Universität der Bundeswehr München

FAKULTÄT FÜR INFORMATIK

A Fresh Look on the Battle of Sexes Paradigm
-- Proofs and Extensions --

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Bericht 2010-01
August 2010

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Abstract

If one tries to model real conflict situations with the help of non-cooperative normal form games, it may happen that strategy combinations have to be considered which are totally unrealistic in practice but which, however may be taken into account in equilibrium with positive probability.
In this paper the battle of sexes paradigm is considered which is the most simple game owning this unrealistic feature. It is shown that a slight modification of the rules of this game remedies the problem: If the mixed equilibrium is agreed upon as solution of the game, and the unrealistic strategy combination would have to be chosen, the game is repeated as long as this happens. It turns out that the expected run length of this new game is only slightly larger than one. In other words, this modification removes the unrealistic feature, but changes only slightly the outcome of the game.
Also the case of altruistic behavior of the players is considered. Here the strange situation occurs that if in the first step the absurd strategy combination would have to be chosen, in the second step the game will be terminated with exactly this strategy combination.

A Fresh Look on the Battle of Sexes Paradigm – Proofs and Extensions –

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3rd August 2010

1 Introduction

Three years ago J. Smilovitz worked about the conflict between the Northern Sudan, which is dominated by Arabs and Muslims, and Southern Sudan, mainly comprised of black Africans of Animist or Christian belief and who has control over indigenous oil resources, see [Smi11].

We discussed with him the possibility of describing this conflict with the help of a non-cooperative normal form game, with four clear possible choices for both parties, namely (1) no autonomy and Khartoum controls oil, (2) shared control of oil and autonomy, (3) South control of oil and autonomy and (4) independence. Northern Sudan's preferences are $(1) \succ (2) \succ (3) \succ (4)$, while Southern Sudan's preferences are $(4) \succ (3) \succ (2) \succ (1)$. But we were not successful, even though we tried hard. The reason was that in our normal form game we had to consider strategy combinations which appeared to be totally unrealistic in practice.

Therefore, we gave up this approach but continued to find out in which way conflict situations were modelled where similar problems occurred. Indeed, in the theory of correlated equilibria situations of this kind are discussed, but there, the existence of a mediator or a noisy channel is required, see, e.g., [vD87] which does not hold for situations like the Sudan conflict. Thus, we looked for an intrinsic solution. For the sake of the argument we consider in the following the most simple conflict situation with this difficulty, namely the well-known Battle of Sexes paradigm.

For this non-cooperative two-person game there exist two equilibria in pure strategies, and one in mixed strategies. If both players agree on the latter one, which provides the same payoff to both players, then with positive probability both players choose independently of each other that strategy they like the least. From now on we call this strategy combination the *absurd* strategy combination.

In the following a modification of this game is proposed such that in case the absurd strategy combination would have to be chosen, the game is repeated as long as this does occur. It will be shown that the expected number of repetitions is only slightly larger than one. In other words, the unrealistic feature of the original Battle of Sexes paradigm can be removed by a slight and in its consequences not important modification of the rules of the game.

Also the case of altruistic behavior of the players is considered. Here the strange situation occurs that if in the first step the absurd strategy combination would have to be chosen, in the second step the game will be terminated with exactly this strategy combination.

We conclude this paper with some remarks about the applicability of these results to more realistic and complicated conflict situations with the above described property.

2 Original Model

Assume that a couple cannot agree how to spend the evening together, see [LR57] and also [Rap74][1]: He wants to attend a boxing fight whereas she wants to go to a ballet. Of course, both would like to spend the evening together. In Figure 1 the normal form of this non-cooperative two-person game is shown.

As shown in the Figure, there are two Nash equilibria in pure strategies and one in mixed strategies: Let M_1 and F_1 be the expected payoffs to both players. Then the mixed Nash equilibrium is given by

$$p_1^* = \frac{3}{5}\,, \quad q_1^* = \frac{2}{5}\,, \quad M_1^* = F_1^* = \frac{1}{5}\,. \tag{1}$$

Two problems characterize this model: First, there are three Nash equilibria none of which can be chosen in a natural way. This should, however, not be considered a weakness of the model, but a representation of reality: Otherwise there would be no quarrel. The bargaining model by Nash, see [Nas50], provides one unique solution, but is totally different.

