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Abstract

We consider an extensive form game with incomplete information which consists of several steps of a kind of regular structure. Since the determination of Nash equilibria of this kind of games is straightforward but gets more and more tedious if the number of steps increases, and since for the purpose of applications to the just mentioned models only those equilibria are interesting which do not imply a premature end of the game at one of the intermediate dead ends, an algorithm is considered the intuitive meaning of which may be described as a learning procedure: If the player, who has only incomplete knowledge of this adversary's type, has reached some information set of the game, he is supposed to know more about the type of the latter one than in the beginning of the game due to the fact that the latter one behaved in some specific way. Working out this idea one can devise a kind of backward induction procedure which, however, contains some recursive elements. It is the purpose of this note to describe this algorithm and furthermore, to show that it leads to *all* Nash equilibria of the game with the property that all information sets are reached during the course of the game.

In addition, it is shown that this algorithm can also be applied to extensive form games with imperfect information which have a similar tree structure as those considered before.

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1 Problem Formulation

We consider an extensive form game with incomplete information which consists of several steps of a kind of regular structure. An example is given in Figure 1 which ist taken from Beetz [Bee05] who discusses various types of arms race and escalation models.

Figure 1 about here.

Since the determination of Nash equilibria of this kind of games with the help of the equilibrium conditions is straightforward but gets more and more tedious if the number of steps increases, and since for the purpose of applications to the just mentioned models only those equilibria are interesting which do not imply a premature end of the game at one of the intermediate dead ends, an algorithm is considered the intuitive meaning of which may be described as a learning procedure: If the player, who has only incomplete knowledge of this adversary's type, has reached some information set of the game, he is supposed to know more about the type of the latter one than in the beginning of the game due to the fact that the latter one behaved in some specific way.

Working out this idea one can devise a kind of backward induction procedure which, however, contains some recursive elements. It is the purpose of this note to describe this algorithm and furthermore, to show that it leads to *all* Nash equilibria of the game with the property that all information sets are reached.

In the following, a simple example is considered which first is analyzed in the standard way. Thereafter, the algorithm is presented at the hand of this example, and it is shown that in this way indeed all Nash equilibria under consideration are obtained. It is also shown, however, that the intuitive idea of learning is not so straightforward as one might expect.

In the subsequent section it is proven that under appropriate assumptions this algorithm works as promised.

In the fourth chapter it is shown that this algorithm can also be applied to extensive form games with imperfect information which have a similar tree structure as those considered before.

In the concluding section we suppose that our algorithm is equivalent to determining the so-called Perfect Bayesian Equilibria (PBE) for games with incomplete information, as defined by Fudenberg and Tirole [FT98] respectively, to determining sequential equilibria as introduced by Kreps and Wilson [KW82].

2 Presentation of the algorithm with the help of an illustrative example

Consider a non-cooperative two-person game with incomplete information the extensive form of which is represented graphically in Figure 2.

Figure 2 about here.

In this game, in the beginning nature chooses type A of the first player with probability γ, and with probability $1-\gamma$ type B. Now, the first player decides either to go left or right. In the first case the game ends; in the second case the second player, who only knows the probabilities for the first player being of type A or B, decides either to go left or right.

It should be mentioned here that the game as given by Figure 2 is not exactly of the same type as that given by Figure 1. In the latter one the second player is represented by two types, and the first player moves first thus, he can update his knowledge on the adversary´s type only when he moves a second time. This is the reason for our change of models: For the purpose of illustration we want to consider as simple a model as possible, i.e., a model with only one non-trivial information set.

Payoffs for all outcomes of the game considered now are given in Figure 2. It is important to realize $0<\gamma<1$ according to general understanding since in the two extreme cases the game is in fact a game with complete information.

Following Harsanyi [Har67] we represent the first player by two agents given by type A and type B. The expected payoffs to the - now - three players in a game with imperfect information are

$$\begin{aligned} H_{1_A}(p_A;q) &= 1+p_A\,(2-5\,q) && (1)\\ H_{1_B}(p_B;q) &= 1+p_B\,(1-q) && (2)\\ H_2(p_A,p_B;q) &= \gamma\,p_A\,[\,-(1-q)+4\,q\,]+(1-\gamma)\,p_B\,[\,2\,(1-q)-2\,q\,]\\ &= 2\,(1-\gamma)\,p_B-\gamma\,p_A+q\,[\,5\,\gamma\,p_A-4\,(1-\gamma)\,p_B\,]\,. && (3)\end{aligned}$$

According to these expected payoffs, Nash equilibria (p_A^*, p_B^*, q^*) of our game are defined by the following inequalities:

$$p_A^* \,(2 - 5q^*) \;\geq\; p_A \,(2 - 5\,q^*) \qquad \forall p_A \tag{4}$$
$$p_B^* \,(1 - q^*) \;\geq\; p_B \,(1 - q^*) \qquad \forall p_B \tag{5}$$
$$q^* \,[\,5\,\gamma\, p_A^* - 4\,(1-\gamma)\, p_B^*\,] \;\geq\; q\,[\,5\,\gamma\, p_A^* - 4\,(1-\gamma)\, p_B^*\,] \qquad \forall q\,. \tag{6}$$

In the following we determine all Nash equilibria first with a systematic conventional method and thereafter with our new algorithm.

2.1 Systematic determination of all Nash equilibria

We determine all Nash equilibria of our game by proceeding in a systematic way:

1. With (4) and (5) the case $q^* = 1$ implies $p_A^* = 0$ and p_B^* arbitrary. From inequality (6) we obtain $-4\,(1-\gamma)\,p_B^* \geq q\,[\,-4\,(1-\gamma)\,p_B^*\,]$ for all q, which implies $p_B^* = 0$. Note that this holds for all $\gamma \in (0,1)$.

