \partial\Omega_R^{reg}\cap\Gamma$$

for $j\in\mathcal{C}$ and $k\in\mathcal{T}$.

Before deriving error estimates we will investigate in the following that the regularity is better in comparison to the unconstrained case as the control cannot tend to infinity at reentrant corners when control constraints are present. However, we also have to show that singularities occurring in the vicinity of the transition points are comparatively weak such that the convergence rate is not affected by these points. We show in the next Lemma that $\bar{u}$ will be indeed active on $\mathcal{A}^\pm$ and that the leading singularity at the transition points vanishes. The proof of the following Lemma is motivated by a similar observation for Dirichlet control problems in $H^{1/2}(\Gamma)$ [71].

Lemma 5.3.1. *Let Assumption 4 be satisfied and assume that $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Then, the solution of (5.14) satisfies*

$$y_f, \bar{y}_u, \bar{p} \in H^{\min\{2,1+\lambda-\varepsilon\}}(\Omega),$$
$$\bar{u} \in L^2(\Gamma),$$

for arbitrary $\varepsilon > 0$. Moreover, there holds $\bar{u} = u_a$ a. e. on $\mathcal{A}^-$ and $\bar{u} = u_b$ a. e. on $\mathcal{A}^+$.

Proof. The stated regularity of y_f follows directly from Theorem 2.2.2. Without loss of generality we assume that only upper control bounds are present, i. e. $u_a = -\infty$. We abbreviate the Lagrange multiplier by

$$\xi := T^\alpha \bar{u} + g = \alpha \bar{y}|_\Gamma + \bar{p}|_\Gamma \in H^{1/2}(\Gamma).$$

Note that due to the definition (5.49) there holds $\xi = 0$ a. e. on $\mathcal{I}$ and $\xi < 0$ a. e. on $\mathcal{A}^+$. The variational inequality (5.48) tested with $u = u_b$ yields $\langle u_b - \bar{u}, \xi\rangle_\Gamma \geq 0$. In contrast to this the control constraints and the fact that $\xi \leq 0$ a. e. on Γ lead to $\langle \bar{u} - u_b, \xi\rangle_\Gamma \geq 0$. This results in the complementarity condition

$$\bar{u} \leq u_b \text{ in } H^{-1/2}(\Gamma), \qquad \xi \leq 0, \text{ a. e. on } \Gamma, \qquad \langle \bar{u} - u_b, \xi\rangle_\Gamma = 0. \tag{5.53}$$

Taking into account that $\xi = 0$ on $\mathcal{I}$ and $\xi \leq 0$ on $\mathcal{A}^+$ leads to a boundary value problem with Signorini conditions on $\mathcal{A}^+$, namely

$$\begin{cases} -\Delta \bar{y}_u + \bar{y}_u = 0 & \text{in } \Omega, \\ \bar{y}_u = -\alpha^{-1}\bar{p} & \text{on } \mathcal{I}, \\ \bar{y}_u \leq -\alpha^{-1}\bar{p}, \quad \partial_n \bar{y}_u \leq u_b, \quad (\bar{y}_u + \alpha^{-1}\bar{p})(\partial_n \bar{y}_u - u_b) = 0 & \text{on } \mathcal{A}^+. \end{cases} \tag{5.54}$$

In the following we will use the decomposition

$$\bar{y}_u = \bar{y}_0 - \alpha^{-1}\bar{p}, \tag{5.55}$$

with some function $\bar{y}_0$ solving a problem with homogeneous Dirichlet conditions on $\mathcal{I}$, namely

$$\begin{cases} -\Delta \bar{y}_0 + \bar{y}_0 = \alpha^{-1}(\bar{y} - y_d) & \text{in } \Omega, \\ \bar{y}_0 = 0 & \text{on } \mathcal{I}, \\ \bar{y}_0 \leq 0, \quad \partial_n \bar{y}_0 \leq u_b, \quad \bar{y}_0(\partial_n \bar{y}_0 - u_b) = 0 & \text{on } \mathcal{A}^+. \end{cases} \tag{5.56}$$

With appropriate embeddings we one can show that $\bar{y} - y_d \in L^p(\Omega)$ for arbitrary $p \in [1,\infty)$ as $\bar{y} \in H^1(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$. The solution $\bar{y}_0$ can hence be decomposed into a regular part $y_R \in W^{2,p}(\Omega)$ and some singular parts that restrict the regularity in the vicinity of corners $x^{(j)}$, $j \in \mathcal{C}$, and transition points $x_T^{(k)}$, $k \in \mathcal{T}$. Away from these points we have higher regularity $\bar{y}_0 \in W^{2,p}(\Omega^{reg}_{R/2})$ which is stated e. g. in [45, Theorem 2.1.4], see also Section 2.7 in this reference.

In the vicinity of corners $x^{(j)}$, $j \in \mathcal{C}$, the leading singularities have the form

$$r^{\lambda_j}\cos(\lambda_j\varphi) \quad \text{and} \quad r^{\lambda_j}\sin(\lambda_j\varphi), \quad \lambda_j := \frac{\pi}{\omega_j},$$

where (r, φ) are polar coordinates centered in $x^{(j)}$ such that $\varphi = 0$ and $\varphi = \omega_j$ coincide with the edges $\Gamma^{(j)}$ and $\Gamma^{(j+1)}$. The second singular functions belong to $W^{2,p}(\Omega_R^j)$ when we choose

$$p \in [1, (1-\lambda_j)^{-1}) \quad \text{if} \quad \lambda_j < 1, \qquad p \in [1, \infty) \quad \text{if} \quad \lambda_j > 1.$$

If Γ_R^j belongs to the Dirichlet boundary $\mathcal{I}$ this can be deduced from [45, Theorem 2.4.3], and otherwise, if Signorini boundary conditions are present, the singular solutions are stated in [68]. Taking into account the regularity of the singular functions which can be computed analytically, and, that the choice $p = 2$ is possible for arbitrary corners we obtain $\bar{y}_0 \in H^{\min\{2, 1+\lambda_j - \varepsilon\}}(\Omega_R^j)$ for all $j \in \mathcal{C}$ and $\varepsilon > 0$.

Finally, we investigate the regularity in the vicinity of the transition points $x_T^{(k)}$, $k \in \mathcal{T}$. Let (r, φ) denote polar coordinates centered at $x_T^{(k)}$, and without loss of generality let $\varphi = 0$ belong to $\mathcal{I}$ and $\varphi = \pi$ to $\mathcal{A}^+$. From Theorem 2.4.3 in [45] it is known that the solution $\bar{y}_0$ admits the decomposition

$$\bar{y}_0(r, \varphi) = y_R(r, \varphi) + B r^\lambda \sin(\lambda \varphi)$$

with a certain coefficient $B \in \mathbb{R}$. Due to Assumption 4 transition points can only occur in the interior of a boundary edge, and hence, the exponent of the leading singularity is $\lambda = 1/2$. The regularity of y_R is restricted by the singularity corresponding to the exponent $3/2$ and hence, $y_R \in W^{2,p}(\tilde{\Omega}_R^k)$ for arbitrary $p \in [1, 4)$. We demonstrate in the following that the singular part must vanish as it fails to satisfy the Signorini boundary conditions. The normal derivative of $\bar{y}_0$ on the boundary part $\mathcal{I}$ can be computed using the chain rule and we obtain the representation

$$\partial_n \bar{y}_0(r, 0) = \partial_n y_R(r, 0) - B \lambda r^{\lambda - 1}.$$

As the solution has to fulfill the control constraints $\bar{u} = \partial_n \bar{y}_0 \le u_b$ there must hold $B \ge 0$ as $r^{\lambda-1}$ $(\lambda = 1/2)$ grows unboundedly towards infinity for $r \to 0$. Moreover, we have for $\varphi = \pi$ the inequality

$$\bar{y}_0(r, \pi) = y_R(r, \pi) + B r^\lambda \stackrel{!}{\le} 0.$$

With the choice $p \in (2, 4)$ we get from the trace theorem [44, Theorem 1.5.1.2] (note that the boundary part $\tilde{\Gamma}_R^k$ is smooth) and the Sobolev embedding theorem

$$y_R, \bar{p} \in W^{2,p}(\tilde{\Omega}_R^k) \hookrightarrow W^{2-1/p,p}(\tilde{\Gamma}_R^k) \hookrightarrow C^1(\tilde{\Gamma}_R^k). \tag{5.57}$$

Thus, we can perform a Taylor expansion of $y_R(r, \pi) + \alpha^{-1} p(r, \pi)$ in the point $r = 0$ with some intermediate point $s \in [0, r]$. Exploiting the fact that $y_R(0, \pi) = -\alpha^{-1} p(0, \pi)$ leads to

$$\begin{aligned}
& y_R(r, \pi) + \alpha^{-1} \bar{p}(r, \pi) + B r^\lambda \le 0 \\
\iff \quad & y_R(0, \pi) + \alpha^{-1} \bar{p}(0, \pi) + r \partial_r \left(y_R + \alpha^{-1} \bar{p}\right)(s, \pi) + B r^\lambda \le 0 \\
\iff \quad & \partial_r \left(y_R + \alpha^{-1} \bar{p}\right)(s, \pi) + B r^{\lambda - 1} \le 0.
\end{aligned}$$

The term $\partial_r \left(y_R + \alpha^{-1} \bar{p}\right)(s, \pi)$ is bounded as y_R and $\bar{p}$ are regular, see (5.57), and thus the inequality can hold in case of $B \le 0$ only. We already stated the condition $B \ge 0$ and consequently, the boundary conditions of problem (5.56) can only be satisfied in case of $B = 0$. The singular part corresponding to $\lambda = 1/2$ hence vanishes and thus

$$\tilde{B} r^{3/2} \sin\left(\frac{3}{2}\varphi\right), \qquad \tilde{B} \in \mathbb{R},$$

is in general the leading singularity and belongs to $H^2(\tilde\Omega_R^k)$.