Second, in case the mixed equilibrium is agreed upon – both players get the same payoffs – with positive probability $q_1^*\,(1 - p_1^*) = (2/5)^2 = 0.16$ the absurd situation occurs that the man attends the ballet and the wife the boxing fight. This is quite unrealistic therefore, in the following we will consider this problem.

[1] Rapoport considers also a Battle of Sexes variant where both players have three possible choices when planning their vacation, namely (1) sea shore, (2) ocean voyage and (3) mountain hiking. Her preferences are (1) $\succ$ (2) $\succ$ (3), while his preferences are (3) $\succ$ (2) $\succ$ (1). This model stands between the Battle of Sexes model considered in this section and that one we discussed together with J. Smilovitz.

		q_1	$1-q_1$
	M \ F	Boxing	Ballet
p_1	Boxing	2 ☆ 1	- 1 - 1
$1-p_1$	Ballet	- 1 - 1	1 ☆ 2

Figure 1: Normal form of the Battle of Sexes paradigm. The arrows indicate the preference directions, the stars denote the two equilibria in pure strategies.

Nevertheless, such absurd situations occur in reality: G. O. Faure, Sorbonne University, attended a meeting of French and German cultural delegations in Paris, see [Ave06]. On the morning of the meeting which was scheduled for 9 am, the French delegation entered the meeting room at 9 sharp and the German delegation at 9:15. The next day the same thing happened again. Obviously, both delegations wanted to show respect for each other. The French, assuming the Germans always arrive in time did not want to let them wait. Conversely the Germany assuming that French are late, did not want to blame them by arriving early. We will return to this case in the fourth section.

Finally, it should be mentioned that this conflict situation could also be modelled in a different way, where the players' choice is not entirely free and the choices of the other player determine the subset to which his selection is restricted, see, e.g., [Deb52]. In that model one would get the game of Figure 1 without the strategy combination Ballet for the man and Boxing for the woman. The modification of the Nash equilibrium concept – the so-called social equilibrium – leads to the two social equilibria: (Boxing,Boxing) and (Ballet,Ballet). Thus it does not provide a solution to the couple's conflict.

3 New model

Let us assume that the couple agrees to choose the mixed Nash equilibrium and furthermore, to repeat the game in case the absurd strategy combination would have to be chosen. One could object that then both better would agree on a random experiment which results in a joint visit either of the boxing fight or the ballet, but this might go too far since it would exclude the separate visit of those events they prefer.

It should be mentioned that this agreement is not part of our mathematical model. It remains an open question if it is possible to formulate a model which takes this agreement into account in an intrinsic way.

Furthermore, it should be remarked that this new game varies from the idea of repeated games, see, e.g., [FT98]: Here we have neither a base game which is repeated finitely or infinitely many times, nor are the payoffs the (eventually discounted) sums of payoffs that the players receive at all rounds of the game.

To begin we consider a game which consists of only two steps. If we represent this new game in extensive form, see Figure 2, then we see that the second step game is a subgame of the total one which means that the equilibrium can be determined recursively with the help of a backward induction, see [Owe82].