2. With (4) and (5) the case $q^* = 2/5$ implies p_A^* arbitrary and $p_B^* = 1$. From inequality (6) we obtain $2/5\,[\,5\,\gamma\,p_A^* - 4\,(1-\gamma)\,] \geq q\,[\,5\,\gamma\,p_A^* - 4\,(1-\gamma)\,]$ for all q, which implies $5\,\gamma\,p_A^* - 4\,(1-\gamma) = 0$. This is equivalent to

$$p_A^* = \frac{4}{5}\,\frac{1-\gamma}{\gamma}\,.$$

   For $p_A^* \leq 1$ it is required $\gamma \geq \frac{4}{9}$.

3. With (4) and (5) the case $\frac{2}{5} < q^* < 1$ implies $p_A^* = 0$ and $p_B^* = 1$. From inequality (6) we obtain $q^*\,[\,-4\,(1-\gamma)\,] \geq q\,[\,-4\,(1-\gamma)\,]$ for all q, which implies $q^* = 0$ and therefore a contradiction to $\frac{2}{5} < q^* < 1$.

4. With (4) and (5) the case $0 < q^* < \frac{2}{5}$ implies $p_A^* = p_B^* = 1$. From inequality (6) we obtain $q^*\,[\,5\,\gamma - 4\,(1-\gamma)\,] \geq q\,[\,5\,\gamma - 4\,(1-\gamma)\,]$ for all q, which implies $5\,\gamma - 4\,(1-\gamma) = 0$ and therefore $\gamma = \frac{4}{9}$.

5. With (4) and (5) the case $q^* = 0$ implies $p_A^* = p_B^* = 1$. From inequality (6) we obtain $0 \geq q\,[\,5\,\gamma - 4\,(1-\gamma)\,]$ for all q, which is fulfilled in case of $\gamma \leq \frac{4}{9}$.

Let us summarize these results by ordering them according to their dependence on the probability γ:

(i) γ arbitrary:

$$p_A^* = p_B^* = 0, q^* = 1 \quad \text{with} \quad H_{1_A}^* = H_{1_B}^* = 1, H_2^* = 0\,, \tag{7}$$

(ii) $\gamma < \frac{4}{9}$:

$$p_A^* = p_B^* = 1, q^* = 0 \quad \text{with} \quad H_{1_A}^* = 3, H_{1_B}^* = 2, H_2^* = 2 - 3\gamma\,, \tag{8}$$

(iii) $\gamma = \frac{4}{9}$:

$$p_A^* = p_B^* = 1, 0 < q^* < \frac{2}{5} \quad \text{with} \quad H_{1_A}^* = 3 - 5\,q^*, H_{1_B}^* = 2 - q^*, H_2^* = 2 - 3\gamma\,, \tag{9}$$

(iv) $\gamma > \frac{4}{9}$:

$$p_A^* = \frac{4}{5}\,\frac{1-\gamma}{\gamma}, p_B^* = 1, q^* = \frac{2}{5} \quad \text{with} \quad H_{1_A}^* = 1, H_{1_B}^* = \tfrac{8}{5}, H_2^* = \tfrac{6}{5}\,(1-\gamma)\,. \tag{10}$$

It should be mentioned that equilibrium (i) is *payoff*-dominated by the other ones.

2.2 The new algorithm

Let us consider now only equilibria with $p_A^* > 0$ and $p_B^* > 0$ and develop our algorithm for the determination of Nash equilibria as follows:

In the beginning it is known to the second player that the first player is of type A with probability γ. If the former one has to make his decision, i.e., only in the case that the latter one decides to go right, he can *update* his knowledge and work with a revised probability β for the first player being of type A:

Let us define the events

A: player 1 is of type A,

s_{11}**:** player 1 chooses s_{11}.

Then we get with the help of the Bayesian formula

$$\beta = prob(A|s_{11}) = \frac{prob(s_{11}|A)\cdot prob(A)}{prob(s_{11})} = \frac{p_A^*\,\gamma}{(1-\gamma)\,p_B^* + \gamma\,p_A^*}\,. \tag{11}$$

Accordingly, $1-\beta$ is the revised probability for the first player being of type B. Thus, we consider now the conditional expected payoff to player 2, i.e., the expected payoff if the game would start in the information set of the second player and the probability of being in the left node is β,

$$H_2(q;\beta) = \beta\,[\,-(1-q) + 4\,q\,] + (1-\beta)\,[\,2\,(1-q) - 2\,q\,] = 2 - 3\,\beta + q\,(9\,\beta - 4)\,, \tag{12}$$

which, of course, he wants to maximize as regards to q, as well as the two players A and B want to maximize their expected payoffs as regards to their strategies.

With (4),(5) and (12) we get immediately conditions for the optimal strategies of all players as follows:

$$q^* = \begin{Bmatrix} 1 \\ \text{arbitrary} \\ 0 \end{Bmatrix} \quad \text{for} \quad \beta \begin{Bmatrix} > 4/9 \\ = 4/9 \\ < 4/9 \end{Bmatrix} \tag{13}$$

$$p_A^* = \begin{Bmatrix} 1 \\ \text{arbitrary} \\ 0 \end{Bmatrix} \quad \text{for} \quad q^* \begin{Bmatrix} < 2/5 \\ = 2/5 \\ > 2/5 \end{Bmatrix} \tag{14}$$

$$p_B^* = \begin{Bmatrix} 1 \\ \text{arbitrary} \end{Bmatrix} \quad \text{for} \quad q^* \begin{Bmatrix} < 1 \\ = 1 \end{Bmatrix} . \tag{15}$$

Now we have to perform consistency checks, keeping in mind the meaning of β as given by (11).

1. $q^* = 1$ implies $\beta > 4/9$, $p_A^* = 0$ and $p_B^* > 0$ ($p_B^* = 0$ excluded by assumption). This is a contradiction to (11).
2. $q^* = 0$ implies $\beta < 4/9$, $p_A^* = p_B^* = 1$ and therefore $\beta = \gamma < 4/9$.
3. $q^* = 2/5$ implies $\beta = 4/9$, $p_A^* > 0$ ($p_A^* = 0$ excluded by assumption) and $p_B^* = 1$ and therefore with (11)

   $$\frac{4}{9} = \frac{\gamma\, p_A^*}{(1-\gamma) + \gamma\, p_A^*} \Longleftrightarrow p_A^* = \frac{4}{5}\,\frac{1-\gamma}{\gamma} .$$

   Since $0 < p_A^* \leq 1$ we obtain $1 > \gamma \geq 4/9$.
4. $0 < q^* < 2/5$ implies $\beta = 4/9$, $p_A^* = p_B^* = 1$ and therefore $\beta = \gamma = 4/9$.
5. $2/5 < q^* < 1$ implies $\beta = 4/9$, $p_A^* = 0$ and $p_B^* = 1$. This is a contradiction to (11).