We have shown that $\bar p, \bar y_0 \in H^{\min\{2,1+\lambda-\varepsilon\}}(\Omega)$ and conclude that $\bar y_u = \bar y_0 - \alpha^{-1}\bar p$ also belongs to this space. A trace theorem finally implies $\bar u \in L^2(\Gamma)$.

As $\bar u$ is a measurable function we can express the complementarity condition (5.53) by means of the inner product in $L^2(\Gamma)$ and find from

$$0 = (\bar u - u_b, \xi)_\Gamma = (\bar u - u_b, \xi)_{\mathcal{A}^+}$$

that $\bar u = u_b$ a. e. on $\mathcal{A}^+$ as $\xi < 0$ on $\mathcal{A}^+$. □

We are now in the position to formulate more accurate regularity results in weighted Sobolev spaces. As singularities can also occur in the vicinity of the transition points $x_T^{(k)}$, $k \in \mathcal{T}$, we additionally introduce the weighted Sobolev spaces $W_\beta^{\ell,q}(\tilde\Omega_R^k)$ with $\ell \in \mathbb{N}_0$, $q \in [1,\infty]$ and $\beta \in \mathbb{R}$, defined as the set of functions with finite norm

$$\|v\|_{W_\beta^{\ell,q}(\tilde\Omega_R^k)} := \begin{cases} \left(\sum_{|\alpha|\le\ell} \int_{\tilde\Omega_R^k} \rho_k(x)^{q\beta} |D^\alpha v(x)|^q \,\mathrm{d}x\right)^{1/q}, & \text{if } q \in [1,\infty), \\ \sum_{|\alpha|\le\ell} \operatorname*{ess\,sup}_{x\in\tilde\Omega_R^k} \rho_k(x)^\beta |D^\alpha v(x)|, & \text{if } q = \infty, \end{cases} \tag{5.58}$$

where $\rho_k(x) := |x - x_T^{(k)}|$. The trace space $W_\beta^{\ell-1/q,q}(\tilde\Gamma_R^k)$ is defined in analogy to (2.22). In order to simplify the notation we use the globally defined spaces $W_{\vec\alpha,\kappa}^{\ell,q}(\Omega)$ for $\ell \in \mathbb{N}_0$, $q \in [1,\infty]$, $\vec\alpha \in \mathbb{R}^d$, $\kappa \in \mathbb{R}$, equipped with the norm

$$\|v\|_{W_{\vec\alpha,\kappa}^{\ell,q}(\Omega)} := \left(\sum_{j\in\mathcal{C}} \|v\|^q_{W_{\alpha_j}^{\ell,q}(\Omega_R^j)} + \sum_{k\in\mathcal{T}} \|v\|^q_{W_\kappa^{\ell,q}(\tilde\Omega_R^k)} + \|v\|^q_{W^{\ell,q}(\Omega_{R/2}^{reg})}\right)^{1/q}$$

if $q < \infty$, and otherwise

$$\|v\|_{W_{\vec\alpha,\kappa}^{\ell,\infty}(\Omega)} := \max\left\{\max_{j\in\mathcal{C}} \|v\|_{W_{\alpha_j}^{\ell,\infty}(\Omega_R^j)}, \max_{k\in\mathcal{T}} \|v\|_{W_\kappa^{\ell,\infty}(\tilde\Omega_R^k)}, \|v\|_{W^{\ell,\infty}(\Omega_{R/2}^{reg})}\right\}.$$

In the usual way the trace spaces $W_{\vec\alpha,\kappa}^{\ell-1/q,q}(\Gamma)$ can be defined.

Lemma 5.3.2. *Let Assumption 4 be satisfied. Assume that $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline\Omega)$ with some $\sigma \in (0,1)$. The weight vectors $\vec\alpha, \vec\beta, \vec\gamma \in \mathbb{R}^d$ are defined as in Theorem 5.2.2. Moreover, $\delta > 0$ is an arbitrary number and $\vec{\tilde\gamma} \in \mathbb{R}^d$ is defined by*

$$\tilde\gamma_j = \begin{cases} 0, & \text{if } j \in \hat{\mathcal{C}}, \\ \gamma_j, & \text{if } j \in \check{\mathcal{C}}. \end{cases}$$

Then, the solutions $(\bar y_u, \bar u, \bar p)$ of (5.14) and y_f of (5.6) satisfy

$$\begin{aligned} y_f &\in W_{\vec\alpha}^{2,2}(\Omega), \\ \bar p &\in W_{\vec\alpha}^{2,2}(\Omega) \cap W_{\vec\beta}^{2,\infty}(\Omega) \cap W_{\vec\gamma}^{2,2}(\Gamma), \\ \bar y_u &\in W_{\vec\alpha}^{2,2}(\Omega) \cap W_{\vec\beta,1/2}^{2,\infty}(\Omega) \cap W_{\vec\gamma,\delta}^{2,2}(\Gamma), \\ \bar u &\in W_{\vec{\tilde\gamma},\delta}^{1,2}(\Gamma). \end{aligned}$$

Proof. The regularity of y_f follows directly from Theorem 2.3.5. The stated regularity of $\bar{p}$ follows from Theorems 2.3.5 and 2.3.6 as well as the embedding

$$W^{2,\infty}_{\vec{\beta}}(\Omega) \hookrightarrow W^{2,\infty}_{\vec{\beta}}(\Gamma) \hookrightarrow W^{2,2}_{\vec{\gamma}}(\Gamma) \tag{5.59}$$

stated in Lemma 2.3.3. Note that Theorem 2.3.6 requires Hölder continuity of the state variable which follows from Lemma 5.3.1 and the embedding

$$\bar{y}_u, y_f \in H^{\min\{2,1+\lambda-\varepsilon\}}(\Omega) \hookrightarrow C^{0,\sigma}(\overline{\Omega}), \qquad \sigma \in (0, \min\{1, \lambda - \varepsilon\}).$$

To transfer the shown regularity of $\bar{p}$ to $\bar{y}_u$ we use again the decomposition (5.55), and show only regularity results for the auxiliary function $\bar{y}_0$ solving the mixed problem

$$\begin{cases} -\Delta y_0 + y_0 = \alpha^{-1}(\bar{y} - y_d) & \text{in } \Omega, \\ y_0 = 0 & \text{on } \mathcal{I}, \\ \partial_n y_0 = u_a & \text{on } \mathcal{A}^-, \\ \partial_n y_0 = u_b & \text{on } \mathcal{A}^+. \end{cases} \tag{5.60}$$

Note that this boundary value problem follows from (5.56) taking into account that $\partial_n y_0 = \bar{u}$ on $\mathcal{A}^\pm$ and $\bar{u} = u_a$ on $\mathcal{A}^-$ as well as $\bar{u} = u_b$ on $\mathcal{A}^+$ which is stated in Lemma 5.3.1.

Taking into account that $\bar{y} - y_d \in C^{0,\sigma}(\overline{\Omega})$ and $u_a, u_b \in C^{1,\sigma}(\Gamma)$ we get from [44, Theorem 6.4.2.5] that the solution of (5.60) can be decomposed in a vicinity of the corner $x^{(j)}$, $j \in \mathcal{C}$, into

$$\bar{y}_0(r,\varphi) = y_R(r,\varphi) + \sum_{\substack{m\in\mathbb{N} \\ \lambda_{j,m}<2+\sigma}} c_m r^{\lambda_{j,m}} \Phi(\lambda_{j,m}\varphi),$$

with a regular part $y_R \in C^{2,\sigma}(\overline{\Omega})$, certain constants $c_m \in \mathbb{R}$, and $\Phi(\cdot) = \sin(\cdot)$ if $x^{(j)}$ belongs to the Dirichlet boundary $\mathcal{I}$, and $\Phi(\cdot) = \cos(\cdot)$ if $x^{(j)}$ belongs to the Neumann boundary $\mathcal{A}^\pm$.

The regular part belongs trivially to $W^{2,\infty}(\Omega^j_R)$ and the regularity of $\bar{y}_0$ is restricted only by the first singularity corresponding to the singular exponent $\lambda_{j,1} := \pi/\omega_j$. By a direct calculation and the embeddings (5.59) one easily confirms that

$$\bar{y}_0 \in W^{2,2}_{\alpha_j}(\Omega^j_R) \cap W^{2,\infty}_{\beta_j}(\Omega^j_R) \cap W^{2,2}_{\gamma_j}(\Gamma^j_R) \qquad \forall j \in \mathcal{C}. \tag{5.61}$$

Let us investigate the regularity of the state $\bar{y}_0$ in the vicinity of transition points $x^{(k)}_T$, $k \in \mathcal{T}$. By a slight abuse of notation (r,φ) are now polar coordinates centered in $x^{(k)}_T$ such that $\varphi = 0$ corresponds with that part of the edge belonging to $\mathcal{I}$ and $\varphi = \pi$ with the part belonging to $\mathcal{A}^+$. From the already shown regularity of $\bar{p}$ we conclude $\bar{p} \in W^{2,\infty}(\tilde{\Omega}^k_R)$ for all $k \in \mathcal{T}$ as the domains $\tilde{\Omega}^k_R$ exclude the corner points due to Assumption 4. In the vicinity of the transition point $x^{(k)}_T$ the solution of (5.60) possesses the decomposition

$$\bar{y}_0(r,\varphi) = y_R + c_1 r^{3/2}\sin(3\varphi/2) + c_2 r^{5/2}\sin(5\varphi/2), \qquad c_1, c_2 \in \mathbb{R}, \tag{5.62}$$

with a regular part $y_R \in C^{2,\sigma}(\overline{\Omega})$. Note that the singularities belonging to the exponents $7/2, 9/2, \ldots$ are regular enough and belong also to y_R. By a direct calculation we can show again that

$$\bar{y}_0 \in H^2(\tilde{\Omega}^k_R) \cap W^{2,\infty}_{1/2}(\tilde{\Omega}^k_R) \cap W^{2,2}_{\delta}(\tilde{\Gamma}^k_R) \qquad \forall k \in \mathcal{T}. \tag{5.63}$$