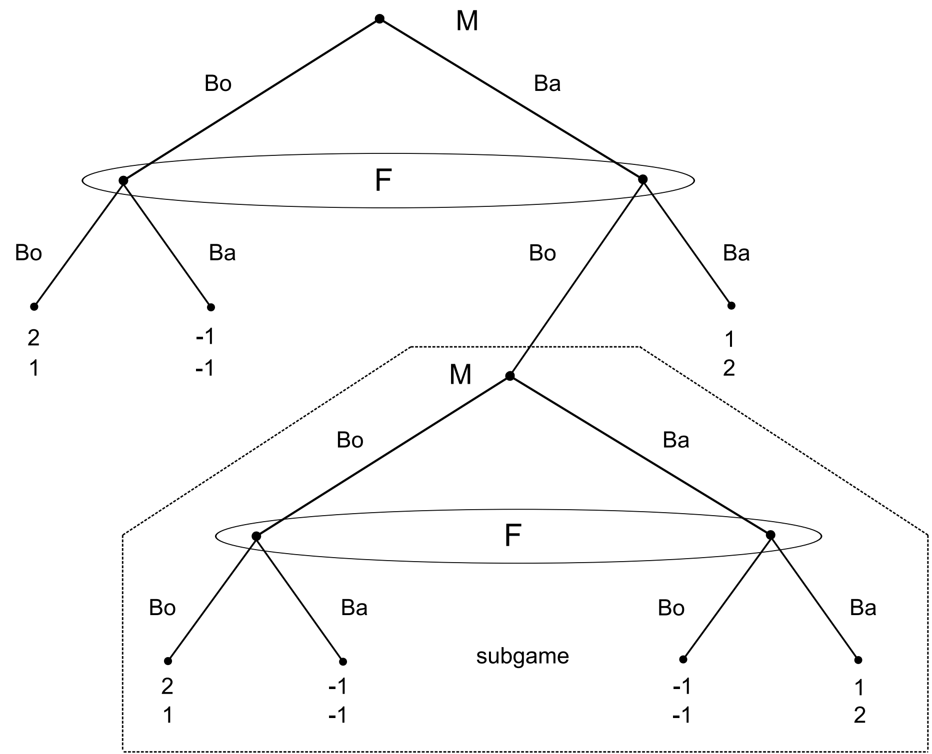

Figure 2: Extensive form of the 2-step game. The upper payoff belongs to the man, the lower one to the woman.

If we continue this way, then we can represent the n-step game, i.e., the game with at most n steps, in a reduced normal form which is shown in Figure 3. This game may end with the realization of the absurd strategy combination and is different from its infinite analogon.

Let M^*_{n-1} and F^*_{n-1} be the expected payoffs to both players for the mixed Nash equilibrium of the $n-1$ step game in case that always the mixed equilibrium is chosen.

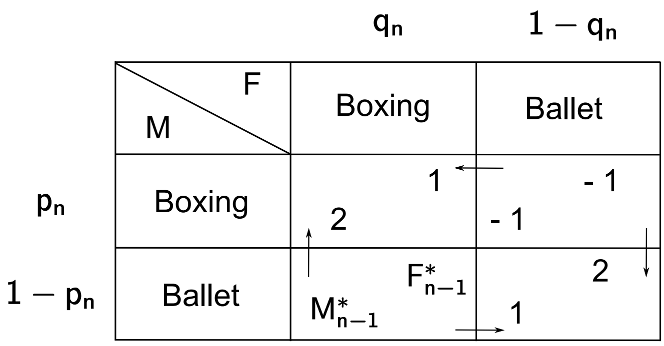

Figure 3: Reduced normal form of the n-step Battle of Sexes paradigm.

Lemma 1. The mixed equilibrium of the n-step game is given by

$$p_n^* = \frac{2-F_{n-1}^*}{4-F_{n-1}^*}, \quad F_n^* = \frac{2+F_{n-1}^*}{4-F_{n-1}^*}, \quad n=1,2,\dots, \quad F_0^* = -1 \tag{2}$$

$$q_n^* = \frac{2}{4-M_{n-1}^*}, \quad M_n^* = \frac{2+M_{n-1}^*}{4-M_{n-1}^*}, \quad n=1,2,\dots, \quad M_0^* = -1\,. \tag{3}$$

Proof. According to Figure 3 both players are indifferent with respect to their own strategies if

$$F_n^* = p_n^* \cdot 1 + (1-p_n^*) \cdot F_{n-1}^* = p_n^* \cdot (-1) + (1-p_n^*) \cdot 2$$

and

$$M_n^* = q_n^* \cdot 2 + (1-q_n^*) \cdot (-1) = q_n^* \cdot M_{n-1}^* + (1-q_n^*) \cdot 1\,,$$

which leads immediately to (2) and (3).

Now $M_n^* < 1$ for $n = 1, 2, \dots$ can be shown by complete induction:

For $n = 1$ it follows from (1).

