Numerical study of PDE eigenvalue problems with IGA

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Approximation of Laplace eigenvalues

We study the numerical approximation of eigenvalues λ for the Dirichlet eigenvalue problem of the Laplace operator

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega\,, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

on the unit square $\Omega = (0,1)^2$ and the unit circular disk $\Omega = B_1(0) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_2 \leq 1\}$ with isogeometric analysis (IGA) using the software package GeoPDEs 3.0 (see [2]). With p being the order and q the regularity of the NURBS basis functions we compare the error and experimental convergence order (EOC) for different choices of p and q.

The table on the right exemplarily shows the EOC for the second eigenvalue of the unit square, where the mesh sizes for the calculation are chosen such that the absolute error is smaller than 10^{-10} for the first time.

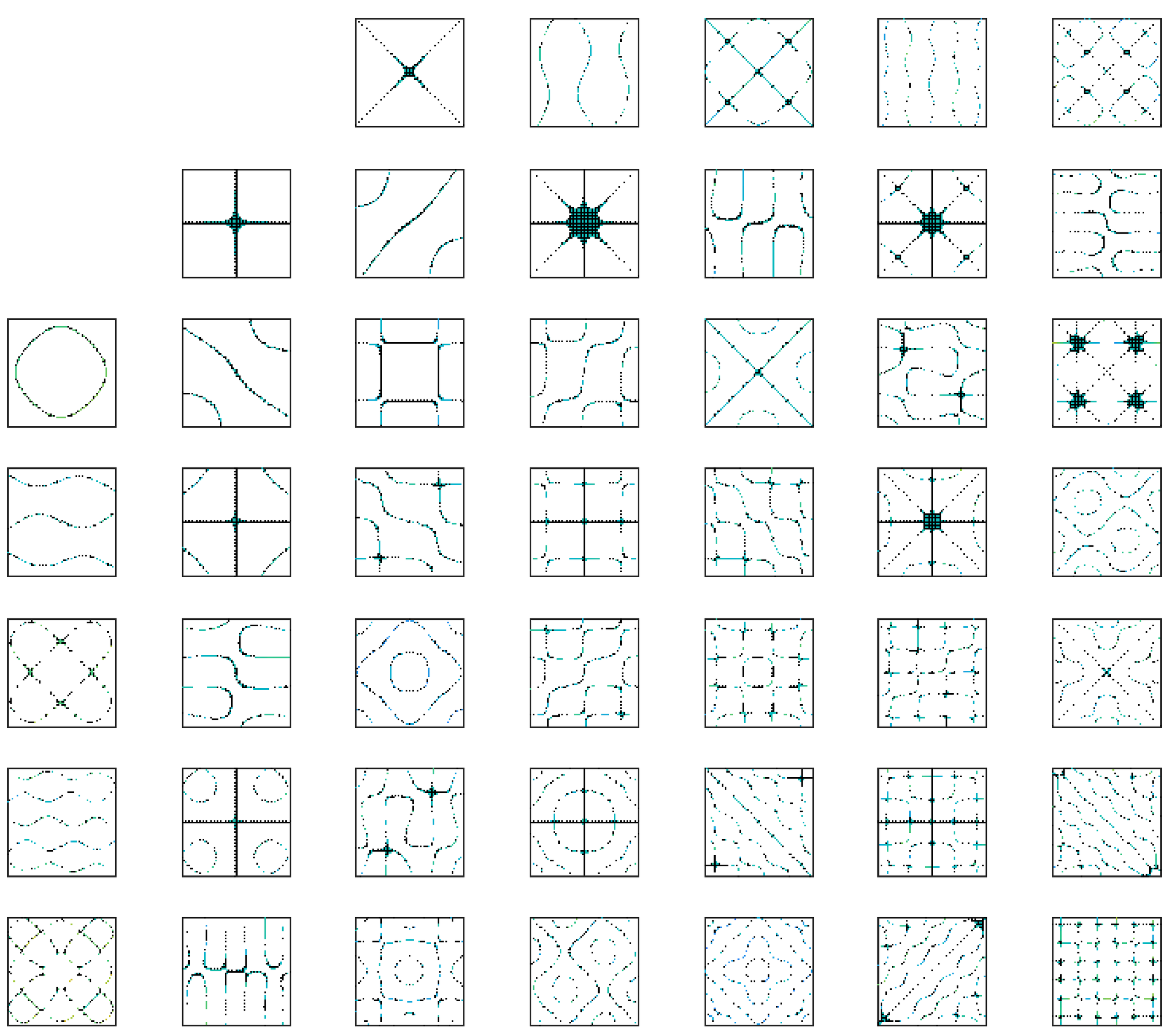
q \ p	3	4	5	6
1	5.78	8.04	9.83	12.30
2	6.17	7.78	10.18	11.27
3	-	8.17	9.64	12.60
4	-	-	10.36	11.08
5	-	-	-	12.94