The regularity results (5.61), (5.63) and the fact that the singularities are of local nature lead to the desired result for $\bar{y}_u$ taking also the regularity of $\bar{p}$ and (5.55) into account. The stated regularity of $\bar{u}$ follows from embedding and trace theorems. In the vicinity of concave corners we obtain even higher regularity. Due to Assumption 4 and Lemma 5.3.1 we have $\bar{u} \equiv u_a$ or $\bar{u} \equiv u_b$ in the vicinity of concave corners, and hence $\bar{u} \in H^1(\Omega_R^j)$ for $j \in \hat{\mathcal{C}}$. Furthermore, by means of (5.62) the control variable can be decomposed in the vicinity of the transition points $x_T^{(k)}$, $k \in \mathcal{T}$, into

$$\bar{u} = u_R + u_S, \quad \text{with} \quad u_R := \partial_n(y_R + r^{5/2}\sin(5\varphi/2)) \in H^1(\Gamma), \quad u_S := \begin{cases} c_1 r^{1/2}, & \text{on } \mathcal{I}, \\ 0, & \text{on } \mathcal{A}^+. \end{cases}$$

A simple calculation moreover yields $u_S \in W_\delta^{1,2}(\tilde{\Gamma}_R^k)$ for arbitrary $\delta > 0$ which leads to the desired regularity result for $\bar{u}$. □

Analogous to the unconstrained case we discretize the optimality condition (5.48) and search a solution in the discrete spaces V_h and U_h. In this section we restrict our considerations to the case that $U_h = U_h^0$ introduced on page 109. The choice of piecewise linear controls $U_h = U_h^1$ considered on page 111 is in principal possible, but the proof of Lemma 5.3.4 cannot be extended to this choice.

The discretized optimality system reads:

Find $\bar{u}_h \in U_{h,ad} := U_h \cap U_{ad}$ and $\bar{y}_{u,h}, \bar{p}_h \in V_h$ such that

$$\begin{aligned} a(\bar{y}_{u,h}, v_h) - \langle \bar{u}_h, v_h\rangle_\Gamma &= 0 && \forall v_h \in V_h, \\ a(\bar{p}_h, v_h) - (\bar{y}_{u,h}, v_h) &= (y_{f,h} - y_d, v_h) && \forall v_h \in V_h, \\ \langle w_h - \bar{u}_h, \alpha \bar{y}_{u,h} + \bar{p}_h\rangle_\Gamma &\geq 0 && \forall w_h \in U_{h,ad}, \end{aligned} \tag{5.64}$$

where $y_{f,h} \in V_h$ can be computed from the equation

$$a(y_{f,h}, v_h) = \langle f, v_h\rangle_\Omega \qquad \forall v_h \in V_h$$

in advance. This system can be rewritten in a rather compact form as

$$\langle u_h - \bar{u}_h, T_h^\alpha \bar{u}_h + g\rangle_\Gamma \geq \qquad \text{for all } u_h \in U_{h,ad}. \tag{5.65}$$

Analogous to the proof of Theorem 5.2.5 we introduce the auxiliary function $\tilde{u}_h \in U_{h,ad}$ which solves

$$\langle u_h - \tilde{u}_h, T^\alpha \tilde{u}_h + g\rangle_\Gamma \geq 0 \qquad \text{for all } u_h \in U_{h,ad}. \tag{5.66}$$

Note that we only approximate the ansatz and test space, but not the operator T^α. We begin with an initial estimate for the discrete approximation of the control.

Lemma 5.3.3. *Let $\Gamma_0 \supseteq \{x \in \Gamma\colon \tilde{u}_h(x) \neq \bar{u}_h(x)\}$. Then the estimate*

$$\begin{aligned} \|\bar{u} - \bar{u}_h\|_{H^{-1/2}(\Gamma)} \leq c\Big(&\|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|(S^* - S_h^*)(y - y_d)\|_{H^{1/2}(\Gamma_0)} \\ &+ \alpha\|(\mathcal{N} - \mathcal{N}_h)\bar{u}\|_{H^{1/2}(\Gamma_0)} + \|y_f - y_{f,h}\|_{L^2(\Omega)} + \|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)}\Big) \end{aligned} \tag{5.67}$$

holds.

Proof. The arguments applied in the proof of Theorem 5.2.5 widely coincide with the control-constrained case and we just outline the differences. We apply again the triangle inequality and get

$$\|\bar{u} - \bar{u}_h\|_{H^{-1/2}(\Gamma)} \leq \|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)} + \|\tilde{u}_h - \bar{u}_h\|_{H^{-1/2}(\Gamma)}, \tag{5.68}$$

where $\tilde{u}_h$ is the solution of (5.66). One easily confirms that (5.30) with $w_h := \tilde{u}_h - \bar{u}_h$ also holds in the control-constrained case when we replace all "=" by "≤". This yields

$$\|w_h\|^2_{H^{-1/2}(\Gamma)} \leq \langle w_h, (T_h^\alpha - T^\alpha)\,(\tilde{u}_h - \bar{u})\rangle_\Gamma + \langle w_h, (T_h^\alpha - T^\alpha)\,\bar{u} - g + g_h\rangle_\Gamma\,. \tag{5.69}$$

The estimate (5.31) remains the same and we have

$$\langle w_h, (T_h^\alpha - T^\alpha)(\tilde{u}_h - \bar{u})\rangle_\Gamma \leq c\|w_h\|_{H^{-1/2}(\Gamma)}\|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)}. \tag{5.70}$$

Moreover, (5.32) becomes

$$\begin{aligned}&\langle w_h, (T_h^\alpha - T^\alpha)\,\bar{u} + g - g_h\rangle_\Gamma\\ &\leq c\|w_h\|_{H^{-1/2}(\Gamma)}\Big(\|S_h^*(S_h - S)\bar{u}\|_{H^{1/2}(\Gamma_0)} + \|(S_h^* - S^*)(y - y_d)\|_{H^{1/2}(\Gamma_0)}\\ &+ \|S_h^*(y_f - y_{f,h})\|_{H^{1/2}(\Gamma_0)} + \alpha\|(\mathcal{N}_h - \mathcal{N})\bar{u}\|_{H^{1/2}(\Gamma_0)}\Big).\end{aligned} \tag{5.71}$$

Dividing by $\|w_h\|_{H^{-1/2}(\Gamma)}$ and exploiting stability properties of S_h^* yields the assertion. □

Deriving error estimates for the term $\|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)}$ requires more effort in the control-constrained case than in the unconstrained case where we merely applied the Céa-Lemma (5.29) and inserted the best-approximation properties from Lemma 5.2.6.

Lemma 5.3.4. *Let Assumption 4 be satisfied and assume that $\bar{u} \in W^{1,2}_{\vec{\tilde{\gamma}},\varepsilon'}(\Gamma)$ with $\vec{\tilde{\gamma}} \in \mathbb{R}^d$ defined in Lemma 5.3.2 and $\varepsilon' \in (0,1)$. Then the error estimate*

$$\|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)} \leq c h^{\min\{3/2,\check{\lambda}\}-\varepsilon}|\bar{u}|_{W^{1,2}_{\vec{\tilde{\gamma}},\varepsilon'}(\Gamma)},$$

holds for sufficiently small $0 < \varepsilon' < \varepsilon$.

Proof. The results of Lemma 5.1.2 allow us to apply the Céa-type lemma from [46, Lemma 7.16] which reads

$$\begin{aligned}\frac{\alpha}{2}\|\bar{u} - \tilde{u}_h\|^2_{H^{-1/2}(\Gamma)} \leq &\inf_{v\in U_{ad}} \langle v - \tilde{u}_h, T^\alpha\bar{u} + g\rangle_\Gamma\\ &+ \inf_{v_h\in U_{h,ad}}\left\{\langle v_h - \bar{u}, T^\alpha\bar{u} + g\rangle_\Gamma + c\|\bar{u} - v_h\|^2_{H^{-1/2}(\Gamma)}\right\}.\end{aligned}$$

In the present situation the first term on the right-hand side vanishes for the choice $v := \tilde{u}_h$, which is possible since $\tilde{u}_h \in U_{ad}$. The second term vanishes if we choose

$$v_h \in \tilde{U}_{h,ad} := \left\{\bar{u}_h \in U_{h,ad} \colon \bar{u}_h = u_a \text{ on } \mathcal{A}^-,\ \bar{u}_h = u_b \text{ on } \mathcal{A}^+\right\},$$

since $v_h - \bar{u} \equiv 0$ on $\mathcal{A}^\pm$ and $T^\alpha\bar{u} + g \equiv 0$ on $\mathcal{I}$. We consequently get

$$\|\bar{u} - \tilde{u}_h\|_{H^{-1/2}(\Gamma)} \leq c \inf_{v_h\in\tilde{U}_{h,ad}} \|\bar{u} - v_h\|_{H^{-1/2}(\Gamma)}. \tag{5.72}$$

We insert the $L^2(\Gamma)$-projection onto U_h as intermediate function and obtain

$$\|\bar{u} - v_h\|_{H^{-1/2}(\Gamma)} \leq \|\bar{u} - P_h^\partial \bar{u}\|_{H^{-1/2}(\Gamma)} + \|P_h^\partial \bar{u} - v_h\|_{H^{-1/2}(\Gamma)}. \tag{5.73}$$

The first term also occurs for unconstrained problems and an estimate is given in Lemma 5.2.6. However, we can exploit that the term $\bar{u} - P_h^\partial \bar{u}$ vanishes in a neighborhood of concave corners and obtain the improved estimate

$$\|\bar{u} - P_h^\partial \bar{u}\|_{H^{-1/2}(\Gamma)} \leq c h^{\min\{3/2-\varepsilon', \check{\lambda}-\varepsilon\}} |\bar{u}|_{W^{1,2}_{\vec{\gamma},\varepsilon'}(\Gamma)}, \tag{5.74}$$

when exploiting the regularity stated in Theorem 5.3.2 . To derive an estimate for the second term in (5.73) we use the modified $L^2(\Gamma)$-projection

$$v_{h|E} = \begin{cases} [P_h^\partial \bar{u}]|_E, & \text{if } E \subset \mathcal{I}, \\ u_a, & \text{if } E \cap \mathcal{A}^- \neq \emptyset, \\ u_b, & \text{if } E \cap \mathcal{A}^+ \neq \emptyset. \end{cases} \tag{5.75}$$