Thus, we get the same equilibria as before ((8) to (10)), and the question arises if we can prove this directly.

In our example this question can be answered easily. According to (3) and (6), we have to maximize the following expected payoff to the second player

$$\gamma\, p_A^* \left[-(1-q) + 4\,q \right] + (1-\gamma)\, p_B^* \left[2\,(1-q) - 2\,q \right]$$

with respect to $q \in [0, 1]$. However, because of our assumptions $p_A^* > 0$ and $p_B^* > 0$, this is equivalent to maximizing the following form

$$\frac{1}{(1-\gamma)\, p_B^* + \gamma\, p_A^*} \left(\gamma\, p_A^* \left[-(1-q) + 4\,q \right] + (1-\gamma)\, p_B^* \left[2\,(1-q) - 2\,q \right] \right) ,$$

with respect to $q \in [0,1]$, which is equivalent to determining q^* such that (12) is maximized.

In the next section we will discuss this issue in a more general way.

Let us close this section with a remark on the updating idea. In our case, we get from (8) to (10):

$$\gamma \leq \frac{4}{9} \qquad \text{implies} \qquad \beta = \gamma$$
$$\gamma > \frac{4}{9} \qquad \text{implies} \qquad \beta = \frac{\gamma \cdot \frac{4}{5} \cdot \frac{1-\gamma}{\gamma}}{1 - \gamma + \gamma \cdot \frac{4}{5} \cdot \frac{1-\gamma}{\gamma}} = \frac{4}{9}\,.$$

In the first case in fact there is no updating. The second case, however, is difficult to understand: Let us assume γ to be close to one. This means that the second player is nearly sure that the first player is of type A. The *updating* nevertheless leads to $\beta = 4/9$, i.e., nearly equal probabilities for the first player to be A or B!

Thus, updating does not mean necessarily, that the a-priori probabilities for the first player's types starting with γ and $1-\gamma$ get closer to 0 and 1 respectively 1 and 0.

3 Theoretical considerations

In this section we show for games like that in Figure 1 that our algorithm leads always to a Nash equilibrium and that every Nash equilibrium in which the last information set is reached with positive probability can be obtained with the help of our algorithm. Let us consider the game in Figure 3 with four stages, which is a special case of the game in Figure 1.

Figure 3 about here.

Let

$$H_{2_A}(y_1, x_2) := (1 - y_1)\, f_1 + y_1\, [(1 - x_2)\, b_2 + x_2\, f_2] \tag{16}$$

be the conditional expected payoff for the second player if in the course of the game node n_1 is reached and

$$H_{2_B}(z_1, x_2) := (1 - z_1)\, h_1 + z_1\, [(1 - x_2)\, d_2 + x_2\, h_2] \tag{17}$$

be the conditional expected payoff for the second player if in the course of the game node n_2 is reached.

Let us describe our algorithm which consists of four steps for this example in a formal way:

1. We consider the first player in his information set I_2 and assume that the probability to be in the left node of I_2 is $\beta_2 \in [0,1]$. The first player determines x_2^* fulfilling

$$H_1^{(I_2)}(\beta_2; x_2^*) = \max_{x_2 \in [0,1]} H_1^{(I_2)}(\beta_2; x_2) \tag{18}$$

with

$$H_1^{(I_2)}(\beta_2; x_2) := \beta_2 \left[(1-x_2)\, a_2 + x_2\, e_2\right] + (1-\beta_2) \left[(1-x_2)\, c_2 + x_2\, g_2\right].$$

The result of this optimization is $x_2^*(\beta_2)$.

2. Now we consider player 2_A and 2_B. For every $\beta_2 \in [0,1]$ and a $x_2^*(\beta_2)$ fulfilling (18) the player chooses strategies y_1^* and z_1^* maximizing the conditional payoffs defined in (16) and (17), i.e.,

$$H_{2_A}(y_1^*, x_2^*(\beta_2)) = \max\nolimits_{y_1 \in [0,1]} H_{2_A}(y_1, x_2^*(\beta_2)) \qquad \text{and}$$
$$H_{2_B}(z_1^*, x_2^*(\beta_2)) = \max\nolimits_{z_1 \in [0,1]} H_{2_B}(z_1, x_2^*(\beta_2)). \tag{19}$$

   The result of this optimization is $y_1^*(x_2^*(\beta_2))$ and $z_1^*(x_2^*(\beta_2))$.

3. In the last step we consider the first player in his information set I_1 and assume that the probability to be in the left node of I_1 is $\beta_1 \in [0,1]$. If we use the abbreviations

$$y_1^* := y_1^*(x_2^*(\beta_2)),\, z_1^* := z_1^*(x_2^*(\beta_2)) \qquad \text{and} \qquad x_2^* := x_2^*(\beta_2) \tag{20}$$

   the first player determines x_1^* fulfilling

$$H_1^{(I_1)}(\beta_1; x_1^*, y_1^*, z_1^*, x_2^*) = \max_{x_1 \in [0,1]} H_1^{(I_1)}(\beta_1; x_1, y_1^*, z_1^*, x_2^*) \tag{21}$$

   with

$$\begin{aligned} &H_1^{(I_1)}(\beta_1; x_1, y_1^*, z_1^*, x_2^*) := \\ &\quad \beta_1 \left[(1-x_1)\, a_1 + x_1 \left\{(1-y_1^*)\, e_1 + y_1^* \left[(1-x_2^*)\, a_2 + x_2^*\, e_2\right]\right\}\right] \\ &\quad +(1-\beta_1) \left[(1-x_1)\, c_1 + x_1 \left\{(1-z_1^*)\, g_1 + z_1^* \left[(1-x_2^*)\, c_2 + x_2^*\, g_2\right]\right\}\right]. \end{aligned} \tag{22}$$

