Numerical analysis for elliptic Neumann boundary control problems on polygonal domains

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Zusammenfassung

Gegenstand dieser Dissertation ist die numerische Analysis von Optimalsteuerungsproblemen mit linearen und semilinearen elliptischen partiellen Differentialgleichungen in polygonalen Gebieten. Die Steuerung wirkt dabei auf den Neumannrand und soll zusätzlich punktweise Ungleichungsbeschränkungen erfüllen. Als Diskretisierungsvarianten werden das Konzept der variationellen Diskretisierung und der Postprocessing-Zugang untersucht. Die numerische Analysis für beide Diskretisierungsvarianten benötigt Finite-Elemente-Fehlerabschätzungen in der L^2-Norm auf dem Rand für das lineare elliptische Randwertproblem. Der erste Beitrag dieser Arbeit ist der Nachweis von quasi-optimalen Fehlerabschätzungen bei Verwendung von quasi-uniformen Netzen. Dabei zeigt sich, dass die Konvergenzordnung durch das Auftreten von Eckensingularitäten ab einem bestimmten Innenwinkel des polygonalen Gebietes abnimmt. Deswegen werden außerdem graduell verfeinerte Netze untersucht, die diesen Effekt nachweislich kompensieren. Des Weiteren wird gezeigt, wie sich die Resultate auf semilineare elliptische Randwertprobleme übertragen lassen. Für das lineare und das semilineare Randsteuerungsproblem werden anschließend unter Verwendung der Fehlerabschätzungen auf dem Rand quasi-optimale Konvergenzraten für beide Diskretisierungsvarianten bewiesen. Die theoretischen Ergebnisse für die Randwertprobleme und die Optimalsteuerungsprobleme werden jeweils durch numerische Experimente bestätigt.

Abstract

Subject of this thesis is the numerical analysis of optimal control problems with linear and semilinear elliptic partial differential equations in polygonal domains. It is assumed that the control acts on the Neumann boundary and additionally fulfills point-wise inequality constraints. As discretization strategies the concept of variational discretization and the postprocessing approach are considered. The numerical analysis for both approaches relies on finite element error estimates in the L^2-norm on the boundary for the linear elliptic boundary value problem. The first contribution of this work is the proof of quasi-optimal error estimates for quasi-uniform meshes. There it turns out that the convergence order decreases in general due to the appearance of corner singularities starting from a certain size of the interior angles of the polygonal domain. Therefore, gradually refined meshes, which compensate this effect, are analyzed in addition. Beyond that, it is demonstrated how these results can be transferred to semilinear elliptic boundary value problems. Then for linear as well as for semilinear elliptic Neumann boundary control problems quasi-optimal convergence rates are proven for both discretization strategies by using the obtained error estimates on the boundary. All theoretical results for the boundary value problems and the optimal control problems are confirmed by numerical examples.

Danksagung

Diese Dissertation ist während meiner Zeit als wissenschaftlicher Mitarbeiter am Institut für Mathematik und Bauinformatik an der Universität der Bundeswehr München entstanden. Zuallererst möchte ich mich bei meinem Betreuer Prof. Dr. Thomas Apel bedanken. Seine Unterstützung und sein hilfsbereiter Rat, aber auch seine stets aufmunternden Worte, haben entscheidend zum Gelingen dieser Arbeit beigetragen. Vielen Dank dafür. Natürlich möchte ich mich auch bei meinen Kolleginnen und Kollegen von der Universität der Bundeswehr München bedanken. Die gute Atmosphäre und die schöne Zeit, auch abseits der Arbeit, werden für mich unvergesslich bleiben.

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Contents

CHAPTER 1

Introduction

In this work we derive discretization error estimates for Neumann boundary control problems governed by linear and semilinear elliptic partial differential equations (PDEs) in polygonal domains with pointwise inequality constraints on the control.

Before specifying the problems which we are dealing with, let us begin with a very general description of optimal control problems. Usually, in the context of PDE constrained optimization one has a state variable y and a control variable u which are coupled by a partial differential equation. The aim is to find a state and a control, which fulfill the partial differential equation and possibly further control and state constraints such that a certain quantity is minimized. In abstract form, these problems can be formulated as

$$\begin{aligned} \min_{y\in Y, u\in U} \quad & F(y,u), \\ \text{subject to} \quad & e(y,u) = 0, \\ & u \in U_{ad}, \end{aligned} \tag{1.1}$$

where Y and U denote the state and the control space, respectively. The objective functional F comprises the aims of the optimal control problem. Equation (1.1) represents the partial differential equation, the so-called state equation, and the set U_{ad} denotes the set of admissible controls or admissible set, respectively. In this work the admissible set will contain pointwise inequality constraints on the control, only. However, it can also handle more general constraints such as pointwise state constraints or gradient constraints on the state, etc.

Problems of that kind naturally arise in many practical applications. Let us mention some of them which have already been treated in the literature. For example, laser surface hardening is applied in practice in order to increase the surface hardness of a workpiece, cf. [65]. A further example can be found in the field of cancer treatment, where local hyperthermia, which is induced by radio frequency radiation, is used to make the cancerous tissue more susceptible for other therapies such as chemotherapy, cf. [41]. Moreover, in the field of constructional engineering, methods are analyzed how to influence the hydration of young concrete with the purpose to avoid the appearance of cracks, cf. [5]. Note that in all these examples the underlying physical and chemical processes are described by partial differential equations of various types.

According to the enormous number of possible applications there is a considerable interest in solving such problems. However, in general it is not possible to state an analytic solution. Therefore, computer-aided methods are employed in practice in order to solve these problems numerically. Consequently, the resulting solution represents an approximation of the exact solution only, since in general these methods imply a procedural inherent error. In the present work we will analyze this error for two numerical methods, where each is applied to two different optimal control problems.

Let us get specific about the problems and the numerical methods, which we have in mind. The objective functional F will be chosen to be a standard tracking type functional,

$$F(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}.$$

Thus, we would like to minimize the distance between the state y and a desired state y_d in the $L^2(\Omega)$-sense having regard to the control costs in $L^2(\Gamma)$ which are weighted by a real and positive parameter ν. Note that the control costs can analytically be interpreted as a regularization term. The state equation (1.1) will either take the form

$$\begin{aligned} -\Delta y + \alpha y &= 0 \quad \text{in } \Omega, \\ \partial_n y &= u \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{1.2}$$

if we consider the linear elliptic case, or

$$\begin{aligned} -\Delta y + d(\cdot, y) &= 0 \quad \text{in } \Omega, \\ \partial_n y &= u \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{1.3}$$

if semilinear elliptic problems are of interest. In both cases the domain Ω is assumed to be polygonal with boundary $\Gamma = \bigcup_{j=1}^m \bar{\Gamma}_j$, corner points $x^{(j)}$ and associated angles ω_j with $j \in \{1, \dots, m\}$. For a more precise definition of all quantities we refer to the particular sections. Furthermore, we are going to seek solutions u which fulfill the inequality constraints