The idea of using such a modification has already been used in [86] for the nodal interpolant and in [57] for the midpoint interpolant. Note that $v_h \in \tilde{U}_{h,ad}$ by construction, and that $P_h^\partial \bar{u} - v_h$ vanishes on all elements

$$E \notin \mathcal{K}_h := \{E \in \mathcal{E}_h \colon \overline{E} \cap \mathcal{A}^\pm \neq \emptyset \ \wedge \ \overline{E} \cap \mathcal{I} \neq \emptyset\}.$$

Due to Assumption 4 the set $\mathcal{K}_h$ contains a finite number of elements, independent of h. Exploiting the orthogonality property of the projection P_h^∂ we get

$$\begin{aligned} \|P_h^\partial \bar{u} - v_h\|_{H^{-1/2}(\Gamma)} &= \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} \sum_{E \in \mathcal{K}_h} (P_h^\partial \bar{u} - v_h, \varphi)_E \\ &= \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} \sum_{E \in \mathcal{K}_h} (P_h^\partial (\bar{u} - v_h), P_h^\partial \varphi)_E \\ &\leq \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} \sum_{E \in \mathcal{K}_h} \|\bar{u} - v_h\|_{L^2(E)} \|P_h^\partial \varphi\|_{L^2(E)}. \end{aligned} \tag{5.76}$$

Note that each $E \in \mathcal{K}_h$ is contained in some $\tilde{\Gamma}_R^k$, $k \in \mathcal{T}$, and hence, we can exploit the $W^{1,2}_{\varepsilon'}(\tilde{\Gamma}_R^k)$-regularity stated in Lemma 5.3.2. As v_h coincides with $\bar{u}$ at the endpoint of E which belongs to $\mathcal{A}^\pm$ we get with the Poincaré-Friedrichs inequality

$$\|\bar{u} - v_h\|_{L^2(E)} \leq c h^{1-\varepsilon'} |\bar{u}|_{W^{1,2}_{\varepsilon'}(E)}, \qquad \text{for } E \in \mathcal{K}_h,$$

for arbitrary $\varepsilon \in (0,1)$, where we exploited the regularity of $\bar{u}$ stated in Lemma 5.3.2. For the second term on the right-hand side of (5.76) we apply the Hölder inequality and stability properties of the projection P_h^∂ and get for arbitrary $q \in [2, \infty)$

$$\|P_h^\partial \varphi\|_{L^2(E)} \leq c h^{1/2-1/q} \|\varphi\|_{L^q(E)}.$$

Hence, (5.76) becomes

$$\begin{aligned}
&\|P_h^{\partial}\bar{u} - v_h\|_{H^{-1/2}(\Gamma)} \\
&\leq c \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} h^{3/2-\varepsilon'-1/q} \sum_{E\in\mathcal{K}_h} |\bar{u}|_{W^{1,2}_{\varepsilon'}(E)} \|\varphi\|_{L^q(E)} \\
&\leq c \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} h^{3/2-\varepsilon'-1/q} \left(\sum_{E\in\mathcal{K}_h} |\bar{u}|^2_{W^{1,2}_{\varepsilon'}(E)}\right)^{1/2} \left(\sum_{E\in\mathcal{K}_h} 1\right)^{1/2-1/q} \|\varphi\|_{L^q(\Gamma)} \\
&\leq ch^{3/2-\varepsilon} \sum_{k\in\mathcal{T}} |\bar{u}|_{W^{1,2}_{\varepsilon'}(\tilde{\Gamma}^k_R)},
\end{aligned}$$

where we exploited that the number of elements in $\mathcal{K}_h$ is independent of h, the embedding

$$\|\varphi\|_{L^q(\Gamma)} \leq c\|\varphi\|_{H^{1/2}(\Gamma)} = c,$$

and we chose q sufficiently large such that $\varepsilon = 1/q + \varepsilon'$. Inserting this together with (5.74) into (5.73) completes the proof. □

The control $\bar{u}$ is in general active in the vicinity of concave corners. In the following lemma we show that this property is transferred also to the discrete solution $\bar{u}_h$, and hence we get $\bar{u} - \bar{u}_h \equiv 0$ near these corners. This is the key idea for the improved error estimates that we will show in Theorem 5.3.6.

Lemma 5.3.5. *Let Assumption 4 be satisfied. Then, some constants $h_0 > 0$ and $R > 0$ exist such that*

$$\bar{u}_h(x) = u_b \quad \text{or} \quad \bar{u}_h(x) = u_a \qquad \text{for all } x \in \Gamma^j_R,\ j \in \hat{\mathcal{C}},$$

provided that $h \leq h_0$.

Proof. Without loss of generality we show the assertion for the case that the upper bound is strictly active, i.e. $T^\alpha\bar{u} + g < -\tau$ within Γ^j_R. The key step is to show pointwise convergence of $T^\alpha_h\bar{u}_h + g_h$ towards $T^\alpha\bar{u} + g$, i. e.

$$\|(T^\alpha\bar{u} + g) - (T^\alpha_h\bar{u}_h + g_h)\|_{L^\infty(\Gamma)} \xrightarrow{h\to 0} 0, \tag{5.77}$$

which then implies $T^\alpha_h\bar{u}_h + g_h < 0$ within Γ^j_R when $h \leq h_0$. By element-wise consideration of the discrete optimality condition (5.48) we conclude that $\bar{u}_h = u_b$ and have shown the assertion.

From the definition (5.11) of T^α and g as well as their discrete analogues (5.24) we get

$$\|(T^\alpha\bar{u} + g) - (T^\alpha_h\bar{u}_h + g_h)\|_{L^\infty(\Gamma)} = \|\alpha(\bar{y}_u - \bar{y}_{u,h}) + (\bar{p} - \bar{p}_h)\|_{L^\infty(\Gamma)}.$$

Let us first derive a pointwise estimate for the state variable. With the triangle inequality and a trace theorem we get

$$\|\bar{y}_u - \bar{y}_{u,h}\|_{L^\infty(\Gamma)} \leq \|\bar{y}_u - S_h\bar{u}\|_{L^\infty(\Omega)} + \|S_h(\bar{u} - \bar{u}_h)\|_{L^\infty(\Omega)}. \tag{5.78}$$

The first term tends to zero as $h \to 0$ due to the pointwise convergence of the finite element method, see e. g. [74, Lemma 3.41]. For the second term, we get with the stability of S_h from

$L^2(\Gamma)$ to $L^\infty(\Omega)$, the triangle inequality, and the inverse inequality from [85, Lemma 10.10] the estimate

$$\begin{aligned} \|S_h(\bar u-\bar u_h)\|_{L^\infty(\Omega)} &\le c\|\bar u-\bar u_h\|_{L^2(\Gamma)} \\ &\le c\|\bar u-P_h^\partial \bar u\|_{L^2(\Gamma)} + h^{-1/2}\left(\|\bar u-P_h^\partial \bar u\|_{H^{-1/2}(\Gamma)} + \|\bar u-\bar u_h\|_{H^{-1/2}(\Gamma)}\right). \end{aligned} \tag{5.79}$$

With the estimates for the $L^2(\Gamma)$-projection of Lemma 5.2.6 we immediately get

$$\|\bar u-P_h^\partial \bar u\|_{L^2(\Gamma)} + h^{-1/2}\|\bar u-P_h^\partial \bar u\|_{H^{-1/2}(\Gamma)} \to 0. \tag{5.80}$$

Moreover from the estimate of Lemma 5.3.3 for $\Gamma_0=\Gamma$ and the error estimates from Theorems 3.3.1, 3.3.3 and Lemma 5.3.4 which yield a rate higher than $h^{1/2}$ on arbitrary domains, we get

$$\begin{aligned} h^{-1/2}\|\bar u-\bar u_h\|_{H^{-1/2}(\Gamma)} \le ch^{-1/2}\Big(&\|(S-S_h)\bar u\|_{L^2(\Omega)} + \|(S^*-S_h^*)(\bar y-y_d)\|_{H^{1/2}(\Gamma)} \\ &+\alpha\|(\mathcal N-\mathcal N_h)\bar u\|_{H^{1/2}(\Gamma)} + \|y_f-y_{f,h}\|_{L^2(\Omega)} + \|\bar u-\tilde u_h\|_{H^{-1/2}(\Gamma)}\Big) \\ \to 0.& \end{aligned} \tag{5.81}$$

Consequently, we get with (5.78), (5.79), (5.80) and (5.81) the pointwise convergence of the state, i. e.

$$\|\bar y_u-\bar y_{u,h}\|_{L^\infty(\Gamma)} \to 0 \quad \text{as} \quad h\to 0. \tag{5.82}$$

It remains to show pointwise convergence of the discrete adjoint state. We use the representations $\bar p|_\Gamma = S^*(\bar y_u+y_f-y_d)$ and $\bar p_h|_\Gamma = S_h^*(\bar y_{u,h}+y_{f,h}-y_d)$, introduce several intermediate functions and get the equivalent formulation

$$\begin{aligned} \bar p|_\Gamma-\bar p_h|_\Gamma &= S^*(S\bar u+y_f-y_d) - S_h^*(S_h\bar u_h+y_{f,h}-y_d) \\ &= (S^*-S_h^*)(S\bar u+y_f-y_d) + S_h^*(S-S_h)\bar u \\ &\quad + S_h^*S_h(\bar u-\bar u_h) + S_h^*(y_f-y_{f,h}). \end{aligned} \tag{5.83}$$

One easily confirms that

$$\begin{aligned} \|(S^*-S_h^*)(S\bar u+y_f-y_d)\|_{L^\infty(\Gamma)} &\le \|(P^*-P_h^*)(S\bar u+y_f-y_d)\|_{L^\infty(\Omega)} \to 0, \\ \|S_h^*(S-S_h)\bar u\|_{L^\infty(\Gamma)} &\le \|(S-S_h)\bar u\|_{L^2(\Omega)} \to 0, \\ \|S_h^*S_h(\bar u-\bar u_h)\|_{L^\infty(\Gamma)} &\le \|S_h(\bar u-\bar u_h)\|_{L^\infty(\Omega)} \to 0, \\ \|S_h^*(y_f-y_{f,h})\|_{L^\infty(\Gamma)} &\le \|y_f-y_{f,h}\|_{L^2(\Omega)} \to 0, \end{aligned}$$

as $h\to 0$, where we exploited that the finite element method converges in the $L^\infty(\Omega)$- and $L^2(\Omega)$-norm to show the first, second and fourth relation. The third relation has been discussed already in (5.79). Together with the reformulation (5.83) and the triangle inequality we arrive at

$$\|\bar p-\bar p_h\|_{L^\infty(\Gamma)} \to 0 \quad \text{as} \quad h\to 0.$$

Together with (5.82) the desired property (5.77) follows. □

We are now in the position to improve the error estimates from Theorem 5.2.8 exploiting the fact that $\bar u-\bar u_h\equiv 0$ in the vicinity of concave corners.