The result of this procedure leads for every pair $(\beta_1, \beta_2) \in [0,1] \times [0,1]$ to the set

$$\begin{aligned} Opt(\beta_1, \beta_2) := \{(x_1^*, y_1^*, z_1^*, x_2^*) \;:\; & x_2^* = x_2^*(\beta_2), y_1^* = y_1^*(x_2^*(\beta_2)), z_1^* = z_1^*(x_2^*(\beta_2)), \\ & x_1^* = x_1^*(\beta_1, y_1^*(x_2^*(\beta_2)), z_1^*(x_2^*(\beta_2)), x_2^*(\beta_2))\}. \end{aligned}$$

We call an element of the set $Opt(\beta_1, \beta_2)$ **consistent** iff

(i) $x_1^*(\beta_1, y_1^*(x_2^*(\beta_2)), z_1^*(x_2^*(\beta_2)), x_2^*(\beta_2)) \in (0,1)$ and $y_1^*(x_2^*(\beta_2)) + z_1^*(x_2^*(\beta_2)) > 0$,

(ii) $\beta_1 = \gamma$ and with $x_1^* := x_1^*(\beta_1, y_1^*(x_2^*(\beta_2)), z_1^*(x_2^*(\beta_2)), x_2^*(\beta_2))$

$$\begin{aligned} \beta_2 &= \frac{\gamma \cdot x_1^* \cdot y_1^*}{\gamma \cdot x_1^* \cdot y_1^*(x_2^*(\beta_2)) + (1-\gamma) \cdot x_1^* \cdot z_1^*(x_2^*(\beta_2))} \\ &= \frac{\gamma \cdot y_1^*(x_2^*(\beta_2))}{\gamma \cdot y_1^*(x_2^*(\beta_2)) + (1-\gamma) \cdot z_1^*(x_2^*(\beta_2))}. \end{aligned}$$

The first condition implies, that the game always reaches the information set I_2 and implies as well that $\beta_2 > 0$. The second condition is what we have called the consistency check: the arbitrary chosen β_1 and β_2 have to fulfill the Bayesian rule. Therefore the last step of our algorithm is

4. We perform the just mentioned consistency check (i) and (ii).

Now we prove

Lemma 1. Let us given a pair $(\beta_1, \beta_2) \in [0,1] \times [0,1]$. Then every consistent element of $Opt(\beta_1, \beta_2)$ is a Nash equilibrium of the entire game.

Proof. The (unconditional) expected payoff for the second player can be written using definitions (16) and (17) as

$$H_2(x_1, y_1, z_1, x_2) := \gamma \left[(1 - x_1)\, b_1 + x_1\, H_{2_A}(y_1, x_2)\right] + (1 - \gamma)\left[(1 - x_1)\, d_1 + x_1\, H_{2_B}(z_1, x_2)\right].$$

Let $(x_1^*, y_1^*, z_1^*, x_2^*)$ be a consistent element of $Opt(\beta_1, \beta_2)$. We obtain from (19) using the abbreviations (20)

$$\begin{aligned} & H_2(x_1^*, y_1^*, z_1^*, x_2^*) \\ & = \gamma \left[(1 - x_1^*)\, b_1 + x_1^*\, H_{2_A}(y_1^*, x_2^*)\right] + (1 - \gamma)\left[(1 - x_1^*)\, d_1 + x_1^*\, H_{2_B}(z_1^*, x_2^*)\right] \\ & \geq \gamma \left[(1 - x_1^*)\, b_1 + x_1^*\, H_{2_A}(y_1, x_2^*)\right] + (1 - \gamma)\left[(1 - x_1^*)\, d_1 + x_1^*\, H_{2_B}(z_1, x_2^*)\right] \\ & = H_2(x_1^*, y_1, z_1, x_2^*) \end{aligned}$$

for all $y_1, z_1 \in [0,1]$, i.e., the Nash condition for the second player is fulfilled. The (unconditional) expected payoff for the first player is defined by

$$\begin{aligned} H_1(x_1, y_1, z_1, x_2) \;:=\; & \gamma \left[(1 - x_1)\, a_1 + x_1 \left\{(1 - y_1)\, e_1 + y_1 \left[(1 - x_2)\, a_2 + x_2\, e_2\right]\right\}\right] \\ & + (1 - \gamma)\left[(1 - x_1)\, c_1 + x_1 \left\{(1 - z_1)\, g_1 + z_1 \left[(1 - x_2)\, c_2 + x_2\, g_2\right]\right\}\right]. \end{aligned} \tag{23}$$

From (ii) we get $\beta_1 = \gamma$ and with (22) and (21)

$$H_1(x_1^*, y_1^*, z_1^*, x_2^*) = H_1^{(I_1)}(\gamma; x_1^*, y_1^*, z_1^*, x_2^*) \geq H_1^{(I_1)}(\gamma; x_1, y_1^*, z_1^*, x_2^*) = H_1(x_1, y_1^*, z_1^*, x_2^*) \tag{24}$$

for all $x_1 \in [0,1]$. Let us define

$$f(\gamma; x_1, y_1, z_1) := \gamma \left[(1 - x_1)\, a_1 + x_1\, (1 - y_1)\, e_1\right] + (1 - \gamma)\left[(1 - x_1)\, c_1 + x_1\, (1 - z_1)\, g_1\right],$$

then we have

$$\begin{aligned} H_1(x_1, y_1, z_1, x_2) \;=\; & f(\gamma; x_1, y_1, z_1) \\ + \; & \gamma\, x_1\, y_1 \left[(1 - x_2)\, a_2 + x_2\, e_2\right] + (1 - \gamma)\, x_1\, z_1 \left[(1 - x_2)\, c_2 + x_2\, g_2\right]. \end{aligned}$$