$$u_a \le u(x) \le u_b \quad \text{for a.a. } x \in \Gamma, \tag{1.4}$$

where u_a and u_b denote two real numbers. Thus, the admissible set can be defined by

$$U_{ad} := \{u \in L^2(\Gamma) : \ u \text{ fulfills (1.4)}\}.$$

Due to the fact that the control u is located at the Neumann boundary in (1.2) and (1.3), respectively, the corresponding control problems are called Neumann boundary control problems. Of course, linear problems can be seen as special cases of the semilinear ones. Nevertheless, we study both problems separately. This has several reasons, which we point out now as well as in the further course of this introductory chapter. As we will see, the linear problems possess a unique global solution since they are strictly convex. On the contrary, in the semilinear context we do not have the unique solvability, although the semilinear boundary value problem possesses a unique solution. Beyond that, we are also faced with locally optimal solutions. In summary this means that the consideration of necessary optimality conditions alone does not suffice. In addition, we have to take into account second order sufficient optimality conditions to ensure local optimality. From the analytic point of view this is very well analyzed and we can refer to the textbooks [107] and [63]. In fact, these textbooks will provide the basis for the discussion of the Neumann boundary control problems on the analytic level. However, from the numerical point of view, especially in the context of the discretization error analysis, there are some gaps which this work will close.

As already mentioned, we are going to apply and analyze two discretization strategies, more precisely, the concept of variational discretization and the postprocessing approach. The first one, the concept of variational discretization, was established in [60] for optimal control problems with distributed control and in [26] for Neumann boundary control problems. In this concept we only discretize the state by linear finite elements, but not the control. The discretization of the control is implicitly given by the first order necessary optimality condition. This fact will simplify the numerical analysis, since we are looking for a control in the continuous admissible set. However, the implementation is more sophisticated. The second discretization strategy will be the postprocessing approach, which was introduced in [87] for distributed control problems and in [77] for problems with Neumann boundary control. In this approach we discretize the state by linear finite elements and the control by piecewise constant functions. Only in a postprocessing step a control is calculated which possesses superconvergence properties and can easily be implemented. However, the numerical analysis in this approach is harder to accomplish. Despite of the advantages and disadvantages of both discretization strategies, they have in common that the discretization error analysis in the linear as well as in the semilinear case requires optimal finite element error estimates in the L^2-norm on the boundary for specific linear elliptic boundary value problems, the so-called adjoint problems, which are also discretized by linear finite elements.

The proof of quasi-optimal discretization error estimates in the $L^2(\Gamma)$-norm for linear elliptic boundary value problems will be the first main result of this work. To elucidate the difficulties and the deficiencies of common approaches let us consider the linear elliptic boundary value problem

$$\begin{aligned} -\Delta y + \alpha y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{1.5}$$

and a discretization of this problem by piecewise linear and continuous ansatz functions. Furthermore, let us denote the largest interior angle of the domain by ω.

As first approach, to get an error estimate in the $L^2(\Gamma)$-norm, one normally applies a trace theorem (such as Theorem 2.10) to reduce the estimate on the boundary to one in the domain, which can be estimated by standard techniques. For quasi-uniform meshes this yields at best an error bound of $ch^{3/2}$ if we assume a convex polygonal domain and data f, g and α which admit a solution in $H^2(\Omega)$. This estimate is sharp in case of a solution which only belongs to $H^2(\Omega)$. As an alternative to this technique, one can also use the Aubin-Nitsche method in $L^2(\Gamma)$ to deduce the same result. However, the Aubin-Nitsche method is favorable when considering non-convex domains. In this case, one gets a convergence order of s in $L^2(\Gamma)$ with some $s < 1/2 + \pi/\omega$, cf. [77]. In general, this estimate cannot be improved, even if we assume arbitrarily regular data, since singular terms appear in the solution, which are caused by the reentrant corners. Note that in non-convex domains the first technique yields a convergence order of $3\pi/2\omega - \epsilon < 1/2 + \pi/\omega$ in $L^2(\Gamma)$ with some arbitrarily small $\epsilon > 0$.

Next, one can ask if 3/2 represents the limiting convergence order in convex domains. To answer this question, let us assume, that the data are chosen such that they admit a solution in $W^{2,p}(\Omega)$ with $p > 2$. It is shown in [77], that the solution of the boundary value problem actually belongs to that space if the parameter p additionally fulfills $p < \infty$ in case of $\omega \leq \pi/2$ and $p < 2/(2 - \pi/\omega)$ in case of $\pi/2 < \omega < \pi$. The additional restrictions of the parameter p are again caused by the corner singularities. Now, having such a regular solution at hand, one can first use the embedding $L^p(\Gamma) \hookrightarrow L^2(\Gamma)$ together with a trace theorem for functions in $W^{1,p}(\Omega)$ such as Theorem 2.10 and then apply finite element error estimates in $L^p(\Omega)$ and $W^{1,p}(\Omega)$, which can be found for convex domains and quasi-uniform triangulations in [96] and [20, Chapter 8], to prove a convergence order of $2 - 1/p$. Similar techniques are used in [77]. Alternatively, one can also apply the Aubin-Nitsche method in $L^p(\Gamma)$ together with the $L^p(\Omega)$ and $W^{1,p}(\Omega)$ estimates to conclude the same result. In summary this means, that we have a convergence order close to two in domains with interior angles smaller than $\pi/2$ and a reduced one, which is definitely greater than 3/2, in domains with interior angles between $\pi/2$ and π.

In the present work we are going to demonstrate that these estimates can still be improved. The approach, which we are going to use, is not straightforward at all and will be the most challenging step in this work. It is based on regularity results in weighted $W^{2,\infty}$-spaces, which we are going to prove for data in weighted Hölder spaces, techniques of [104, 105, 8] and local finite element error estimates as described in [103, 109, 39]. By this we will show a quasi-optimal error bound of $ch^2|\ln h|^{1+\varrho}$ with some $\varrho \in [0, 1/2]$ for domains with interior angles smaller than $\pi/(2 - \varrho)$. In domains, where the largest interior angle ω is greater than or equal to $\pi/(2 - \varrho)$, we will see that the error is bounded by $ch^\rho|\ln h|^{1+\varrho}$ with some arbitrary $\rho < \varrho + \pi/\omega$. Hence, in domains with interior angles smaller than $2\pi/3$ the error is definitely bounded by $ch^2|\ln h|^{3/2}$. Otherwise, we have a convergence order of almost $1/2+\pi/\omega$, which fits to the aforementioned estimates in non-convex domains. A closely related result can be found in [83], where quasi-optimal discretization error estimates in the L^2-norm are proven on a strip at the boundary with width h. We will comment on this result in more detail in Section 3.2.3. Let us also remark, that all of these estimates hold for quasi-uniform triangulations.