Theorem 5.3.6. *Let $\bar u \in U_{ad}$ and $\bar u_h \in U_{h,ad}$ be the solutions of (5.48) and (5.65), respectively. Let Assumption 4 be satisfied and assume that the input data satisfy $f \in L^2(\Omega)$ and $y_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Then, the error estimates*

$$\|\bar u - \bar u_h\|_{H^{-1/2}(\Gamma)} + h^{1/2}\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le c h^{\min\{3/2,\check\lambda,2\lambda-1/2\}-\varepsilon} |\ln h|^{3/2}\eta,$$
$$\|\bar y - \bar y_h\|_{H^1(\Omega)} \le c\max\{h, h^{\lambda-\varepsilon}|\ln h|^{3/2}\}\eta$$

hold for sufficiently small $0 < \varepsilon' < \varepsilon$, where

$$\eta := |\bar u|_{W^{1,2}_{\vec{\tilde\gamma},\varepsilon'}(\Gamma)} + \|f\|_{L^2(\Omega)} + \|\bar y_u\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|\bar y_u\|_{W^{2,\infty}_{\vec\beta,1/2}(\Omega)} + |\bar y_u|_{W^{2,2}_{\vec\gamma}(\Gamma)}$$
$$+ \|\bar p\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|\bar p\|_{W^{2,\infty}_{\vec\beta}(\Omega)} + |\bar p|_{W^{2,2}_{\vec\gamma}(\Gamma)},$$

with the weight vectors $\vec\alpha, \vec\beta, \vec\gamma \in \mathbb{R}^d$ from Theorem 5.2.2, and $\vec{\tilde\gamma} \in R^d$ from Lemma 5.3.2.

Proof. Consider again the domains introduced in (5.50). Due to Assumption 4 and Lemma 5.3.5 there exists some $R > 0$ such that $\bar u_h(x) = \bar u(x) \in \{u_a, u_b\}$ for all $x \in \Gamma^j_R$ and $j \in \hat{\mathcal{C}}$. Since $\tilde u_h$ behaves like the best-approximation of $\bar u$ (see (5.72)), we also get $\tilde u_h(x) \in \{u_a, u_b\}$ for all $x \in \Gamma^j_R$, $j \in \hat{\mathcal{C}}$. In the following we write

$$\Omega^0_R := \Omega \setminus \left(\bigcup_{j\in\hat{\mathcal{C}}} \Omega^j_R\right), \qquad \Gamma^0_R := \partial\Omega^0_R \cap \Gamma.$$

By construction the term $\tilde u_h - \bar u_h$ vanishes on $\Gamma \setminus \Gamma^0_R$ and the assumptions of Lemma 5.3.3 are satisfied. In order to show the estimate in the $H^{-1/2}(\Gamma)$-norm we have to discuss the five terms on the right-hand side of the estimate in Lemma 5.3.3.

First, we get from Theorem 3.3.1 that

$$\|(S - S_h)\bar u\|_{L^2(\Omega)} + \|y_f - y_{f,h}\|_{L^2(\Omega)} \le c h^{\min\{2,2\lambda-\varepsilon\}} \left(\|\bar y_u\|_{W^{2,2}_{\vec\alpha}(\Omega)} + \|f\|_{L^2(\Omega)}\right). \tag{5.84}$$

For the second term on the right-hand side of (5.67) we write $\bar p|_\Gamma := S^*(\bar y - y_d)$ and $p^h|_\Gamma := S^*_h(\bar y - y_d)$ and an inverse inequality yields

$$\|\bar p - p^h\|_{H^{1/2}(\Gamma^0_R)} \le \|\bar p - I^\partial_h \bar p\|_{H^{1/2}(\Gamma^0_R)} + h^{-1/2}\left(\|\bar p - I^\partial_h \bar p\|_{L^2(\Gamma^0_R)} + \|\bar p - p^h\|_{L^2(\Gamma^0_R)}\right). \tag{5.85}$$

The terms depending on the interpolation error have been discussed in Lemma 3.2.4 and with $\gamma_j = \max\{0, 3/2 - \lambda_j + \varepsilon\}$ for $j \in \mathcal{C}$ we get

$$\|\bar p - I_h \bar p\|_{H^{1/2}(\Gamma^0_R)} + h^{-1/2}\|\bar p - I_h\bar p\|_{L^2(\Gamma^0_R)} \le c h^{\min\{3/2,\check\lambda-\varepsilon\}} \|\bar p\|_{W^{2,2}_{\vec\gamma}(\Gamma)}. \tag{5.86}$$

We also exploited that Γ^0_R excludes neighborhoods of concave corners. For the finite-element error on the boundary we exploit Corollary 3.65 in [74] which states that if $\bar p \in W^{2,\infty}_{\beta_j}(\Omega^j_{2R})$ with $\beta_j = \max\{1/2, 2 - \lambda_j + \varepsilon\}$, the error estimate

$$\|\bar p - p^h\|_{L^2(\Gamma^j_R)} \le c\left(h^{\min\{2,1/2+\lambda_j-\varepsilon\}}|\ln h|^{3/2}|\bar p|_{W^{2,\infty}_{\beta_j}(\Omega^j_{2R})} + \|\bar p - p^h\|_{L^2(\Omega^j_{2R})}\right) \tag{5.87}$$

holds for all $j \in \check{\mathcal{C}}$. In [74, Equation (3.130)] an estimate on the regular part of the boundary

$$\hat{\Omega}_R^0 := \Omega \setminus \left(\bigcup_{j\in\mathcal{C}} \Omega_R^j \right), \qquad \hat{\Gamma}_R^0 := \partial\hat{\Omega}_R^0 \cap \Gamma,$$

is proved which reads in our situation

$$\|\bar{p} - p^h\|_{L^2(\hat{\Gamma}_R^0)} \le c \left(h^2 |\ln h| |\bar{p}|_{W^{2,\infty}(\hat{\Omega}_{R/2}^0)} + \|\bar{p} - p^h\|_{L^2(\Omega)} \right). \tag{5.88}$$

Furthermore, we use the finite element error estimate in $L^2(\Omega)$ from Theorem 3.3.1 to get

$$\|\bar{p} - p^h\|_{L^2(\Omega)} \le c h^{\min\{2,2\lambda-\varepsilon\}} \left(\|\bar{y}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} \right). \tag{5.89}$$

Combining the estimates (5.87), (5.88) and inserting (5.89) leads to

$$\begin{aligned} \|\bar{p} - p^h\|_{L^2(\Gamma_0)} &\le \left(\sum_{j\in\check{\mathcal{C}}} \|\bar{p} - p^h\|^2_{L^2(\Gamma_R^j)} + \|\bar{p} - p^h\|^2_{L^2(\hat{\Gamma}_R^0)} \right)^{1/2} \\ &\le c h^{\min\{2,1/2+\check{\lambda}-\varepsilon,2\lambda-\varepsilon\}} |\ln h|^{3/2} \left(\|\bar{p}\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)} \right). \end{aligned} \tag{5.90}$$

Inserting now (5.86) and (5.90) into (5.85) leads to the estimate

$$\|\bar{p} - p^h\|_{H^{1/2}(\Gamma_0)} \le c h^{\min\{3/2,\check{\lambda}-\varepsilon,2\lambda-1/2-\varepsilon\}} |\ln h|^{3/2} \left(\|\bar{p}\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)} + |\bar{p}|_{W^{2,2}_{\vec{\gamma}}(\Gamma)} \right). \tag{5.91}$$

It remains to derive an estimate for the third term on the right-hand side of (5.67). Additional singularities occur now in a neighborhood of the transition points. The optimal state possesses the regularity $\bar{y}_u \in W^{2,\infty}(\Omega_R^{reg})$, and according to Lemma 5.3.2 in the vicinity of corner and transition points only

$$\bar{y}_u \in W^{2,\infty}_{1/2}(\tilde{\Omega}_{2R}^k),\ k \in \mathcal{T}, \qquad \bar{y}_u \in W^{2,\infty}_{\beta_j}(\Omega_{2R}^j),\ j \in \mathcal{C},$$

with $\beta_j = \max\{1/2, 2 - \lambda_j + \varepsilon\}$. Thus, the estimates (5.87) and (5.88) can be applied again and we obtain for $\bar{y}_{u|\Gamma} = \mathcal{N}\bar{u}$ and $y^h_{u|\Gamma} = \mathcal{N}_h\bar{u}$ the estimate

$$\begin{aligned} \|\bar{y}_u - y_u^h\|^2_{L^2(\Gamma_0)} &= \sum_{j\in\check{\mathcal{C}}} \|\bar{y}_u - y_u^h\|^2_{L^2(\Gamma_R^j)} + \sum_{k\in\mathcal{T}} \|\bar{y}_u - y_u^h\|^2_{L^2(\tilde{\Gamma}_R^k)} + \|\bar{y}_u - y_u^h\|^2_{L^2(\Gamma_R^{reg})} \\ &\le c\Big(h^{2\min\{2,1/2+\check{\lambda}-\varepsilon\}} |\ln h|^3 \sum_{j\in\check{\mathcal{C}}} |\bar{y}_u|^2_{W^{2,\infty}_{\beta_j}(\Omega_{2R}^j)} + h^4 |\ln h|^3 \sum_{k\in\mathcal{T}} |\bar{y}_u|^2_{W^{2,\infty}_{1/2}(\tilde{\Omega}_{2R}^k)} \\ &\quad + h^4 |\ln h|^2 |\bar{y}_u|^2_{W^{2,\infty}(\Omega_{R/2}^{reg})} + \|\bar{y}_u - y_u^h\|^2_{L^2(\Omega)} \Big) \\ &\le c h^{2\min\{2,1/2+\check{\lambda}-\varepsilon,2\lambda-\varepsilon\}} |\ln h|^{3/2} \left(\|\bar{y}_u\|^2_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|\bar{y}_u\|^2_{W^{2,\infty}_{\vec{\beta},1/2}(\Omega)} \right). \end{aligned} \tag{5.92}$$