Assume $M_{n-1}^* < 1$. Then with (3) we get

$$M_n^* = \frac{2+M_{n-1}^*}{4-M_{n-1}^*} < \frac{2+1}{4-1} = 1\,,$$

which completes the induction. Therefore, since the same hold for F_n^*, the preference directions in Figure 3 holds for any n as shown there and thus, (2) and (3) represent indeed the mixed equilibrium of the n-step game. □

Furthermore, the probability to get the absurd strategy combination on the n-th step of the game is

$$w_n^* = q_n^* \cdot (1-p_n^*) = \left(\frac{2}{4-M_{n-1}^*}\right)^2, \quad n=1,2,\dots, \tag{4}$$

and the probability that with the n-th step the game is terminated, i.e., for the run length L of the game to be n is

$$P(L=n)=\prod_{i=1}^{n} w_{i-1}^* \cdot (1-w_n^*)\,, \quad n=1,2,\ldots, \quad w_0^*:=1\,. \tag{5}$$

Special values of M_n^*, w_n^* and $P(L=n)$ are given in Table 1.

step n	M_n^*	w_n^*	$P(L=n)$
1	0.2	0.16	0.84
2	0.58	0.28	0.12
3	0.75	0.34	0.029
4	0.85	0.38	0.009
5	0.91	0.40	0.003
∞	1	0.45	0

Table 1: Special values of M_n^*, w_n^* and $P(L=n)$.

We see that $P(L=n)$ decreases rapidly with increasing n which means that the probability for more than two steps is very small.

The explicit solution of the recursive relations (2) and (3) is given by the following

Lemma 2. The mixed equilibrium payoffs to the two players of the n-step game are

$$M_n^*=F_n^*=2-\frac{1}{1-(2/3)^{n+1}}\,, \qquad n=1,2,\ldots\,. \tag{6}$$

Proof. Relation (6) is proven by verifying (2) resp. (3). □

A constructive proof of Lemma 2 is given the Annex.

4 Expected Run Length and Variance

The mass function of the run length distribution, given by (5), represents a complete system of probabilities. We get – if we delete the stars –

$$\begin{aligned}\sum_{n=1}^{m} P(L=n) &= \sum_{n=1}^{m}\prod_{i=1}^{n} w_{i-1}\cdot(1-w_n)\\ &= 1-w_1+w_1\cdot(1-w_2)+w_1\cdot w_2\cdot(1-w_3)+\ldots\\ &\qquad +w_1\cdot\ldots\cdot w_{m-1}\cdot(1-w_m)\\ &= 1-w_1\cdot\ldots\cdot w_m\to 1 \qquad \text{for} \qquad m\to\infty\,.\end{aligned}$$

Therefore, the expected run length $E(L)$ is given by (with $w_0 = 1$)

$$\begin{aligned} E(L) &= \sum_{n=1}^{\infty} n \cdot \prod_{i=1}^{n} w_{i-1} \cdot (1 - w_n) \\ &= 1 - w_1 + 2 \cdot w_1 \cdot (1 - w_2) + 3 \cdot w_1 \cdot w_2 \cdot (1 - w_3) + \cdots \\ &\qquad + n \cdot w_1 \cdot \ldots \cdot w_{n-1} \cdot (1 - w_n) + \ldots \\ &= 1 + w_1 + w_1 \cdot w_2 + w_1 \cdot w_2 \cdot w_3 + \ldots \\ &= 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} w_i \,. \end{aligned}$$

With (4) we get

$$E(L) = 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left(\frac{2}{4 - M^*_{i-1}} \right)^2 . \tag{7}$$

An explicit expression of $E(L)$ is given by

Lemma 3. The expected run length $E(L)$ of our recursive game is

$$E(L) = \frac{1}{4} \cdot \sum_{n=1}^{\infty} \frac{1}{((3/2)^n - 1)^2} \,.$$

Proof. From (3) we get – if we delete the stars –

$$1 - M_n = 1 - \frac{2 + M_{n-1}}{4 - M_{n-1}} = 2 \cdot \frac{1 - M_{n-1}}{4 - M_{n-1}} \,, \quad n = 1, 2, \ldots \,, \quad M_0 = -1 \,,$$

therefore,

$$\frac{1 - M_n}{1 - M_{n-1}} = \frac{2}{4 - M_{n-1}} \,, \quad n = 1, 2, \ldots \,, \quad M_0 = -1 \,.$$