As we will see, the convergence order is only reduced due to singular terms in the solution coming from the corners $x^{(j)}$, where the associated angle ω_j is greater than $\pi/(2-\varrho)$. For that reason we will be able to use mesh grading techniques in order to compensate this lowering effect. More precisely, in the neighborhood of a corner $x^{(j)}$, where the associated angle ω_j is greater than $\pi/(2 - \varrho)$, we will gradually refine the mesh towards that corner depending on a mesh grading parameter $\mu_j \in (0, 1]$, that determines the strength of the grading, see Section 3.2.1 for details. In particular, for $\mu_j = 1$ the mesh will be quasi-uniform and only for $\mu_j < 1$ a graded one. For such graded meshes we are going to show that the quasi-optimal error bound $ch^2|\ln h|^{1+\varrho}$ can be retained, if the mesh grading parameters μ_j are chosen smaller than $(\varrho + \pi/\omega_j)/2$. In particular, we get an error bound of $ch^2|\ln h|^{3/2}$ in domains with angles $\omega_j \geq 2\pi/3$ if the corresponding mesh grading parameters μ_j fulfill $\mu_j < 1/4 + \pi/(2\omega_j)$.

All the convergence orders for the different approaches are summarized in Figure 1.1 depending on the interior angles of the domain.

Following an idea of [84], we will also transfer these results to semilinear elliptic boundary value problems, where we only assume standard requirements for the nonlinearity. However, for the numerical analysis of the semilinear Neumann boundary control problems, we will need

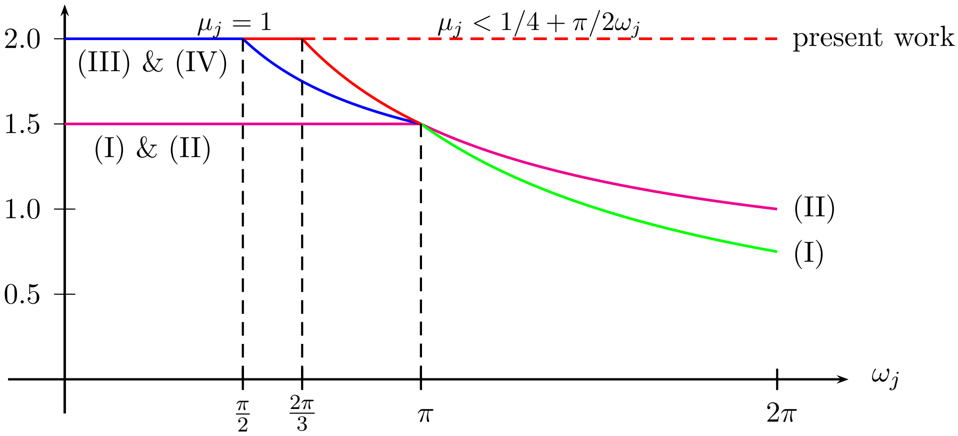

Figure 1.1: Convergence rates of the finite element error in $L^2(\Gamma)$ for the different approaches depending on the interior angles ω_j: (I) trace theorem in $L^2(\Gamma)$, (II) Aubin-Nitsche method in $L^2(\Gamma)$, (III) trace theorem in $L^p(\Gamma)$, (IV) Aubin-Nitsche method in $L^p(\Gamma)$. Solid lines: quasi-uniform meshes. Dashed lines: graded meshes.

finite element error estimates in the domain for semilinear elliptic boundary value problems. For that reason, we are going to extend the discretization error estimates in the domain of [25], which are proven there for quasi-uniform meshes, to the case of gradually refined meshes. We will see, in order to achieve the full order of convergence in $L^2(\Omega)$ and $H^1(\Omega)$, we require a mesh grading with $\mu_j < \pi/\omega_j$ if the corresponding interior angle ω_j is greater than π.

Now, let us return to the numerical analysis of Neumann boundary control problems. This represents the second main topic of this work. As we have already mentioned, we are going to apply and analyze the concept of variational discretization and the postprocessing approach for linear and semilinear problems. Before we state our discretization error estimates, let us review known results for Neumann boundary control problems. We will focus on convergence rates in $L^2(\Gamma)$.

The first approach, we would like to address, is the full discretization, i.e., the state as well as the control are discretized. This approach was already applied to linear Neumann boundary control problems in [52]. Furthermore, it was analyzed in [27] for semilinear Neumann boundary control problems using the finite element error estimates of [25], and in [24] for quasilinear problems employing the finite element error estimates of [23]. In all these papers the state is discretized by linear finite elements and the control by piecewise constant functions. For such a discretization a convergence order of one in $L^2(\Gamma)$ for the control is proven in all these papers assuming convex polygonal domains and quasi-uniform triangulations. Moreover, in [24] also estimates in non-convex domains are discussed. There, the authors could prove a convergence rate of 1/2 in $L^2(\Gamma)$. Alternatively to the approximation of the control by piecewise constant functions, one can also discretize the control by piecewise linear and globally continuous functions. This was done in [26] for semilinear problems in convex domains using quasi-uniform meshes. This approach yields in $L^2(\Gamma)$ a superlinear convergence and a convergence order of 3/2 under a structural assumption on the control.

Next, let us discuss the results from the literature for the variational discretization concept and the postprocessing approach. The former one was applied to semilinear Neumann boundary control problems in [26] and to linear ones in [77, 61, 6]. The latter approach is also well known for linear problems with Neumann boundary control, see [77, 6]. However, to the best of our knowledge there is no reference, where this approach is analyzed for semilinear problems, even in case of a distributed control. But let us come back to the known results. In all the papers mentioned above, different convergence rates in $L^2(\Gamma)$ are proven for the control. This is mainly a consequence of the different finite element error estimates on the boundary, which we have seen before. In [26] an error bound in $L^2(\Gamma)$ of $ch^{3/2-\epsilon}$ with some arbitrary $\epsilon > 0$ is established for the variational discretization concept applied to semilinear problems assuming convex domains and quasi-uniform meshes. For linear problems a convergence rate in $L^2(\Gamma)$ of $3/2$ and an error bound of $ch^{2-2/p}|\ln h|$ in $L^\infty(\Gamma)(\hookrightarrow L^2(\Gamma))$ with the parameter p from above is proven in [61] under the same assumptions on the domain and the triangulations. An improved estimate for the variational discretization concept and the postprocessing approach can be found in [77]. There, the authors proved for convex domains an approximation rate of $2 - 1/p$ with the parameter p as before and a rate of $\min_j(1/2+\pi/\omega_j)$ for non-convex domains, where in each case a quasi-uniform triangulation is used. They further established better estimates using higher order finite elements for the discretization of the state and adjoint state. But this is not scope of this work and might be analyzed in the future. Finally, we remark that all the results, stated so far for non-convex domains, have in common that the convergence rates are lower than $3/2$ in this case. In [6] it is proven that graded meshes with mesh grading parameters $\mu_j < \pi/\omega_j$ can be used for the concept of variational discretization and the postprocessing approach to maintain a convergence order of $3/2$ in non-convex domains.