In analogy to (5.86) we also get

$$\|\bar{y}_u - I_h\bar{y}_u\|_{H^{1/2}(\Gamma_0)} + h^{-1/2} \|\bar{y}_u - I_h\bar{y}_u\|_{L^2(\Gamma_0)} \le c h^{\min\{3/2,\check{\lambda}-\varepsilon\}} \|\bar{y}_u\|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}$$

and with an argument like (5.85) this implies

$$\|\bar{y}_u - y_u^h\|_{H^{1/2}(\Gamma_0)} \\ \le c h^{\min\{3/2, \check{\lambda}-\varepsilon, 2\lambda-1/2-\varepsilon\}} |\ln h|^{3/2} \left(\|\bar{y}_u\|_{W^{2,2}_{\vec{\alpha}}(\Omega)} + \|\bar{y}_u\|_{W^{2,\infty}_{\vec{\beta},1/2}(\Omega)} + \|\bar{y}_u\|_{W^{2,2}_{\vec{\gamma}}(\Gamma)} \right). \tag{5.93}$$

The estimates (5.84), (5.91) and (5.93) together with Lemma 5.3.4 and Lemma 5.3.3 yield the desired estimate in the $H^{-1/2}(\Gamma)$-norm. The estimate for the control in $L^2(\Gamma)$ and for the state in $H^1(\Omega)$ follow with the same arguments like in the proof of Theorem 5.2.8. □

5.4 Numerical experiments

In this section we want to confirm the theoretically predicted results with numerical experiments. Therefore, we constructed a benchmark problem on the family of domains

$$\Omega^\omega := (-1,1)^2 \setminus \{(r\cos\varphi, r\sin\varphi) \colon r \ge 0,\ \varphi \in [0, 2\pi - \omega]\} \quad \text{for } \omega \in \left[\frac{\pi}{2}, 2\pi\right).$$

By construction ω is always the largest interior angle of Ω^ω if $\omega \ge \pi/2$. We want to compute the solution of the optimal control problem

$$J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\langle u, \mathcal{N} u\rangle_\Gamma \to \min! \tag{5.94}$$

subject to

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \partial_n y = u & \text{on } \Gamma, \end{cases} \tag{5.95}$$

$$u \in U_{ad} := \{v \in H^{-1/2}(\Gamma) \colon u_a \le v \le u_b\}, \tag{5.96}$$

using the discretization discussed in Section 5.2 for unconstrained problems (i. e. $u_a = -\infty$ and $u_b = \infty$), and Section 5.3 for problems involving control constraints.

Let us briefly discuss how the discrete problem (5.24) can be solved efficiently. We may represent the unknown functions by means of a linear combination of basis functions. Therefore, introduce the bases

$$V_h = \operatorname{span}\{\varphi_i\}_{i=1}^{N_\Omega}, \qquad U_h = \operatorname{span}\{\psi_j\}_{j=1}^{N_\Gamma},$$

which allow us to define the isomorphisms

$$\vec{u} \leftrightarrow u_h, \qquad \vec{y}_u \leftrightarrow y_{u,h}, \qquad \vec{p} \leftrightarrow p_h.$$

Let now A denote the standard finite element stiffness matrix related to the operator $-\Delta + I$, M the mass matrix and $\tilde{M} := (m_{ij}) \in \mathbb{R}^{N_\Omega \times N_\Gamma}$ a transformation matrix, having entries

$$m_{ij} := \int_\Gamma \varphi_i(x)\psi_j(x)\,\mathrm{d}s_x, \qquad i \in \{1,\dots,N_\Omega\},\ j \in \{1,\dots,N_\Gamma\}.$$

As a consequence the optimality system (5.22) reads in matrix-vector notation

$$\begin{pmatrix} 0 & A & \tilde{M} \\ A & -M & 0 \\ \tilde{M}^\top & \alpha\tilde{M}^\top & 0 \end{pmatrix} \begin{pmatrix} \vec{p} \\ \vec{y}_u \\ -\vec{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{f} \\ 0 \end{pmatrix}. \tag{5.97}$$

The vector $\vec{f}$ corresponds to the right-hand side of the adjoint equation and its components are defined by $f_i = \int_\Omega (y_{f,h} - y_d)\varphi_i$. Note, that the system (5.97) can be transformed into a symmetric one by adding α times the first row to the second one. Instead of solving the discretized optimality system (5.97) directly, we computed the solution of the Schur complement system. The resulting linear equation reads

$$\left(\tilde{M}^\top A^{-1} M A^{-1} \tilde{M} + \alpha \tilde{M}^\top A^{-1} \tilde{M}\right) \vec{u} = \tilde{M}^\top A^{-1} \tilde{M} \vec{f}. \tag{5.98}$$

As it is very inefficient to compute the system matrix explicitly, we solve (5.98) with an iterative solver – in our experiments we used the GMRES method. In each iteration we have to solve two linear equation systems governed by the system matrix A. Since the system matrix will not change during the computation, one can compute a Cholesky factorization of A in advance. The solution of the linear system $Ax = b$ is then obtained by forward-backward substitution which is comparatively cheap. To compute the Cholesky factorization we used the solvers UMFPACK or MUMPS. The latter one runs even parallel using MPI (Message Passing Interface) which shortens the runtime on a high-performance machine significantly.

In order to compute the error norms we computed a reference solution on a much finer grid having mesh parameter $h = 2^{-10}$ for the convex domains $\Omega^{\pi/2}$ and $\Omega^{3\pi/4}$, and $h = 2^{-9}$ for the non-convex domains $\Omega^{5\pi/4}$, $\Omega^{3\pi/2}$ and $\Omega^{7\pi/4}$. To improve the accuracy of the reference solution on the non-convex domains we refined the mesh also locally in the vicinity of the reentrant corner according to the refinement condition (3.57). In the present experiments the parameters $\mu = 0.5$ and $R = 0.2$ were used. As refinement strategy the *newest vertex bisection algorithm* from [17] was applied. This strategy generates hierarchical meshes which allow us to prolongate the solution from a coarse grid to a finer one. We denote by y_h^{ref} the reference solution and by y_h some arbitrary finite element function prolongated to the reference mesh. The corresponding vectors containing the nodal values are denoted by $\vec{y}_{ref}$ and $\vec{y}$, respectively. Then, the $H^1(\Omega)$-norm of the difference between both functions can be computed by means of

$$\|y_h - y_h^{ref}\|_{H^1(\Omega)} = \sqrt{(\vec{y} - \vec{y}^{ref})^\top A (\vec{y} - \vec{y}^{ref})},$$

where A denotes the finite element system matrix on the fine mesh. If the matrix A is replaced by the mass matrix $M_{bd} \in \mathbb{R}^{N_\Gamma \times N_\Gamma}$ having entries $m_{ij} = \int_\Gamma \psi_i \psi_j$ we get a representation of the $L^2(\Gamma)$-norm for discrete functions on the boundary in U_h.

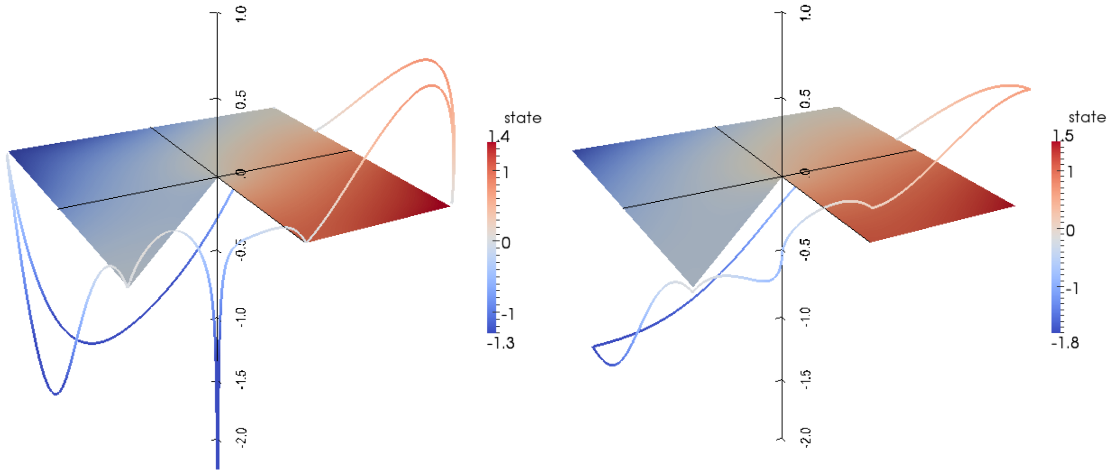
Example 5.4.1. *In the first experiment we compute the problem* (5.94)–(5.96) *without control constraints, i. e.* $U_{ad} = H^{-1/2}(\Gamma)$. *We choose the input data*

$$f \equiv 1, \quad \alpha = 0.1, \quad y_d(x_1, x_2) = x_1 + x_2.$$

As described in Section 5.2 we discretize state and adjoint state with piecewise linear finite elements. For the control we used either piecewise constant functions on the dual mesh (see Figure 5.1b) or piecewise linear functions which are continuous on each boundary edge (see Figure 5.1c). The optimal state and control are illustrated in Figure 5.2. For comparison the solution of the model problem using $L^2(\Gamma)$*-regularization instead was also computed.*

Theoretically, we would expect that the error estimates

$$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \le c \max\{h, h^{\lambda-\varepsilon} |\ln h|^{3/2}\},$$
$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \le c h^{\min\{1, \lambda - 1/2 - \varepsilon\}} |\ln h|^{3/2},$$



(a) $H^{-1/2}(\Gamma)$-regularization (b) $L^2(\Gamma)$-regularization

Figure 5.2: Optimal state (solid surface) and optimal control (boundary line) of the model problem (5.94)–(5.96) for energy and $L^2(\Gamma)$-regularization.

hold, as we have proven in Theorem 5.2.8. The singular exponent is defined by $\lambda = \pi/\omega$ where ω is the largest interior angle of the corner points of Ω. We summarized the measured error norms and the corresponding experimental convergence rates in Table 5.1 for piecewise constant controls on the dual mesh and in Table 5.2 for piecewise linear controls. The computed rates are always slightly better then the theoretically predicted rates which is a consequence of the strategy we used to compute the error norms. However, on very fine grids we observe that theory and numerics almost coincide which confirms the quasi-optimality of the results proven in Theorem 5.2.8.