This leads with (7) to

$$\begin{aligned} E(L) &= 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left(\frac{1 - M_i}{1 - M_{i-1}} \right)^2 = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n} (\, 1 - M_i \,)^2}{\prod_{i=1}^{n} (\, 1 - M_{i-1} \,)^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(1 - M_n)^2}{(1 - M_0)^2} = 1 + \frac{1}{4} \cdot \sum_{n=1}^{\infty} (1 - M_n)^2 \,. \end{aligned}$$

With (6) we get

$$E(L) = 1 + \frac{1}{4} \cdot \sum_{n=1}^{\infty} \frac{1}{((3/2)^{n+1} - 1)^2}\,.$$

With $n + 1 = i$ we get

$$E(L) = 1 + \frac{1}{4} \cdot \sum_{i=2}^{\infty} \frac{1}{((3/2)^{i} - 1)^2} = 1 + \frac{1}{4} \cdot \sum_{i=1}^{\infty} \frac{1}{((3/2)^{i} - 1)^2} - \frac{1}{4} \cdot \frac{1}{(3/2 - 1)^2}\,,$$

which completes the proof. □

In a similar way we determine the second moment:

Lemma 4. The second moment of the run length L is

$$E(L^2) = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{n}{((3/2)^{n} - 1)^2} - E(L)\,. \tag{8}$$

Proof. From (5) we get – if we delete the stars –

$$\begin{aligned} E(L^2) &= \sum_{n=1}^{\infty} n^2 \cdot \prod_{i=1}^{n} w_{i-1} \cdot (1 - w_n) \\ &= 1 - w_1 + 4 \cdot w_1 \cdot (1 - w_2) + 9 \cdot w_1 \cdot w_2 \cdot (1 - w_3) + \ldots \end{aligned}$$

which leads to

$$E(L^2) = 1 + \sum_{n=1}^{\infty} (2 \cdot n + 1) \cdot \prod_{i=1}^{n} w_i\,.$$

With (4) we get

$$E(L^2) = 1 + \sum_{n=1}^{\infty} (2 \cdot n + 1) \cdot \prod_{i=1}^{n} \left(\frac{2}{4 - M^*_{i-1}} \right)^2$$

and therefore, in the same way as in the proof of Lemma 3

$$E(L^2) = 1 + \frac{1}{4} \cdot \sum_{n=1}^{\infty} (2 \cdot n + 1) \cdot (1 - M_n)^2 = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{n}{((3/2)^{n+1} - 1)^2} - E(L)\,,$$

which leads again with $n + 1 = i$ to (8). □

Numerically, by use of an appropriate computer program we obtain

$$\begin{aligned} E(L) &\approx 1.2292 \\ E(L^2) &\approx 1.9095 \\ Var(L) &= E(L^2) - (E(L))^2 \approx 0.3984 \\ S(L) &= \sqrt{Var(L)} \approx 0.6312\,. \end{aligned}$$

This confirms what we mentioned already when looking at Table 1, that in general the game will be terminated not later than after the second step.

5 Generalizations

It looks strange that we talked about an absurd strategy combination even though another combination – both go separately to their preferred events – leads to the same payoffs. The reason for the set of payoffs given in Figure 1 was that we wanted to maintain the original payoffs by Luce and Raiffa and by Rapoport. One can see immediately, however, that our results, in particular the recursive formula, would be the same if we chose worse payoffs to both players in the absurd combination than in the one in which both go separately to their preferred events.

The situation changes completely, if we assume altruistic behavior of both players. Let us consider Figure 4.