In the present work, by using the error estimates on the boundary of the first part, we are going to show quasi-optimal discretization error estimates for the concept of variational discretization and the postprocessing approach, each applied to linear as well as to semilinear elliptic Neumann boundary control problems. More precisely, we will see in each case that the error of the control in $L^2(\Gamma)$ is bounded by $ch^2|\ln h|^{3/2}$ on quasi-uniform meshes if the interior angles fulfill $\omega_j < 2\pi/3$. For larger interior angles we will get a convergence rate of almost $\min_j(1/2 + \pi/\omega_j)$. Furthermore, we are going to prove, that graded meshes with mesh grading parameters $\mu_j < 1/4+\pi/(2\omega_j)$ can be used to maintain the error bound $ch^2|\ln h|^{3/2}$ in case that the interior angles ω_j are greater than or equal to $2\pi/3$. As we have already mentioned we are going to prove each result for linear as well as for semilinear problems separately. This is due to the non-convex character of the semilinear problems. In particular, for the linear problem we will be able to rely on first order necessary optimality conditions to prove the convergence rates for the globally optimal solution. On the contrary, in the semilinear case we have to employ second order sufficient optimality conditions within the proofs in addition. More precisely, assuming that the mesh size is already sufficiently small, we are going to show in a preliminary step that for every local solution of the continuous problem, which fulfills the second order sufficient optimality condition, there is a local solution of the corresponding discrete problem which converges to the local continuous solution with some suboptimal rate. Based on this we are going to prove that both discretization strategies possess the quasi-optimal convergence rates mentioned above. However, another special feature will arise for the postprocessing approach. Normally, in the context of this approach, one has to assume a specific structure of the optimal control in order to derive the error estimates. Roughly speaking, one has to assume that the optimal control has only a finite number of kinks with the control constraints. In

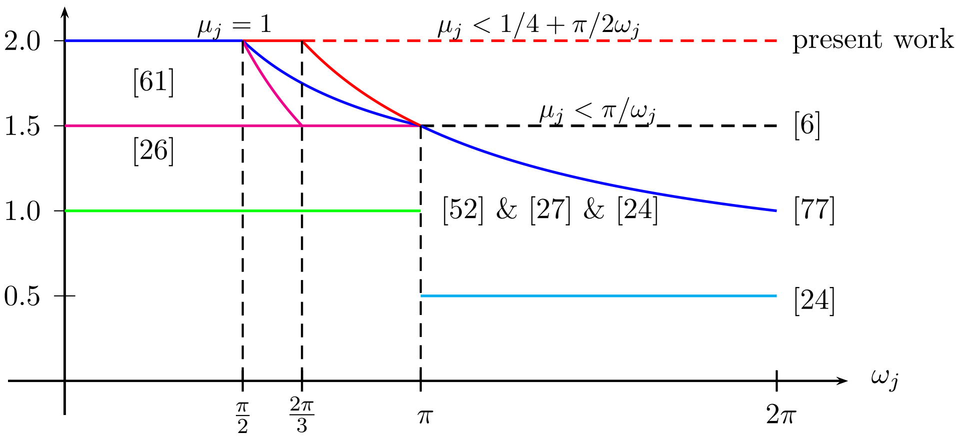

Figure 1.2: Convergence rates of the control in $L^2(\Gamma)$ depending on the interior angles ω_j. Solid lines: quasi-uniform meshes. Dashed lines: graded meshes.

the semilinear case we will need a slightly stronger assumption. There, we will assume that the number of points are finite, where the control constraints become active in general. This means that we additionally limit the number of points, where the control intersects smoothly the control constraints.

In Figure 1.2 one can find the addressed convergence rates for the different approaches depending on the interior angles of the domain.

Before we outline the structure of this work, let us give an overview of further relevant literature about discretization error estimates for elliptic optimal control problems. Since the literature is very comprehensive, the following overview can never be exhaustive and we refer to [64] for a more detailed survey. First of all, we would like to mention the very early contribution [49]. In this paper discretization error estimates for a class of elliptic optimal control problems are established employing a full discretization of the problems. In particular, the author of that paper focused on a discretization of the control by piecewise constant functions. Later on, the topic on discretization error estimates came back into focus by [14]. There, estimates for semilinear elliptic control problems with distributed control are derived using a full discretization of the problem with linear finite elements for the state and piecewise constant functions for the control. Error estimates for a full discretization with piecewise linear and continuous functions for the control can be found in [86, 99] in case of linear problems and in [22] for semilinear problems. For the variational discretization concept, applied to linear control problems with distributed control, we refer to [60, 8]. The postprocessing approach is analyzed for such problems in [87, 9, 8]. We emphasize that in [9, 8] graded meshes in polygonal domains are used to maintain the full order of convergence in different norms. Extensions to polyhedral domains can be found in [13] and [12]. More precisely, in the former one isotropic refinements are considered, whereas in the latter one anisotropic meshes are used. So far we have restricted our considerations to linear and semilinear problems. For the more general quasilinear problems with distributed control we refer to [30, 31]. In these papers not only the full discretization with a piecewise constant and piecewise linear discretization of the control is applied, but also the variational discretization concept. Next, let us address another type of control problems.

Besides the results for problems with distributed control and Neumann boundary control, there is a multitude of papers about Dirichlet boundary control problems. Here, we mention the papers [28], [78] and [37]. In [28] discretization error estimates are derived for semilinear problems in convex polygonal domains using a full discretization with linear and continuous functions for the state as well as for the control. The corresponding linear problem without control constraints is analyzed in [78] with special focus on error estimates in negative norms, which can be used to improve the known estimates for the state and adjoint state. In [37] the variational discretization concept is applied to linear problems in smooth domains. Besides error estimates for quasi-uniform meshes, the authors of [37] obtain improved error estimates for superconvergence meshes. Finally, for discretization error estimates of state constrained problems we refer to [21, 38, 85, 73, 62].

Now, let us come to the outline of this work. In Chapter 2 we provide the basis for the discussion of the boundary value problems and the Neumann boundary control problems. There, we introduce classical and weighted spaces in the domain and on the boundary. We discuss in detail the relation between both, which we summarize in various trace theorems. Furthermore, we state several important properties of these spaces such as embedding theorems and equivalent norms.