Example 5.4.2. *In a second example additional control constraints are taken into account. We bound the control by means of*

$$u \in U_{ad} := \{v \in H^{-1/2}(\Gamma)\colon\ -2.0 \leq v\}.$$

For the remaining input data we choose

$$y_d(x_1, x_2) := x_1^2 + x_2^2, \quad \alpha = 10^{-2}, \quad f \equiv 1.$$

A lower control bound is sufficient to achieve that the control is active in the vicinity of the reentrant corner, compare also the behavior of the optimal control of the unconstrained problem in Figure 5.2. In Figure 5.3 the optimal state and control of the constrained problem are plotted and we observe that the control is active and hence constant near the reentrant corner. From our theory we would expect that the error estimates

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq c h^{\min\{1, \check{\lambda}-1/2-\varepsilon, 2\lambda-1-\varepsilon\}} |\ln h|^{3/2} \eta,$$
$$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \leq c \max\{h, h^{\lambda-\varepsilon} |\ln h|^{3/2}\} \eta,$$

hold. For our family of domains Ω^ω with $\omega \in \{5\pi/4, 3\pi/2, 7\pi/4\}$ the convergence rate for the control is dominated by $2\lambda - 1$ since the largest convex angle is always $\pi/2$ and hence $\check{\lambda} = 2$.

h	# DOF	#DOF BD	$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ (EOC)	$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$ (EOC)
$\Omega^{\pi/2}$ – largest angle 90°			**Expected: 1.00**	**1.00**
2^{-4}	545	60	2.42e-02 (1.03)	7.53e-02 (0.85)
2^{-5}	2113	124	1.16e-02 (1.06)	3.70e-02 (1.03)
2^{-6}	8321	252	5.65e-03 (1.04)	1.69e-02 (1.13)
2^{-7}	33025	508	2.77e-03 (1.03)	7.42e-03 (1.19)
2^{-8}	131585	1020	1.34e-03 (1.04)	3.26e-03 (1.19)
2^{-9}	525313	2044	6.00e-04 (1.16)	1.56e-03 (1.06)
$\Omega^{3\pi/4}$ – largest angle 135°			**Expected 1.00**	**0.80**
2^{-4}	817	92	2.53e-02 (1.01)	7.97e-02 (0.80)
2^{-5}	3169	188	1.24e-02 (1.03)	4.32e-02 (0.88)
2^{-6}	12481	380	6.12e-03 (1.02)	2.32e-02 (0.90)
2^{-7}	49537	764	3.03e-03 (1.02)	1.26e-02 (0.88)
2^{-8}	197377	1532	1.47e-03 (1.04)	7.15e-03 (0.82)
2^{-9}	787969	3068	6.60e-04 (1.16)	4.48e-03 (0.67)
$\Omega^{5\pi/4}$ – largest angle 225°			**Expected 0.80**	**0.30**
2^{-4}	1345	123	5.14e-02 (0.93)	1.08e-01 (0.78)
2^{-5}	5249	251	2.74e-02 (0.91)	7.14e-02 (0.60)
2^{-6}	20737	507	1.48e-02 (0.89)	5.29e-02 (0.43)
2^{-7}	82433	1019	8.05e-03 (0.88)	4.15e-02 (0.35)
2^{-8}	328705	2043	4.41e-03 (0.87)	3.31e-02 (0.33)
2^{-9}	1312769	4091	2.39e-03 (0.88)	2.65e-02 (0.32)
$\Omega^{3\pi/2}$ – largest angle 270°			**Expected 0.67**	**0.17**
2^{-4}	1601	122	9.44e-02 (0.78)	2.45e-01 (0.44)
2^{-5}	6273	250	5.64e-02 (0.74)	2.00e-01 (0.29)
2^{-6}	24833	506	3.43e-02 (0.72)	1.72e-01 (0.22)
2^{-7}	98817	1018	2.11e-02 (0.70)	1.50e-01 (0.20)
2^{-8}	394241	2042	1.30e-02 (0.70)	1.31e-01 (0.21)
2^{-9}	1574910	4090	8.15e-03 (0.88)	1.11e-01 (0.22)
$\Omega^{7\pi/4}$ – largest angle 315°			**Expected 0.57**	**0.08**
2^{-4}	1873	154	1.26e-01 (0.66)	5.82e-01 (0.17)
2^{-5}	7329	314	8.21e-02 (0.62)	5.33e-01 (0.13)
2^{-6}	28993	634	5.41e-02 (0.60)	4.92e-01 (0.12)
2^{-7}	115329	1274	3.60e-02 (0.59)	4.54e-01 (0.12)
2^{-8}	460033	2554	2.40e-02 (0.59)	4.16e-01 (0.12)
2^{-9}	1837569	5114	1.61e-02 (0.59)	3.80e-01 (0.13)

Table 5.1: Numerical results for the unconstrained problem, piecewise constant controls.

h	# DOF	#DOF BD	$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ (EOC)	$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$ (EOC)
$\Omega^{\pi/2}$ – largest angle 90°			**Expected: 1.00**	**1.00**
2^{-4}	545	60	2.37e-02 (1.02)	8.31e-02 (0.85)
2^{-5}	2113	124	1.15e-02 (1.04)	4.07e-02 (1.03)
2^{-6}	8321	252	5.61e-03 (1.03)	1.85e-02 (1.14)
2^{-7}	33025	508	2.76e-03 (1.02)	8.06e-03 (1.20)
2^{-8}	131585	1020	1.34e-03 (1.04)	3.41e-03 (1.24)
2^{-9}	525313	2044	5.99e-04 (1.16)	1.45e-03 (1.23)
$\Omega^{3\pi/4}$ – largest angle 135°			**Expected 1.00**	**0.80**
2^{-4}	817	92	2.49e-02 (1.01)	9.27e-02 (0.81)
2^{-5}	3169	188	1.23e-02 (1.02)	5.00e-02 (0.89)
2^{-6}	12481	380	6.09e-03 (1.01)	2.65e-02 (0.91)
2^{-7}	49537	764	3.02e-03 (1.01)	1.42e-02 (0.90)
2^{-8}	197377	1532	1.47e-03 (1.04)	7.73e-03 (0.88)
2^{-9}	787969	3068	6.59e-04 (1.16)	4.34e-03 (0.83)
$\Omega^{5\pi/4}$ – largest angle 225°			**Expected 0.80**	**0.30**
2^{-4}	1345	123	5.11e-02 (0.92)	1.18e-01 (0.79)
2^{-5}	5249	251	2.73e-02 (0.91)	7.71e-02 (0.62)
2^{-6}	20737	507	1.48e-02 (0.89)	5.66e-02 (0.45)
2^{-7}	82433	1019	8.04e-03 (0.88)	4.42e-02 (0.36)
2^{-8}	328705	2043	4.40e-03 (0.87)	3.52e-02 (0.33)
2^{-9}	1312769	4091	2.38e-03 (0.88)	2.82e-02 (0.32)
$\Omega^{3\pi/2}$ – largest angle 270°			**Expected 0.67**	**0.17**
2^{-4}	1601	122	9.39e-02 (0.77)	2.54e-01 (0.46)
2^{-5}	6273	250	5.62e-02 (0.74)	2.05e-01 (0.30)
2^{-6}	24833	506	3.42e-02 (0.71)	1.76e-01 (0.23)
2^{-7}	98817	1018	2.10e-02 (0.70)	1.53e-01 (0.20)
2^{-8}	394241	2042	1.30e-02 (0.69)	1.32e-01 (0.20)
2^{-9}	1574910	4090	8.13e-03 (0.69)	1.14e-01 (0.21)
$\Omega^{7\pi/4}$ – largest angle 315°			**Expected 0.57**	**0.08**
2^{-4}	1873	154	1.26e-01 (0.65)	5.99e-01 (0.18)
2^{-5}	7329	314	8.19e-02 (0.62)	5.48e-01 (0.13)
2^{-6}	28993	634	5.40e-02 (0.60)	5.06e-01 (0.12)
2^{-7}	115329	1274	3.59e-02 (0.59)	4.67e-01 (0.12)
2^{-8}	460033	2554	2.40e-02 (0.58)	4.29e-01 (0.12)
2^{-9}	1837569	5114	1.60e-02 (0.58)	3.92e-01 (0.13)

Table 5.2: Numerical results for the unconstrained problem, piecewise linear controls.