We want to prove the inequality

$$H_1(x_1, y_1^*, z_1^*, x_2^*) \geq H_1(x_1, y_1^*, z_1^*, x_2)$$

for all $x_1, x_2 \in [0,1]$. First we assume that $x_1 > 0$. With

$$N := N(x_1, y_1^*, z_1^*) := \frac{1}{\gamma\, x_1\, y_1^* + (1-\gamma)\, x_1\, z_1^*} \quad (>0)$$

we obtain with (ii)

$$N\,\gamma\, x_1\, y_1^* = \frac{\gamma\, x_1\, y_1^*}{\gamma\, x_1\, y_1^* + (1-\gamma)\, x_1\, z_1^*} = \beta_2$$

and

$$N\,(1-\gamma)\, x_1\, z_1^* = \frac{(1-\gamma)\, x_1\, z_1^*}{\gamma\, x_1\, y_1^* + (1-\gamma)\, x_1\, z_1^*} = 1-\beta_2\,.$$

Using (18) we get

$$\begin{aligned} & N\,H_1(x_1, y_1^*, z_1^*, x_2^*) \\ &= N\,f(\gamma; x_1, y_1^*, z_1^*) \\ &\quad + N\,\{\gamma\, x_1\, y_1^*\,[(1-x_2^*)\,a_2 + x_2^*\,e_2] + (1-\gamma)\,x_1\,z_1^*\,[(1-x_2^*)\,c_2 + x_2^*\,g_2]\} \\ &= N\,f(\gamma; x_1, y_1^*, z_1^*) + \beta_2\,[(1-x_2^*)\,a_2 + x_2^*\,e_2] + (1-\beta_2)\,[(1-x_2^*)\,c_2 + x_2^*\,g_2] \\ &= N\,f(\gamma; x_1, y_1^*, z_1^*) + H_1^{(I_2)}(\beta_2; x_2^*) \\ &\geq N\,f(\gamma; x_1, y_1^*, z_1^*) + H_1^{(I_2)}(\beta_2; x_2) \\ &= N\,H_1(x_1, y_1^*, z_1^*, x_2) \end{aligned}$$

for all $x_2 \in [0,1]$ and $x_1 > 0$. If $x_1 = 0$ then $H_1(x_1, y_1^*, z_1^*, x_2^*) = f(\gamma; x_1, y_1^*, z_1^*) = H_1(x_1, y_1^*, z_1^*, x_2)$ for all $x_2 \in [0,1]$. Finally we obtain with (24)

$$H_1(x_1^*, y_1^*, z_1^*, x_2^*) \geq H_1(x_1, y_1^*, z_1^*, x_2^*) \geq H_1(x_1, y_1^*, z_1^*, x_2)$$

for all $x_1, x_2 \in [0,1]$, i.e., the Nash condition for the first player is also fulfilled. This shows that $(x_1^*, y_1^*, z_1^*, x_2^*)$ is a Nash equilibrium of the entire game. □

Since the application of our algorithm to the game in Figure 3 leads always to a Nash equilibrium, the question arises, if there are other Nash equilibria in which the information set I_2 is reached with positive probability and which cannot be determined with our algorithm. The next Lemma gives the answer.

Lemma 2. Let $(x_1^*, y_1^*, z_1^*, x_2^*)$ be a Nash equilibrium with $x_1^* > 0$ and $y_1^* + z_1^* > 0$. Then

$$(x_1^*, y_1^*, z_1^*, x_2^*) \in Opt(\beta_1, \beta_2) \quad \text{with} \quad \beta_1 = \gamma \quad \text{and} \quad \beta_2 = \frac{\gamma\, y_1^*}{\gamma\, y_1^* + (1-\gamma)\, z_1^*}\,. \tag{25}$$

Proof. Let β_2 be as in (25). Since

$$H_1(x_1^*, y_1^*, z_1^*, x_2^*) \geq H_1(x_1^*, y_1^*, z_1^*, x_2)$$

for all $x_2 \in [0,1]$, we get with

$$M := \frac{1}{\gamma\, x_1^*\, y_1^* + (1-\gamma)\, x_1^*\, z_1^*} \quad (>0)$$

the inequality

$$\begin{aligned} M\,f(\gamma;x_1^*,y_1^*,z_1^*) + H_1^{(I_2)}(\beta_2;x_2^*) &= M\,H_1(x_1^*,y_1^*,z_1^*,x_2^*) \\ &\geq M\,H_1(x_1^*,y_1^*,z_1^*,x_2) \\ &= M\,f(\gamma;x_1^*,y_1^*,z_1^*) + H_1^{(I_2)}(\beta_2;x_2)\,. \end{aligned}$$

It follows $H_1^{(I_2)}(\beta_2;x_2^*) \geq H_1^{(I_2)}(\beta_2;x_2)$ for all $x_2 \in [0,1]$. This implies that x_2^* fulfills the condition (18) with β_2 from (25). For the second player we obtain from the Nash condition

$$\begin{aligned} & \gamma\,[(1-x_1^*)\,b_1 + x_1^*\,H_{2_A}(y_1^*,x_2^*)] + (1-\gamma)\,[(1-x_1^*)\,d_1 + x_1^*\,H_{2_B}(z_1^*,x_2^*)] \\ = \;& H_2(x_1^*,y_1^*,z_1^*,x_2^*) \\ \geq \;& H_2(x_1^*,y_1,z_1^*,x_2^*) \\ = \;& \gamma\,[(1-x_1^*)\,b_1 + x_1^*\,H_{2_A}(y_1,x_2^*)] + (1-\gamma)\,[(1-x_1^*)\,d_1 + x_1^*\,H_{2_B}(z_1^*,x_2^*)] \end{aligned}$$