In the first part of Chapter 3 we elaborate regularity results for the generalized solution of linear elliptic boundary value problems. We start with results in classical Sobolev Slobodeckij spaces, which only depend on the regularity of the data. Then we continue with the proof of regularity results in various weighted Sobolev spaces. In particular, based on regularity results in weighted Hölder spaces we show that the solution belongs to certain weighted $W^{2,\infty}$-spaces. This regularity is especially required for the derivation of the finite element error estimates on the boundary in Section 3.2.4. Afterwards, in Section 3.1.2, we transfer all these results to the generalized solution of semilinear elliptic boundary value problems by employing the corresponding results of the linear problem and the assumptions on the nonlinearity. Furthermore, we derive for each problem Lipschitz estimates, which are frequently used in Section 4.4. In Section 3.2.1, the beginning of the main part of Chapter 3, we introduce a family of gradually refined triangulations in the domain and on the boundary. Then, as a preliminary step, we derive in Section 3.2.2 error estimates for several interpolation operators defined on such triangulations. Afterwards, in Section 3.2.3, we discretize the linear elliptic boundary value problem by linear finite elements and derive discretization error estimates in various norms in the domain by means of standard techniques, where we focus on quasi-uniform as well as on graded triangulations. Furthermore, we state the finite element error estimates in the $L^2(\Gamma)$-norm. The proof can be found in Section 3.2.4. It is based on a certain dyadic decomposition of the domain, local finite element error estimates in various norms and the regularity results in weighted $W^{2,\infty}$-spaces. As a by-product, we transfer in Section 3.2.6 all these results to semilinear elliptic boundary value problems by means of the assumptions on the nonlinearity and the corresponding results of the linear problem. As in the continuous case, we also state for both problems, the linear and the semilinear, Lipschitz estimates, which are required for the discussion of the semilinear control problems in Section 4.4. Furthermore, we present numerical examples for the linear and semilinear problem in Section 3.2.5 and Section 3.2.7, respectively, which confirm our theoretical findings.

In Chapter 4 we analyze the concept of variational discretization and the postprocessing approach, each applied to linear as well as to semilinear elliptic Neumann boundary control problems. The consideration of linear problems can be found in Section 4.1 and Section 4.2, whereas the analysis for the semilinear ones is presented in Section 4.3 and Section 4.4. More precisely, in Section 4.1 we discuss the first order necessary optimality conditions for linear Neumann boundary control problems, which are also sufficient for these problems. As a key point for the subsequent numerical analysis we also prove optimal regularity in various weighted Sobolev spaces for the solution of the optimal control problem. The discretization error estimates for the variational discretization concept and the postprocessing approach on quasi-uniform and graded meshes can be found in Section 4.2.1 and Section 4.2.2, respectively. We emphasize, that the optimal control and the desired state are always separated from the constants in all estimates, which is not possible for general semilinear problems. The first order necessary and second order sufficient optimality conditions for semilinear Neumann boundary control problems are presented in Section 4.3. Furthermore, regularity results for the local solutions of the semilinear problems are proven in this section. The numerical analysis for the variational discretization concept and the postprocessing approach can be found in Section 4.4.1 and Section 4.4.2, respectively. For both approaches we first show that there exists a mesh size such that for every local solution of the continuous problem, which fulfills the second order sufficient optimality condition, there is a local solution of the corresponding discrete problems, which converges to the continuous solution by a certain suboptimal rate for decreasing mesh parameters. Afterwards, we prove that the concept of variational discretization and the postprocessing approach admit the quasi-optimal convergence rates on quasi-uniform and graded triangulations as in the linear case. Numerical examples for the postprocessing approach, which confirm our theoretical results, are presented for linear and semilinear problems in Section 4.2.3 and Section 4.4.3, respectively.

In the final chapter we conclude our work and give an outlook on possible extensions.

Finally, let us remark that the results of this work have already been published in several papers. The finite element error estimates in the domain for graded meshes are contained in [6]. Furthermore, suboptimal discretization error estimates for the linear Neumann boundary control problem are derived in that reference. The finite element error estimates on the boundary for linear elliptic problems and the quasi-optimal error estimates for linear Neumann boundary control problems can be found in [7]. The transfer of the results to semilinear Neumann boundary control problems is established in [95].

CHAPTER 2

Function spaces

This chapter provides the basis for the forthcoming discussion of the boundary value problems and the Neumann boundary control problems. All functions in the sequel belong to classical function spaces but also to weighted function spaces. Therefore, we start with the definition of continuous, Hölder continuous and Lipschitz continuous functions. Then we proceed with the introduction of Lebesgue spaces, classical Sobolev spaces and its trace spaces. Furthermore, we state some important results concerning these spaces, such as embedding theorems and trace theorems. Afterwards we address the definition of weighted Sobolev and weighted Hölder spaces. In the final part of this chapter we collect selected properties of the introduced weighted spaces, which are needed for the analysis in the sequel.

2.1 Classical function spaces

Throughout this section we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ. For the precise definition of Lipschitz boundaries and boundaries of class $C^{k,\sigma}$ with $k \in \mathbb{N}_0$ and $\sigma \in [0,1]$ we refer to Definition 1.2.1.1 and Definition 1.4.5.1 of [54]. Furthermore, we will denote by $\bar{E}$ or $\mathrm{cl}(E)$ the closure of a set E. Now, let us recall some Banach spaces which are used in the sequel.

Definition 2.1 (Continuous functions in the domain)**.** Let $k \in \mathbb{N}_0$ and let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ be a multi-index. The space $C^k(\Omega)$ is defined as the set of all functions v on Ω with continuous derivatives $D^\alpha v$ up to order k. Further, we set $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$ and $C_0^k(\Omega) = \{v \in C^k(\Omega) : \mathrm{supp}(v) \subset \Omega \text{ is compact}\}$. The space $C^k(\bar{\Omega})$ denotes the set of all functions v on Ω with bounded and uniformly continuous derivatives $D^\alpha v$ up to order k, i.e., the derivatives $D^\alpha v$ can continuously be extended to $\bar{\Omega}$ for $|\alpha| \le k$. The norm in $C^k(\bar{\Omega})$ is defined by

$$\|v\|_{C^k(\bar{\Omega})} := \sum_{|\alpha| \le k} \sup_{x \in \Omega} |(D^\alpha v)(x)|. \tag{2.1}$$

Functions belonging to the Hölder space $C^{k,\sigma}(\bar{\Omega})$ additionally possess bounded derivatives of order k which are Hölder continuous with exponent $\sigma \in (0,1)$. The norm in the Hölder space $C^{k,\sigma}(\bar{\Omega})$ is given by

$$\|v\|_{C^{k,\sigma}(\bar{\Omega})} := \|v\|_{C^k(\bar{\Omega})} + \sum_{|\alpha|=k} \sup_{x_1,x_2\in\Omega} \frac{|(D^\alpha v)(x_1) - (D^\alpha v)(x_2)|}{|x_1 - x_2|^\sigma}. \tag{2.2}$$

For $\sigma = 1$ the derivatives $D^\alpha v$ of order $|\alpha| = k$ are called Lipschitz continuous and for $\sigma = 0$ we set $C^{k,0}(\bar{\Omega}) := C^k(\bar{\Omega})$.