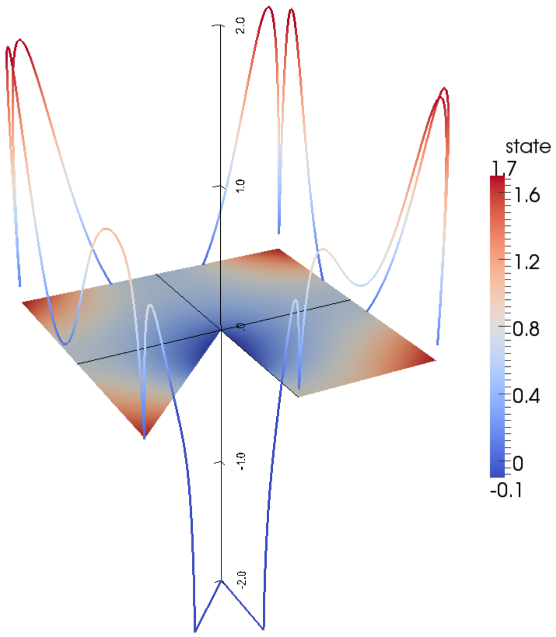


Figure 5.3: Optimal state (solid surface) and optimal control (boundary line) of the problem from Example 5.4.1 with control constraints.

The numerically determined convergence rates are summarized in Table 5.3, and we observe that the proven error estimates are not sharp. Both, the convergence rate for the discrete state and the discrete control are approximately one, but the proven result from Theorem 5.3.6 is too pessimistic.

h	# DOF	#DOF BD	$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ (EOC)	$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$ (EOC)
$\Omega^{5\pi/4}$ – largest angle 225°			**Expected 0.80**	**0.60**
2^{-3}	353	59	1.83e-01 (0.87)	4.34e-01 (0.37)
2^{-4}	1345	123	9.67e-02 (0.92)	2.99e-01 (0.54)
2^{-5}	5249	251	4.80e-02 (1.01)	1.66e-01 (0.85)
2^{-6}	20737	507	2.35e-02 (1.03)	8.30e-02 (1.00)
2^{-7}	82433	1019	1.15e-02 (1.03)	3.88e-02 (1.10)
2^{-8}	328705	2043	5.58e-03 (1.05)	1.79e-02 (1.12)
2^{-9}	1312769	4091	2.49e-03 (1.17)	8.52e-02 (1.07)
$\Omega^{3\pi/2}$ – largest angle 270°			**Expected 0.67**	**0.33**
2^{-3}	417	58	2.12e-01 (0.83)	4.73e-01 (0.33)
2^{-4}	1601	122	1.13e-01 (0.91)	3.31e-01 (0.51)
2^{-5}	6273	250	5.64e-02 (1.00)	1.91e-01 (0.79)
2^{-6}	24833	506	2.77e-02 (1.03)	9.77e-02 (0.97)
2^{-7}	98817	1018	1.36e-02 (1.03)	4.55e-02 (1.10)
2^{-8}	394241	2042	6.57e-03 (1.05)	2.08e-02 (1.13)
2^{-9}	1574913	4090	2.93e-03 (1.16)	1.03e-02 (1.02)
$\Omega^{7\pi/4}$ – largest angle 315°			**Expected 0.57**	**0.14**
2^{-3}	489	74	2.38e-01 (0.83)	5.38e-01 (0.35)
2^{-4}	1873	154	1.25e-01 (0.92)	3.61e-01 (0.57)
2^{-5}	7329	314	6.29e-02 (1.00)	2.02e-01 (0.84)
2^{-6}	28993	634	3.11e-02 (1.02)	1.01e-01 (1.01)
2^{-7}	115329	1274	1.54e-02 (1.02)	4.69e-02 (1.10)
2^{-8}	460033	2554	7.48e-03 (1.04)	2.17e-02 (1.11)
2^{-9}	1837569	5114	3.35e-03 (1.16)	1.08e-02 (1.01)

Table 5.3: Numerical results for the control-constrained problem from Example 5.4.2.

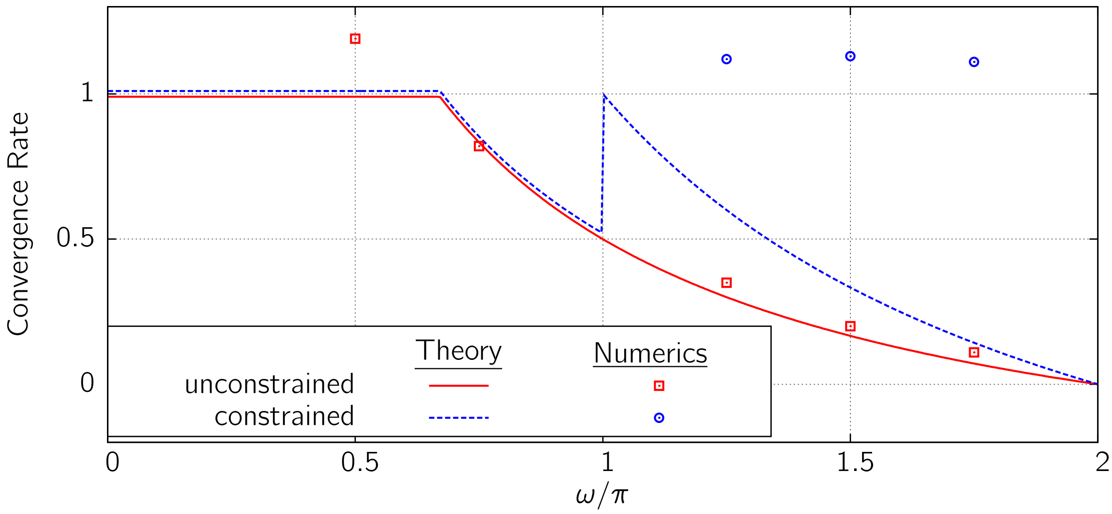


Figure 5.4: Theoretically predicted convergence rates from Theorems 5.2.8 and 5.3.6 compared to the numerically computed rates obtained in Examples 5.4.1 and 5.4.2 in dependence of the largest angle ω of the domain Ω.

CHAPTER 6

Outlook

The main results of this thesis are error estimates for the numerical approximation of both the Neumann boundary value problem for the Yukawa equation, and Neumann boundary control problems on domains having corners and edges. We have shown that the singularities that occur reduce the convergence rate for the control approximation of the optimal control problems when the corresponding singular exponents are smaller than $3/2$. This is for instance the case when the interior angle at an edge is larger than $120°$. Moreover, we have observed that, as a remedy, local mesh refinement can be applied and we derived bounds for the refinement parameter such that optimal convergence is guaranteed.

The results obtained for three-dimensional problems are proven for the case that each edge and corner is refined according to the same refinement parameter, i. e. we refined at each singular point as strong as it is actually necessary for the singularity with the strongest influence. Some edges and corners would be over-refined although it is not required. The reason why we discussed only the very simple refinement criterion (3.56) is that the proof of the finite element error on the boundary presented in Section 3.4 cannot be extended to more complex refinement strategies without substantial modifications. An isotropic refinement strategy which allows to use a different refinement parameter for each edge and corner is presented in [61] where error estimates in $H^1(\Omega)$ and $L^2(\Omega)$ are proven. The local estimates derived Section 3.2 remain valid for this strategy, but the proof of the error estimate in $L^2(\Gamma)$ requires major modifications. A rather advanced strategy exploiting higher regularity of the derivative in the direction parallel to singular edges is anisotropic mesh refinement. A construction of such meshes is presented in [6] where error estimates in $H^1(\Omega)$ are presented. Estimates in $L^2(\Omega)$ using this strategy are discussed in [5]. An extension of the results presented in this thesis seems to be more complicated as some techniques we used to prove Theorem 3.4.14 are not valid when the family of meshes is not locally quasi-uniform. In particular the local maximum norm estimate (3.106) is an open problem for anisotropic meshes, but recently developed techniques which are used to prove maximum norm estimates for parabolic partial differential equations from [59] could solve this issue.

In Chapter 5 we investigated the energy regularization approach for Neumann control problems and restricted our considerations to two-dimensional polygonal domains. Possible extensions of the results obtained in the present thesis are optimal error estimates for control-constrained problems. In the numerical experiments we have observed a higher convergence rate than predicted in Theorem 5.3.6. When tracing through its proof we easily confirm that suboptimal estimates were used in the steps (5.90) and (5.92). More precisely, instead of a local estimate for the finite element error in $L^2(\Omega_R^{reg})$ we used a global estimate in $L^2(\Omega)$ which is bounded by $h^{2\lambda-\varepsilon}$. To the best of our knowledge it is not possible to derive an improved error estimate in $L^2(\Omega_0)$ on some subset $\Omega_0 \subset \Omega$, see also the inverse estimate in [39, Theorem 2.3].

Another open problem is the numerical analysis of the energy regularization approach for Neumann boundary control problems on three-dimensional polyhedral domains. Possible choices for the discrete control space which satisfy Assumption 3 are presented in [20, 22, 53]. The general approach we used to prove estimates on polygonal domains can be applied also for polyhedral domains when carefully outlining the influence of edge and corner singularities as we have done for Neumann control problems in $L^2(\Gamma)$. Moreover, it remains to derive error estimates for the best-approximation of the discrete control space using the framework in weighted Sobolev spaces developed in the present thesis.

In this thesis we considered only the case that the Laplace operator is the principal part of the state equation. However, the techniques developed in this thesis are also applicable to other types of partial differential equations where the structure of the singularities is known in advance. Investigations on the asymptotic behavior of solutions already exist for problems of linear elasticity where the Lamé operator is the leading differential operator, see e. g. [77, 79]. The singular solutions in a vicinity of edges and corners are moreover known for the Stokes problem [34] and the Maxwell equations [30]. For all these problems finite element error estimates on the boundary have not been proved so far, but this is required in order to derive sharp error estimates for the approximate solution of Neumann control problems involving these models.

Another interesting observation is that the refinement conditions we derived can be satisfied for arbitrary singularities as $\lambda > 1/2$ for the Yukawa equation with pure Neumann conditions. However, in the presence of jumping coefficients or mixed boundary conditions there might also occur singularities with smaller exponents $\lambda > 1/4$. In the worst case the criterion $\mu < 1/4+\lambda/2$ would then yield $\mu = 3/8$ which is a contradiction to the general assumption $\mu > 1/3$ we used for the refinement of three-dimensional domains. Then, an optimal convergence with respect to the number of degrees of freedom cannot be achieved. Investigating this behavior in detail could also be subject of further research.

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