for all $y_1 \in [0,1]$ and

$$\begin{aligned} & \gamma\,[(1-x_1^*)\,b_1 + x_1^*\,H_{2_A}(y_1^*,x_2^*)] + (1-\gamma)\,[(1-x_1^*)\,d_1 + x_1^*\,H_{2_B}(z_1^*,x_2^*)] \\ = \;& H_2(x_1^*,y_1^*,z_1^*,x_2^*) \\ \geq \;& H_2(x_1^*,y_1^*,z_1,x_2^*) \\ = \;& \gamma\,[(1-x_1^*)\,b_1 + x_1^*\,H_{2_A}(y_1^*,x_2^*)] + (1-\gamma)\,[(1-x_1^*)\,d_1 + x_1^*\,H_{2_B}(z_1,x_2^*)] \end{aligned}$$

for all $z_1 \in [0,1]$. From the first inequality we obtain $H_{2_A}(y_1^*,x_2^*) \geq H_{2_A}(y_1,x_2^*)$ for all $y_1 \in [0,1]$ and from the second inequality $H_{2_B}(z_1^*,x_2^*) \geq H_{2_B}(z_1,x_2^*)$ for all $z_1 \in [0,1]$. Both inequalities are equivalent to the maximization in (19). Since $\beta_1 = \gamma$ we obtain from (23) and (22) that x_1^* solves the problem (21). □

The Lemma shows that indeed all Nash equilibria with the property that all information sets are reached with positive probability can be obtained with our algorithm.

4 Imperfect Information

Even though our algorithm is designed for games with incomplete information, it can also be applied to games with genuinely imperfect information, i.e., not only to those which are originally games with incomplete information and have been transformed into those with imperfect information.

In order to demonstrate this, let us consider the extensive form game with imperfect information as given in Figure 4.

Figure 4 about here.

It is very close to the game given by Figure 2, the decisive difference being, that we have only two players, 1 and 2. The first player first decides to choose A or B with probability

p and $1-p$, and thereafter left or right with probabilities $1-p_A$ and p_A in case he chooses A, and with probabilities $1-p_B$ and p_B in case he chooses B. The second player decides in the same way as in the original game.

Again, we determine the Nash equilibria of this game with the conventional systematic method and thereafter, with the new algorithm.

4.1 Systematic determination of all Nash equilibria

According to Figure 4 the expected payoffs to both players are given by

$$\begin{aligned} H_1(p, p_A, p_B; q) &= p\,[\,(1-p_A) + p_A\,((1-q)\,3 + q\,(-2))\,] \\ &\quad +(1-p)\,[\,(1-p_B) + p_B\,((1-q)\,2 + q\,] \\ &= p\,[\,1 + p_A\,(2-5\,q)\,] + (1-p)\,[\,1 + p_B\,(1-q)\,] \end{aligned} \tag{26}$$

and

$$\begin{aligned} H_2(p, p_A, p_B; q) &= p\,p_A\,[\,(1-q)\,(-1) + q\,4\,] + (1-p)\,p_B\,[\,(1-q)\,2 + q\,(-2)\,] \\ &= -p\,p_A + 2\,(1-p)\,p_B + q\,[\,5\,p\,p_A - 4\,(1-p)\,p_B\,]\,. \end{aligned} \tag{27}$$

The Nash condition for the equilibria $(p^*, p_A^*, p_B^*; q^*)$ of this game are therefore given by

$$p^*\,p_A^*\,(2-5\,q^*) + (1-p^*)\,p_B^*\,(1-q^*) \geq p\,p_A\,(2-5\,q^*) + (1-p)\,p_B\,(1-q^*) \tag{28}$$

for all p, p_A, p_B, and

$$q^*\,[\,5\,p^*\,p_A^* - 4\,(1-p^*)\,p_B^*\,] \geq q\,[\,5\,p^*\,p_A^* - 4\,(1-p^*)\,p_B^*\,] \tag{29}$$

for all q. From (29) we get immediately

$$q^* = \left\{\begin{matrix} 1 \\ \text{arbitrary} \\ 0 \end{matrix}\right\} \quad \text{for} \quad 5\,p^*\,p_A^* - 4\,(1-p^*)\,p_B^* \left\{\begin{matrix} >0 \\ =0 \\ <0 \end{matrix}\right\}. \tag{30}$$

In case of (28) we have to consider the cases

$$2-5\,q^* > 1-q^*\,, \quad 2-5\,q^* = 1-q^* \quad \text{and} \quad 2-5\,q^* < 1-q^*$$

which is equivalent to

$$q^* < 1/4\,, \quad q^* = 1/4 \quad \text{and} \quad q^* > 1/4\,.$$

We get from (28)

1. for $q^* < 1/4$: $p^* = p_A^* = 1$ and p_B^* arbitrary,
2. for $q^* = 1/4$: $p^*\,p_A^* + (1-p^*)\,p_B^* = 1$, and

3. for $q^* > 1/4$: $p^* = 0$, $p_B^* = 1$ and p_A^* arbitrary.

We compare this with (30):

1. For $q^* < 1/4$ we get $5\,p^*\,p_A^* - 4\,(1-p^*)\,p_B^* = 5$, which is a contradiction to (30).
2. For $q^* = 1/4$ we get
$$p^*\,p_A^* + (1-p^*)\,p_B^* = 1$$
and from (30) we get
$$5\,p^*\,p_A^* - 4\,(1-p^*)\,p_B^*$$
which implies
$$(1-p^*)\,p_B^* = \frac{5}{9} \quad \text{and} \quad p^*\,p_A^* = \frac{4}{9}$$
and finally, as can be shown by contradiction,
$$p^* = \frac{4}{9}\,, \quad p_A^* = p_B^* = 1 \quad \text{and} \quad q^* = \frac{1}{4}\,. \tag{31}$$
3. For $q^* > 1/4$ we get $5\,p^*\,p_A^* - 4\,(1-p^*)\,p_B^* = -4$, which is a contradiction to (30).

Thus, there is just one unique equilibrium (31). Also it should be noted that $p_A^* = p_B^* = 0$ is *not* a Nash equilibrium.