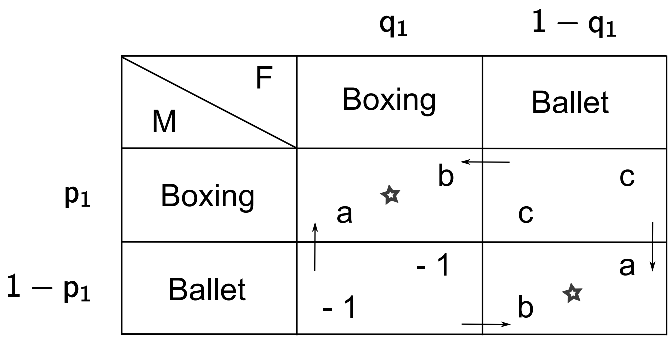

Figure 4: Normal form of the Battle of Sexes paradigm with generalized payoffs.

In order to maintain the equilibrium structure of Figure 1 we assume $-1 < a$ and $c < b$. Then the mixed equilibrium payoffs are

$$M_1^* = F_1^* = \frac{a \cdot b + c}{a + b - c + 1}\,. \tag{9}$$

Let us furthermore assume $-1 < c$ in order to maintain the absurd strategy combination to have the worst preference for both players. Then there remain three cases namely

$$-1 < c < b < a\,, \tag{10}$$

$$-1 < c < a < b\,, \tag{11}$$

$$-1 < a < c < b\,. \tag{12}$$

Now let us consider the same procedure as before, namely that in case the absurd strategy combination would have to be chosen, the second step game is entered with payoffs in the lower left box of the reduced normal form given by (9), and so forth. Then it can be shown that in the first two cases (10) and (11) the game develops in the same way as described in the third and fourth section:

Lemma 5. Under the assumptions (10) and (11) the payoffs of the mixed equilibrium of the n-step game are given by (9) for $n = 1$, and by

$$M_n^* = \frac{a \cdot b - c \cdot M_{n-1}^*}{a + b - c - M_{n-1}^*} = F_n^*\,, \quad n = 2, 3, \ldots\,. \tag{13}$$

Proof. Both (9) and (13) are determined in the same way as (2) and (3) by use of the indifference argument.

Furthermore, we have – if we delete the stars –

$$M_n < \min(a, b)$$

for $n = 1, 2, \ldots$, which is shown by complete induction:

For $n = 1$ it follows from (9).

Assume $M_{n-1} < \min(a, b)$. With (13) the relation $M_n < \min(a, b)$ is equivalent to

$$\frac{a \cdot b - c \cdot M_{n-1}}{a + b - c - M_{n-1}} < \min(a, b)\,,$$

which, with some elementary algebraic manipulations by use of the not commonly known relation $\min(a, b) \cdot (a + b) - a \cdot b = (\min(a, b))^2$, is equivalent to the induction assumption.

Therefore, since the same hold for F_n^*, the preference directions in Figure 4 holds for any n as shown there and thus, (13) represent indeed the payoffs of the mixed equilibrium of the n-step game. □

Let us mention that for the preference directions in Figure 4 to hold also for $n > 1$, we need only $M_n^* = F_n^* < a$. The proof, however, shows a more general upper limit for M_n^* and F_n^*.

In case (12) the game develops completely differently. Since in this case we have

$$a < \frac{a \cdot b + c}{a + b - c + 1} \qquad \text{or equivalently} \qquad a < c\,,$$

the only equilibrium of the 2-step game is the absurd strategy combination! In this case the man likes the boxing fight less than ballet, when considering the boxing fight, and the woman likes the ballet less than the boxing fight, when considering the ballet. Thus, it is not surprising that these payoffs may lead to the strange result indicated above.

This way one interpret the meetings of the French and German delegations mentioned in the second section: Since both delegations behaved altruistically, and since they played in the first morning the (so-called) absurd strategy combination, they played in the second morning the same namely the only equilibrium strategy combination!

6 Concluding Remarks

Only in rare cases it is possible to describe a serious conflict with the help of a normal form game, not only because then unrealistic strategy combinations may have to be considered. If this happens however, then these games may at best describe the initial conflict situation for which no one-shot solution can be given, if no mediator is accepted and no bargaining solution whatsoever can be agreed upon.