Definition 2.2 (Continuous functions on the boundary)**.** Let $\alpha \in \mathbb{N}_0$. Furthermore, let either Γ_s be an open subset of Γ of class $C^{k,\sigma}$ with $k \in \mathbb{N}_0$ or $\Gamma_s = \Gamma$ and $k = 0$. The space $C^k(\bar{\Gamma}_s)$ consists of all functions v on Γ_s with bounded and uniformly continuous tangential derivatives $\partial_t^\alpha v$ up to order k. The space $C^{k,\sigma}(\bar{\Gamma}_s)$ denotes the subspace of functions belonging to $C^k(\bar{\Gamma}_s)$ whose derivatives of order k are additionally Hölder continuous with exponent $\sigma \in (0,1)$ or Lipschitz continuous in case of $\sigma = 1$. The norms in $C^k(\bar{\Gamma}_s)$ and $C^{k,\sigma}(\bar{\Gamma}_s)$ are defined analogously to (2.1) and (2.2), respectively. Moreover, we set $C^{k,0}(\bar{\Gamma}_s) := C^k(\bar{\Gamma}_s)$

Definition 2.3 (L^p-spaces)**.** Let $\mathcal{G}$ be the domain Ω, its boundary Γ or a subset Γ_s of the boundary Γ with $|\Gamma_s| > 0$. The Lebesgue space $L^p(\mathcal{G})$, $1 \le p < \infty$, is the space of all Lebesgue measurable functions v such that the norm

$$\|v\|_{L^p(\mathcal{G})} := \left(\int_{\mathcal{G}} |v|^p\right)^{1/p}$$

is finite. The Lebesgue space $L^\infty(\mathcal{G})$ is the space of all essentially bounded and Lebesgue measurable functions v. The norm in $L^\infty(\mathcal{G})$ is given by

$$\|v\|_{L^\infty(\mathcal{G})} := \operatorname{ess\,sup}_{\mathcal{G}} |v|.$$

Remark 2.4. The space $L^2(\mathcal{G})$ is a Hilbert space with the scalar product

$$(u,v)_{L^2(\mathcal{G})} := \int_{\mathcal{G}} uv.$$

Definition 2.5 ($W^{s,p}(\Omega)$-spaces)**.** Let $1 \le p \le \infty$ and $k \in \mathbb{N}_0$. Furthermore, let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ be a multi-index. The space $W^{k,p}(\Omega)$ (or $H^k(\Omega)$ for $p = 2$) is the space of all functions $v \in L^p(\Omega)$ whose weak derivatives $D^\alpha v$ exist and belong to $L^p(\Omega)$ for $|\alpha| \le k$. The space $W^{k,p}(\Omega)$ is equipped with the norm

$$\|v\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\le k} \|D^\alpha v\|^p_{L^p(\Omega)}\right)^{1/p} \quad \text{if } 1 \le p < \infty,$$
$$\|v\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha|\le k} \|D^\alpha v\|_{L^\infty(\Omega)}.$$

Corresponding seminorms are given by

$$|v|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|= k} \|D^\alpha v\|^p_{L^p(\Omega)}\right)^{1/p} \quad \text{if } 1 \le p < \infty,$$
$$|v|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha|= k} \|D^\alpha v\|_{L^\infty(\Omega)}.$$

For every non-integer $s > 0$ and $1 \le p < \infty$ we set $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $0 < \sigma < 1$ and denote by $W^{s,p}(\Omega)$ the space of all functions which belong to $W^{k,p}(\Omega)$ and satisfy

$$|v|_{W^{s,p}(\Omega)} := \left(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|(D^\alpha v)(x_1) - (D^\alpha v)(x_2)|^p}{|x_1 - x_2|^{2+\sigma p}} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \right)^{1/p} < \infty.$$

The norm in $W^{s,p}(\Omega)$ is given by

$$\|v\|_{W^{s,p}(\Omega)} := \left(\|v\|^p_{W^{k,p}(\Omega)} + |v|^p_{W^{s,p}(\Omega)} \right)^{1/p}.$$

Definition 2.6 ($W^{s,p}(\Gamma_s)$-spaces)**.** Let $1 \le p \le \infty$ and $\alpha \in \mathbb{N}_0$. Furthermore, let either Γ_s be an open subset of Γ of class $C^{k-1,1}$ with $k \in \mathbb{N}$ or $\Gamma_s = \Gamma$ and $k = 1$. The space $W^{k,p}(\Gamma_s)$ (or $H^k(\Gamma_s)$ for $p = 2$) is the space of all functions $v \in L^p(\Gamma_s)$ whose weak tangential derivatives $\partial_t^\alpha v$ exist and belong to $L^p(\Gamma_s)$ for $|\alpha| \le k$. The norm in the space $W^{k,p}(\Gamma_s)$ is given by

$$\|v\|_{W^{k,p}(\Gamma_s)} := \left(\sum_{|\alpha| \le k} \|\partial_t^\alpha v\|^p_{L^p(\Gamma_s)} \right)^{1/p} \quad \text{if } 1 \le p < \infty,$$
$$\|v\|_{W^{k,\infty}(\Gamma_s)} := \sum_{|\alpha| \le k} \|\partial_t^\alpha v\|_{L^\infty(\Gamma_s)}.$$

The seminorm $|\cdot|_{W^{k,p}(\Gamma_s)}$ is defined as in Definition 2.5. Furthermore, we set $W^{0,p}(\Gamma_s) := L^p(\Gamma_s)$. For every non-integer $s \in (0, k)$ and $1 \le p < \infty$ we set $s = m + \sigma$ with $m \in \mathbb{N}_0$ and $0 < \sigma < 1$. The space $W^{s,p}(\Gamma_s)$ denotes the set of all functions which belong to $W^{m,p}(\Gamma_s)$ and fulfill

$$|v|_{W^{s,p}(\Gamma_s)} := \left(\sum_{|\alpha|=m} \int_{\Gamma_s} \int_{\Gamma_s} \frac{|(\partial_t^\alpha v)(x_1) - (\partial_t^\alpha v)(x_2)|^p}{|x_1 - x_2|^{1+\sigma p}} \, \mathrm{d}s_{x_1} \, \mathrm{d}s_{x_2} \right)^{1/p} < \infty.$$

The space $W^{s,p}(\Gamma_s)$ is equipped with the norm

$$\|v\|_{W^{s,p}(\Gamma_s)} := \left(\|v\|^p_{W^{m,p}(\Gamma_s)} + |v|^p_{W^{s,p}(\Gamma_s)} \right)^{1/p}.$$

Now, let us recall a classical embedding theorem, cf. e.g. [3, Chapter 8], [1, Chapters 5, 6 and 7] or [54, Section 1.4].

Theorem 2.7. *Let $\mathcal{G}$ be the domain Ω, its boundary Γ or a subset Γ_s of the boundary Γ as in Definition 2.6 and let n be the dimension of $\mathcal{G}$. Furthermore, let $1 \le p, q < \infty$ and let $s, t \ge 0$ be real numbers and k a nonnegative integer possibly further restricted as in Definition 2.2 and Definition 2.6 to get the well-posedness of the spaces on the boundary. Then the following assertions hold:*

(i) Let $s - n/p = t - n/q$ and $s \ge t$. Then the continuous embedding $W^{s,p}(\mathcal{G}) \hookrightarrow W^{t,q}(\mathcal{G})$ is valid.

(ii) Let $s - n/p > t - n/q$ and $s > t$. Then the compact embedding $W^{s,p}(\mathcal{G}) \overset{c}{\hookrightarrow} W^{t,q}(\mathcal{G})$ holds.