4.2 The new algorithm

In the same way as in section 2.2 we argue that player 2 knows, once he has to make a decision, that he is at the left node of his information set with probability β, where

$$\beta = \frac{p_A\,p}{(1-p)\,p_B + p\,p_A} \tag{32}$$

and accordingly with probability $1-\beta$ at the right node. Thus the expected payoff, conditioned to β, is

$$\begin{aligned} H_2(\beta, q) &= \beta\,[\,(1-q)\,(-1) + q\,4\,] + (1-\beta)\,[\,(1-q)\,2 + q\,(-2)\,] \\ &= 2 - 3\,\beta + q\,(9\,\beta - 4)\,. \end{aligned} \tag{33}$$

In equilibrium, therefore, we have

$$q^* = \left\{ \begin{matrix} 1 \\ \text{arbitrary} \\ 0 \end{matrix} \right\} \quad \text{for} \quad \beta \left\{ \begin{matrix} > 4/9 \\ = 4/9 \\ < 4/9 \end{matrix} \right\}\,. \tag{34}$$

For the first player we get again the first three cases in the previous subsection which lead us to the following consistency check:

1. For $q^* < 1/4$ we get $\beta^* = 1$ which implies with (34) $q^* = 1$, which is a contradiction.
2. For $q^* = 1/4$ we get $\beta^* = p^* \, p_B^* = 4/9$ which implies $(1 - p^*) \, p_B^* = 5/9$.
3. For $q^* > 1/4$ we get $\beta^* = 0$ which implies $q^* = 0$, which is a contradiction.

Thus, we are led again to the same equilibrium (31), which we obtained before.

It should be noted that in this game the new algorithm works very similar to the conventional method thus, for other and more complicated games it remains to be shown what the relative advantages, e.g., as regards to computational effort, of both methods are.

5 Conclusions

The considerations in the third section can be generalized to an arbitrary numbers of stages. In essence the statements of Lemma 1 and Lemma 2 can be formulated appropriately. In the proof we can use a theorem given in van Damme [vD87].

To work out the proofs of the generalized Lemmata, however, is strenuous, let alone the notational problems. Before tackling this task, some conceptual problems have to be clarified which will be mentioned now.

In this paper we develop for the class of game theoretical models which is described in the introduction, an algorithm for the determination of all Nash equilibria with the property that all information sets are reached during the course of the game. Our algorithm represents a kind of backward induction with the surprising property, that in the course of the calculations information sets are cut - which is healed with the help of the consistency check.

We suppose that this algorithm determines all so-called *Perfect Bayesian Equilibria* (PBE) as, e.g., defined by Fudenberg and Tirole [FT98]. We hesitate to assert this, since other authors use slightly different or only verbal definitions of the PBE, see e.g., Osborne and Rubinstein [OR94], and furthermore, since there exist so many different equilibrium refinements also for games in extensive form, that only experts in this highly specialized field can give a satisfying answer.

We also showed that this algorithm can be applied to extensive form games with imperfect information in general. It is again a question to specialists in the field whether or not this algorithm then determines all *Sequential Equilibria* as defined by Kreps and Wilson [KW82].

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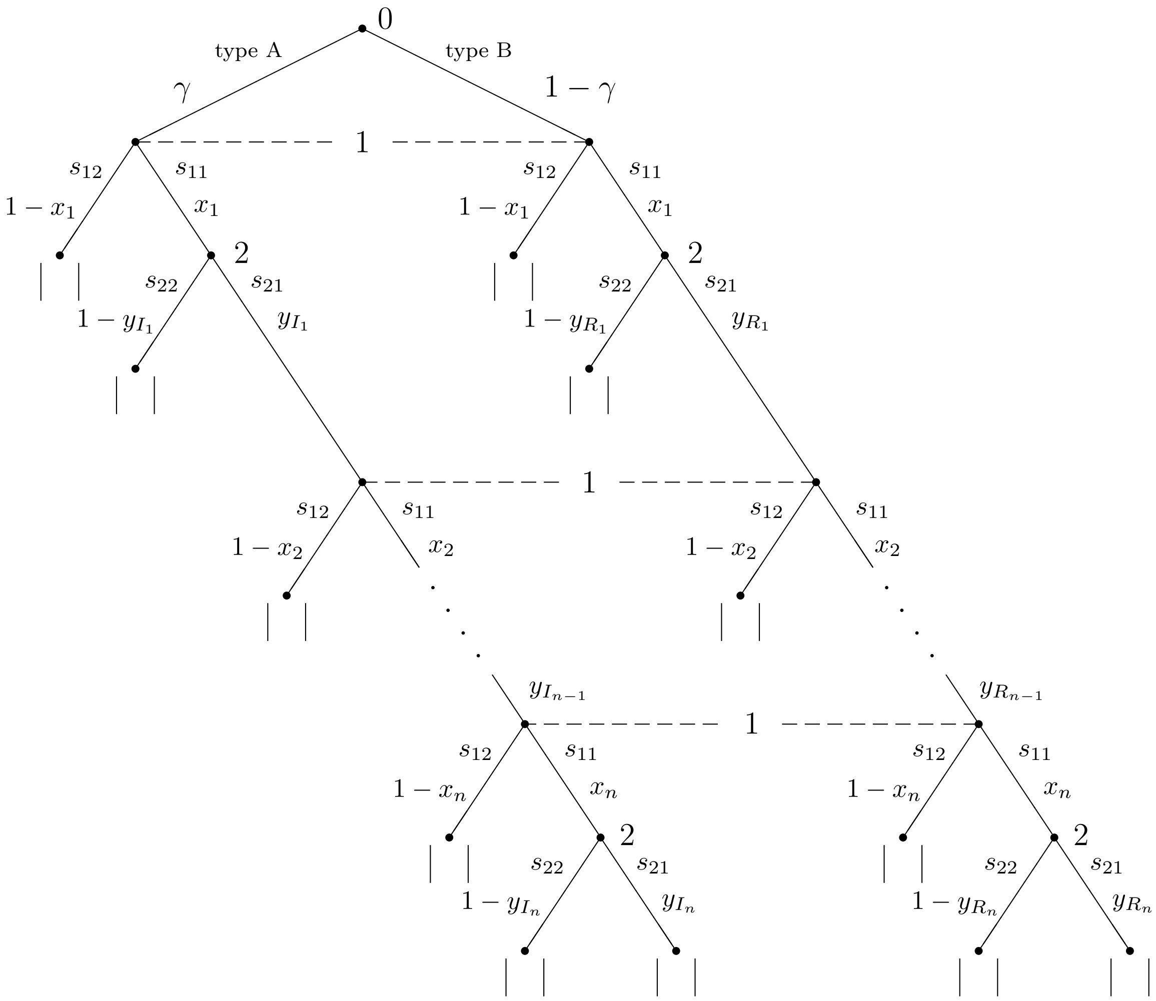

Figure 1: The structure of the extensive form of an armament model with incomplete information. The second player is either of type A or of type B. The first player does not know the type of the second player, and he has information sets with two elements.