Thus one has to model this situation as a game over time. It depends on the concrete conflict if it is appropriate to describe the effort for its solution as a repeated game (in the sense we did it), or if discounting elements have to be introduced, or new moves eventually of a random nature. In any case, as the Sudan conflict showed it may become very difficult to obtain all information necessary for such a more realistic modelling.

7 Acknowledgement

The authors would like to thank the ITIS GmbH for providing the possibility to work on the problems discussed here.

References

[Ave06] R. Avenhaus. French-german official meeting. *PINPoints, IIASA Laxenburg*, 26:11, 2006.

[Deb52] G. Debreu. A social equilibrium exostence theorem. *Proc. Nat. Acad. Sci. USA*, 38:886–893, 1952.

[FT98] D. Fudenberg and J. Tirole. *Game Theory*. The MIT Press, Cambridge, 6th edition, 1998.

[LR57] R. Luce and H. Raiffa. *Games and Decisions*. John Wiley & Sons, New York, 1957.

[Nas50] J. F. Nash. The Bargaining Problem. *Econometrica*, 18:155 – 162, 1950.

[Owe82] G. Owen. *Game Theory*. Academic Press, New York, 2nd edition, 1982.

[Rap74] A. Rapoport. *Fights, Games and Debates*. The University of Michigan Press, Ann Arbor, 1974.

[Smi11] J. Smilovitz. Identity and Mediation. In M. Anstey, P. Meertz, and I. W. Zartman, editors, *Reducing Identity Conflicts and Preventing Genocide*. 2011.

[vD87] E. van Damme. *Stability and Perfection of Nash Equilibria*. Springer-Verlag, Berlin Heidelberg, 1987.

Annex: Constructive proof of Lemma 2

According to (3) we have – if we delete the stars –

$$M_n = \frac{2 + M_{n-1}}{4 - M_{n-1}}\,, \quad n = 1, 2, \ldots\,, \quad M_0 = -1\,.$$

We define

$$T_n := M_n - 2\,, \quad n = 1, 2, \ldots\,, \quad T_0 = -3 \tag{14}$$

and get

$$T_n + 2 = \frac{2 + 2 + T_{n-1}}{4 - T_{n-1} - 2} = \frac{4 + T_{n-1}}{2 - T_{n-1}}$$

or

$$T_n = \frac{4 + T_{n-1} - 4 + 2 \cdot T_{n-1}}{2 - T_{n-1}} = \frac{3 \cdot T_{n-1}}{2 - T_{n-1}}$$

or

$$\frac{1}{T_n} = \frac{2 - T_{n-1}}{3 \cdot T_{n-1}} = \frac{2}{3} \cdot \frac{1}{T_{n-1}} - \frac{1}{3}\,. \tag{15}$$

With

$$S_n := \frac{1}{T_n}\,, \quad n = 1, 2, \ldots\,, \quad S_0 = -\frac{1}{3} \tag{16}$$

we obtain with (15)

$$S_n = \frac{2}{3} \cdot S_{n-1} - \frac{1}{3}\,. \tag{17}$$

With

$$V_n := \left(\frac{3}{2}\right)^n \cdot S_n\,, \quad n = 1, 2, \ldots\,, \quad V_0 = -\frac{1}{3} \tag{18}$$

we obtain with (17)

$$V_n = V_{n-1} - \frac{1}{3} \cdot \left(\frac{3}{2}\right)^n\,, \quad n = 1, 2, \ldots\,.$$

This leads immediately to

$$V_n = -\frac{1}{3} \cdot \sum_{i=0}^{n} \left(\frac{3}{2}\right)^i\,, \quad n = 1, 2, \ldots\,.$$

Therefore, we get with (18)

$$S_n = -\frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \cdot \sum_{i=0}^{n} \left(\frac{3}{2}\right)^i = -\frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \cdot \frac{1-(3/2)^{n+1}}{1-3/2} = \left(\frac{2}{3}\right)^{n+1} - 1\,.$$

With (14) and (16) we finally obtain

$$M_n = \frac{1}{S_n} + 2 = 2 - \frac{1}{1-(2/3)^{n+1}}\,, \quad n = 1, 2, \dots\,,$$

i.e., formula (6).