Knowing that the convergence order is expected to be $2p$, we observe the following pattern for $p \geq 3$ and $q \geq 1$:

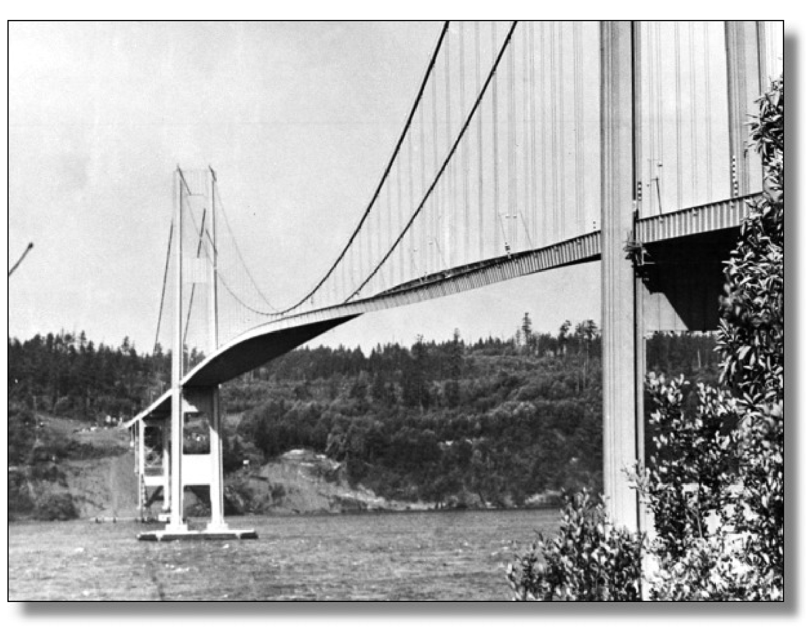
$$\text{EOC} \begin{cases} > 2p & \text{if } p+q \text{ is odd,} \\ < 2p & \text{if } p+q \text{ is even.} \end{cases}$$

Eigenfunctions of the Kirchhoff Love Model

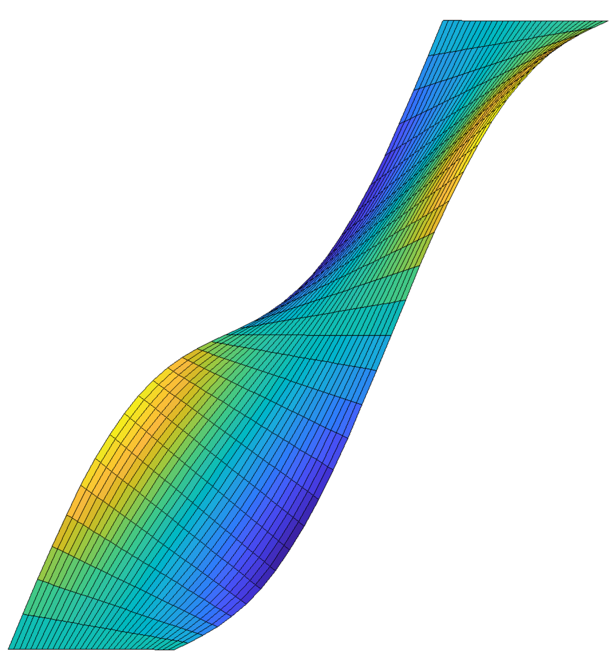
We further compute eigenfunctions of the Kirchhoff Love Model for thin plates with IGA. This enables us to reproduce the famous Chladni figures as illustrated in the following.