(iii) Let $s - n/p = k + \sigma$ and $0 < \sigma < 1$. Then there is the continuous embedding $W^{s,p}(\mathcal{G}) \hookrightarrow C^{k,\sigma}(\bar{\mathcal{G}})$.

(iv) *Let $s - n/p > k + \sigma$ and $0 \le \sigma \le 1$. Then the compact embedding $W^{s,p}(\mathcal{G}) \overset{c}{\hookrightarrow} C^{k,\sigma}(\bar{\mathcal{G}})$ is valid.*

The next theorem shows that functions belonging to the space $W^{k,p}(\Omega)$ with $k \ge 1$ have in some sense boundary values on Γ, cf. e.g. [91, Theorem 4.7 and Theorem 4.8].

Theorem 2.8. *Let $1 \le q \le p/(2-p)$ if $1 \le p < 2$ or $1 \le q < \infty$ if $p = 2$. Then there exists exactly one bounded and linear mapping $\tau : W^{1,p}(\Omega) \to L^q(\Gamma)$ such that for $v \in W^{1,p}(\Omega) \cap C^0(\bar{\Omega})$ there holds $(\tau v)(x) = v(x)$ a.e. on Γ. For $p > 2$ the operator τ is a bounded and linear mapping from $W^{1,p}(\Omega)$ to $C^0(\Gamma)$.*

Remark 2.9. Instead of τv we will often write $v_{|\Gamma}$ or simply v, where it is obvious that we mean the trace of v.

Let us also recall Theorem 1.6.6 of [20].

Theorem 2.10. *Let $1 \le q \le \infty$ and let $v \in W^{1,q}(\Omega)$. Then there is the inequality*

$$\|v\|_{L^q(\Gamma)} \le c\|v\|_{L^q(\Omega)}^{1-1/q}\|v\|_{W^{1,q}(\Omega)}^{1/q}$$

with a positive constant c independent of v.

The space $L^q(\Gamma)$ as in Theorem 2.8 is larger than the trace space of $W^{1,p}(\Omega)$. A natural way to define the trace space is given in the following definition, cf. [74, Section 0.3].

Definition 2.11 ($\tilde{W}^{k-1/p,p}(\Gamma)$-spaces)**.** Let $1 \le p < \infty$ and $k \in \mathbb{N}$. Furthermore, let $\mathring{W}^{k,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. The trace space $\tilde{W}^{k-1/p,p}(\Gamma)$ (or $\tilde{H}^{k-1/2}(\Gamma)$ for $p = 2$) of $W^{k,p}(\Omega)$ is defined as the quotient space

$$\tilde{W}^{k-1/p,p}(\Gamma) := W^{k,p}(\Omega)/\mathring{W}^{k,p}(\Omega)$$

and endowed with the norm

$$\|v\|_{\tilde{W}^{k-1/p,p}(\Gamma)} := \inf\left\{\|u\|_{W^{k,p}(\Omega)} : u \in W^{k,p}(\Omega), v - u \in \mathring{W}^{k,p}(\Omega)\right\}.$$

The previous definition of the trace space of $W^{k,p}(\Omega)$ is rather formal and does not give a characterization of the trace space which allows to simply check if a given function belongs to this space or not. For $k = 1$ we can identify the trace space with the space $W^{1-1/p,p}(\Gamma)$, which relies on the following theorem, see e.g. [54, Theorem 1.5.1.2] or [91, Theorem 5.5 and Theorem 5.7].

Theorem 2.12. *Let $1 < p < \infty$. The trace operator τ is a bounded and linear operator from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\Gamma)$. This operator has a right continuous inverse $E_\tau : W^{1-1/p,p}(\Gamma) \to W^{1,p}(\Omega)$.*

Remark 2.13. One can also show that the trace operator τ is a bounded and linear operator from $H^s(\Omega)$ to $H^{s-1/2}(\Gamma)$ for $s \in (1/2, 3/2)$ and has right continuous inverse for $s \in (1/2, 1)$, see e.g. [82, Theorem 3.37 and Theorem 3.38], [35] and [43].

A direct consequence of Theorem 2.12 is the following corollary.

Corollary 2.14. *The trace space* $\tilde{W}^{1-1/p,p}(\Gamma)$ *can be identified with the space* $W^{1-1/p,p}(\Gamma)$ *and the corresponding norms are equivalent.*

Remark 2.15. An analogous characterization of the trace space of $W^{k,p}(\Omega)$ for $k > 1$ does not make sense since the boundary Γ can only be described by Lipschitz continuous functions. Instead one can show in case of a polygonal domain (see Definition 2.17), that functions in the trace space of $W^{k,p}(\Omega)$ belong on the smooth parts of the boundary to the space $W^{k-1/p,p}(\Gamma_s)$ and satisfy additional compatibility conditions at the singular points, cf. e.g. [54, Section 1.5 and 5.1]. For a characterization of the trace space in case of a general Lipschitz domain we refer to [47], [53] and the references therein.

Finally, let us recall the following result for $k = 2$, which does not make a statement about the range of the trace operator, cf. e.g. [91, Theorem 4.11].

Theorem 2.16. *Let* $1 \leq q \leq p/(2-p)$ *if* $1 \leq p < 2$ *or* $1 \leq q < \infty$ *if* $p = 2$*. Then the trace operator* τ *is a bounded and linear operator from* $W^{2,p}(\Omega)$ *to* $W^{1,q}(\Gamma)$*.*

2.2 Weighted function spaces

For the introduction of weighted spaces in the sequel we restrict ourselves to the consideration of polygonal domains which we are going to define first. For the case of more general domains we refer to e.g. [90, 71, 79].

Definition 2.17 (Polygonal domains)**.** Let $m \in \mathbb{N}$ and $\mathcal{S} = \{1, \ldots, m\}$. Furthermore, let Ω be a bounded domain in $\mathbb{R}^2$. We say that Ω is a polygonal domain or the boundary Γ is a polygon if Γ can be decomposed into a finite number m of line segments Γ_j, $j \in \mathcal{S}$, such that $\Gamma = \bigcup_{j\in\mathcal{S}} \bar{\Gamma}_j$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i, j \in \mathcal{S}$ and $i \neq j$. For convenience we count the boundary parts Γ_j counterclockwise. Then we can introduce the corners of the polygonal domain by $x^{(j)} = \bar{\Gamma}_j \cap \bar{\Gamma}_{j+1}$ with the modification $x^{(m)} = \bar{\Gamma}_m \cap \bar{\Gamma}_1$. Moreover, the inner angle between Γ_j and Γ_{j+1} is denoted by ω_j with an analogous modification for ω_m. As usual, we denote by r_j and φ_j the polar coordinates located at the point $x^{(j)}$.

Remark 2.18. Any domain satisfying the requirements of Definition 2.17 has a Lipschitz boundary, cf. [54, Definition 1.2.1.1 and Definition 1.4.5.1].

We will also need an additional partitioning of the domain in the neighborhood of every corner.