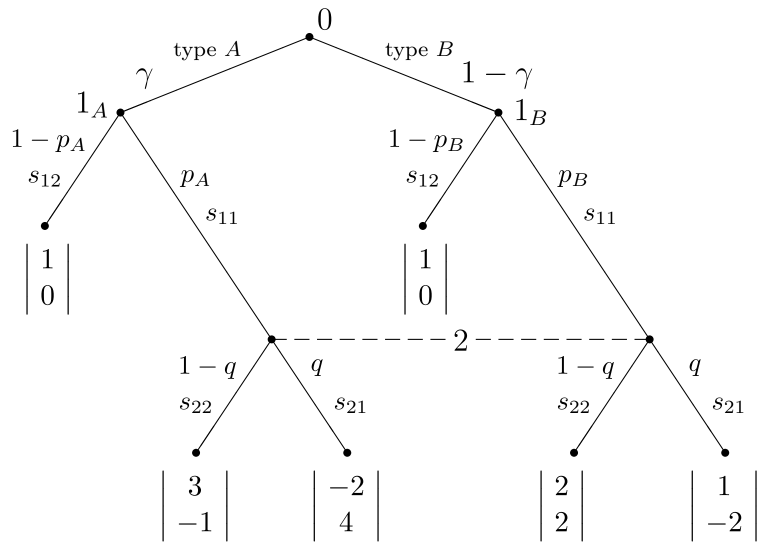

Figure 2: Extensive form of a two-person game with incomplete information. The dashed line represents the information set of the second player.

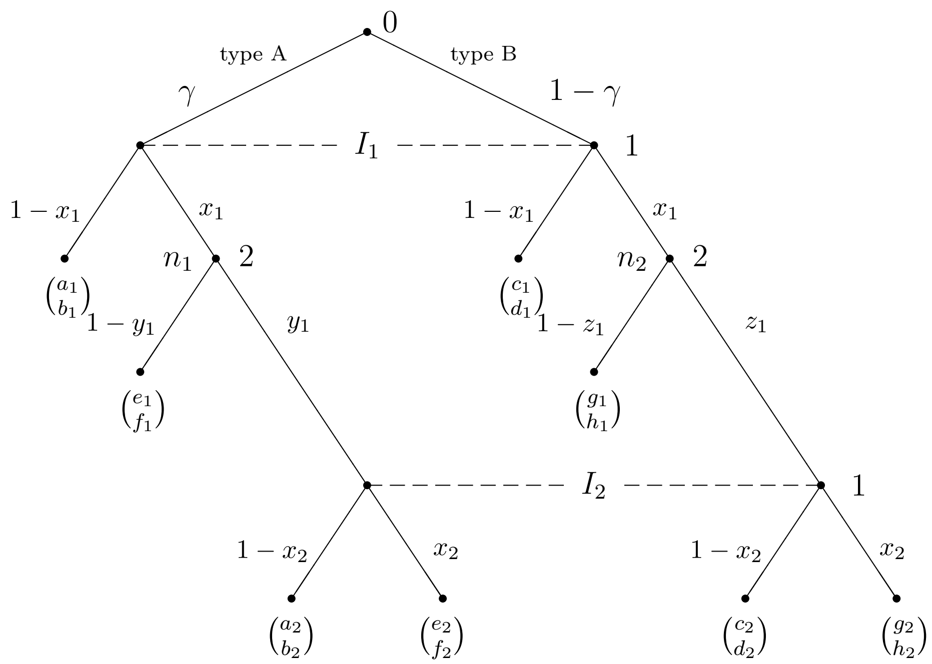

Figure 3: A special case of the game in Figure 1 (We can assume that the second player has determined optimal strategies y_2^* and z_2^* so that the game can appropriately be reduced.).

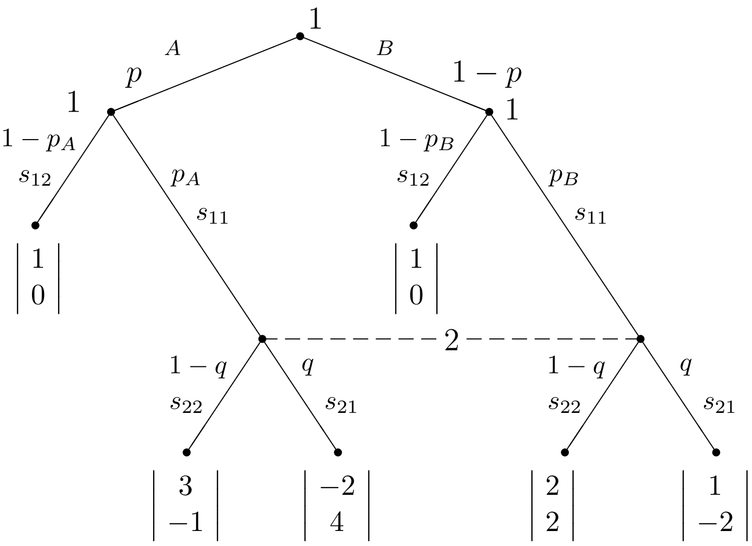

Figure 4: Extensive form of a two-person game with imperfect information.