This way we can also simulate the fatal eigenvibrations which caused the collapse of the Tacoma bridge in 1940, see below. These problems have already been computed in [1] with FEM, and the results are consistent with our outputs.



[Source: https://www.bernd-nebel.de/bruecken/index.html]



Eigenfunctions of domains with cracks

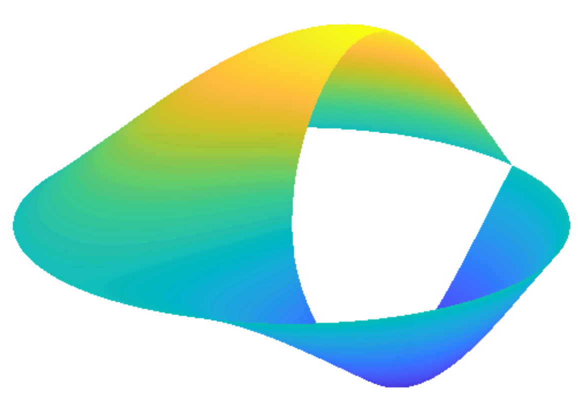
Next, we investigate the Laplace eigenvalues and eigenfunctions of domains with cracks, i.e., we solve the problem

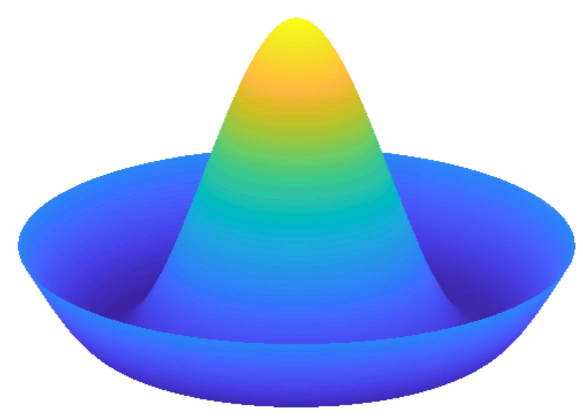
$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \setminus \Gamma_N\,, \\ u &= 0 && \text{on } \Gamma_D\,, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma_N\,, \end{aligned}$$

where Ω is again the unit square or circular disk, $\Gamma_D = \partial\Omega$ is its boundary and Γ_N is an arbitrary crack in the domain which we consider as additional Neumann boundary. For instance, we choose $\Omega = B_1(0) \subset \mathbb{R}^2$ and $\Gamma_N = B_1(0) \cap \{x \geq 0\}$ as we can express the exact eigenfunctions for this domain in polar coordinates through Bessel functions:

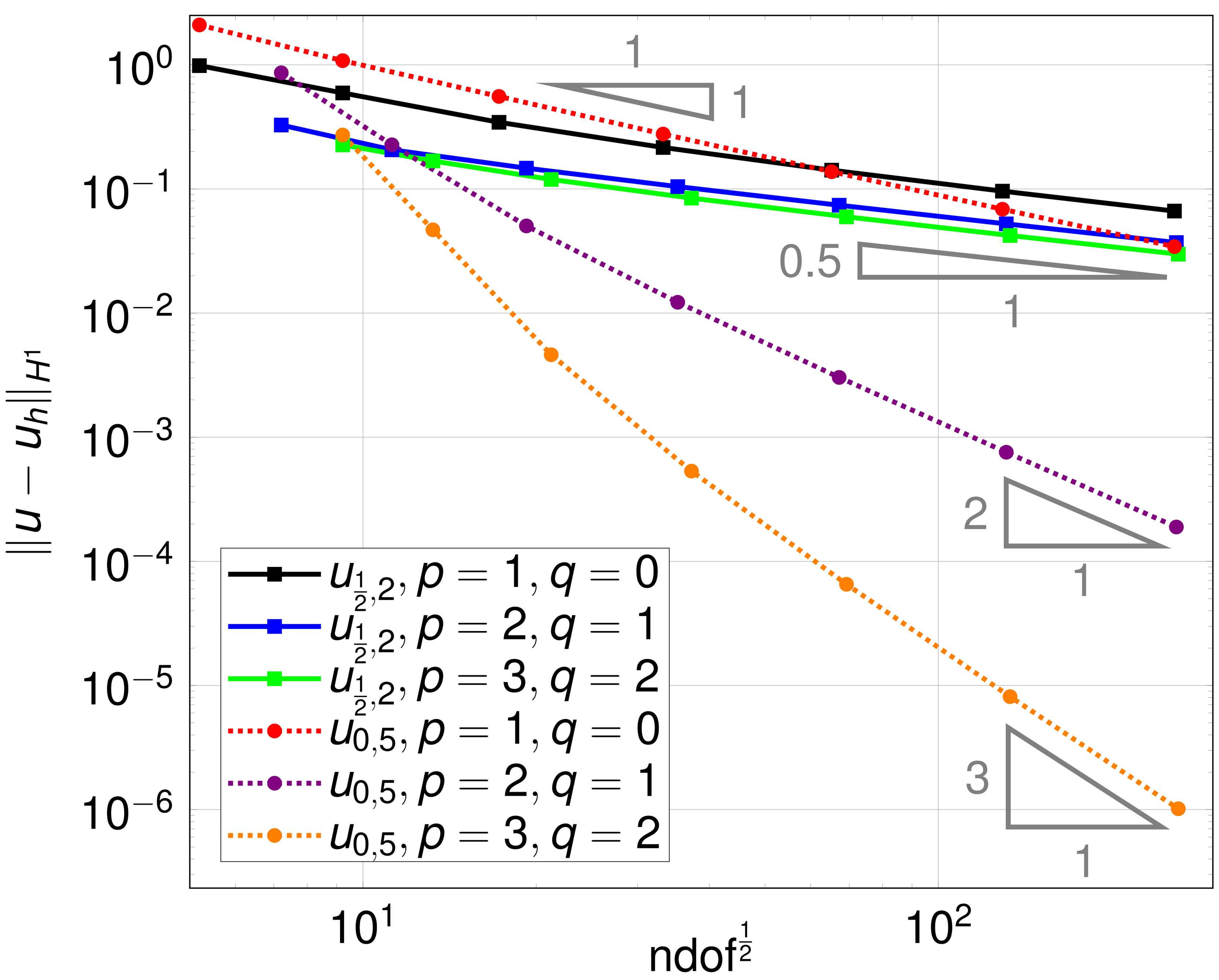
$$u_{n,m}(r,\theta) = J_n(\mu_{n,m} r)(\cos(n\theta) + \sin(n\theta))\,,$$

where J_n is the Bessel function of the first kind to the order $n \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$ and $\mu_{n,m}$ is the m-th root of J_n. We plot the eigenfunctions $u_{\frac{1}{2},2}$ and $u_{0,5}$ and remark that there is a singularity of type $r^{\frac{1}{2}}$ at the crack tip in $u_{\frac{1}{2},2}$, while $u_{0,5}$ is smooth.





We use a single-patch IGA discretization to approximate the eigenfunctions and get optimal EOCs in the H^1-norm. Due to the singularity of $u_{\frac{1}{2},2}$ at the crack tip, we can not expect a better convergence order than 0.5 for any order p and regularity q of the NURBS basis functions, whereas $u_{0,5}$ converges with order p with respect to $\text{ndof}^{\frac{1}{2}}$. Those results are illustrated in the following error plot.



References

[1] M. J. Gander and F. Kwok. Chladni figures and the Tacoma bridge: motivating PDE eigenvalue problems via vibrating plates. *SIAM Review* 54.3 (2012), pp. 573–596.

[2] R. Vázquez. A new design for the implementation of isogeometric analysis in Octave and Matlab: GeoPDEs 3.0. *Comput. Math. Appl.* 72.3 (2016), pp. 523–554.