Definition 2.19 (Partitioning of the domain around the corners)**.** Let Ω be a domain according to Definition 2.17 with its boundary Γ. The subdomains Ω_{R_j} are defined as the intersection of the domain Ω with a circle which is centered at the corner $x^{(j)}$ and has the radius R_j. The radius R_j can be chosen arbitrarily with the only restriction that the circular sectors Ω_{R_j} do not overlap. The sides of the circular sectors Ω_{R_j}, which coincide with the boundary Γ locally, are denoted by Γ_j^+ ($\varphi_j = \omega_j$) and Γ_j^- ($\varphi_j = 0$). We set $\Gamma_j^\pm := \Gamma_j^+ \cup \Gamma_j^-$. Analogously we define the domains $\Omega_{R_j/i}$ as circular sectors with radii R_j/i, $i \in \{2,4,8,16,32,64\}$. The intersection of the boundary of $\Omega_{R_j/i}$ with the boundary Γ is denoted by $\Gamma^+_{R_j/i}$ for $\varphi_j = \omega_j$ and by $\Gamma^-_{R_j/i}$ for $\varphi_j = 0$. The union of both is $\Gamma^\pm_{R_j/i}$. Moreover, we set

$$\tilde{\Omega}^0 := \Omega \setminus \bigcup_{j=1}^m \Omega_{R_j/16}, \quad \tilde{\Gamma}^0 := \Gamma \cap \mathrm{cl}(\tilde{\Omega}^0),$$
$$\check{\Omega}^0 := \Omega \setminus \bigcup_{j=1}^m \Omega_{R_j/32}, \quad \check{\Gamma}^0 := \Gamma \cap \mathrm{cl}(\check{\Omega}^0),$$
$$\Omega^0 := \Omega \setminus \bigcup_{j=1}^m \Omega_{R_j/64}, \quad \Gamma^0 := \Gamma \cap \mathrm{cl}(\Omega^0).$$

Now we are in the position to introduce weighted Sobolev spaces.

Definition 2.20 ($W^{k,p}_{\beta_j}(\Omega_{R_j})$- and $V^{k,p}_{\beta_j}(\Omega_{R_j})$-spaces)**.** Let $1 \le p \le \infty$, $k \in \mathbb{N}_0$, $j \in \{1,\dots,m\}$, $\beta_j \in \mathbb{R}$ and let $\alpha \in \mathbb{N}_0^2$ be a multi-index. The weighted Sobolev spaces $W^{k,p}_{\beta_j}(\Omega_{R_j})$ and $V^{k,p}_{\beta_j}(\Omega_{R_j})$ denote the set of all functions v on Ω_{R_j} whose weak derivatives $D^\alpha v$ exist for $|\alpha| \le k$ and fulfill

$$\|v\|_{W^{k,p}_{\beta_j}(\Omega_{R_j})} := \left(\sum_{|\alpha|\le k} \|r_j^{\beta_j} D^\alpha v\|^p_{L^p(\Omega_{R_j})} \right)^{1/p} < \infty,$$
$$\|v\|_{V^{k,p}_{\beta_j}(\Omega_{R_j})} := \left(\sum_{|\alpha|\le k} \|r_j^{\beta_j-k+|\alpha|} D^\alpha v\|^p_{L^p(\Omega_{R_j})} \right)^{1/p} < \infty$$

for $1 \le p < \infty$ and

$$\|v\|_{W^{k,\infty}_{\beta_j}(\Omega_{R_j})} := \sum_{|\alpha|\le k} \|r_j^{\beta_j} D^\alpha v\|_{L^\infty(\Omega_{R_j})} < \infty,$$
$$\|v\|_{V^{k,\infty}_{\beta_j}(\Omega_{R_j})} := \sum_{|\alpha|\le k} \|r_j^{\beta_j-k+|\alpha|} D^\alpha v\|_{L^\infty(\Omega_{R_j})} < \infty$$

for $p = \infty$, respectively. The seminorms

$$|\cdot|_{W^{k,p}_{\beta_j}(\Omega_{R_j})} \text{ and } |\cdot|_{V^{k,p}_{\beta_j}(\Omega_{R_j})}$$

are defined by setting $|\alpha| = k$ in the definition of the norms.

Definition 2.21 ($W^{k,p}_{\vec{\beta}}(\Omega)$- and $V^{k,p}_{\vec{\beta}}(\Omega)$-spaces)**.** Let $1 \le p \le \infty$, $k \in \mathbb{N}_0$, $\vec{\beta} = (\beta_1,\dots,\beta_m)^T \in \mathbb{R}^m$ and let $\alpha \in \mathbb{N}_0^2$ be a multi-index. The weighted Sobolev spaces $W^{k,p}_{\vec{\beta}}(\Omega)$ and $V^{k,p}_{\vec{\beta}}(\Omega)$ denote the set of all functions v on Ω whose weak derivatives $D^\alpha v$ exist for $|\alpha| \le k$ and fulfill

$$\|v\|_{W^{k,p}_{\vec{\beta}}(\Omega)} := \|v\|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^m \|v\|_{W^{k,p}_{\beta_j}(\Omega_{R_j})} < \infty,$$
$$\|v\|_{V^{k,p}_{\vec{\beta}}(\Omega)} := \|v\|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^m \|v\|_{V^{k,p}_{\beta_j}(\Omega_{R_j})} < \infty,$$

respectively. Corresponding seminorms are defined by

$$|v|_{W^{k,p}_{\vec\beta}(\Omega)} := |v|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} |v|_{W^{k,p}_{\beta_j}(\Omega_{R_j})},$$
$$|v|_{V^{k,p}_{\vec\beta}(\Omega)} := |v|_{W^{k,p}(\Omega^0)} + \sum_{j=1}^{m} |v|_{V^{k,p}_{\beta_j}(\Omega_{R_j})}.$$

The following remark is with respect to the notation, which will simplify the demonstrations in the sequel.

Remark 2.22. For every subset G of a subdomain Ω_{R_j} and $v \in W^{k,p}_{\beta_j}(\Omega_{R_j})$ we set

$$\|v\|_{W^{k,p}_{\beta_j}(G)} := \left(\sum_{|\alpha|\le k} \|r_j^{\beta_j} D^\alpha v\|^p_{L^p(G)} \right)^{1/p} \qquad \text{if } 1 \le p < \infty,$$
$$\|v\|_{W^{k,\infty}_{\beta_j}(G)} := \sum_{|\alpha|\le k} \|r_j^{\beta_j} D^\alpha v\|_{L^\infty(G)},$$

i.e., for every subset G of Ω_{R_j} the weight in the norm is related to the corner $x^{(j)}$. An analogous modification is also made for the space $V^{k,p}_{\beta_j}(\Omega_{R_j})$ and for all other weighted spaces introduced below.

We will also need weighted Sobolev spaces on the boundary analogously to those given in Definition 2.6.

Definition 2.23 (Weighted Sobolev spaces on the boundary parts Γ_j^+ and Γ_j^-)**.** Let $1 \le p \le \infty$, $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0$, $j \in \{1, \dots, m\}$, $\beta_j \in \mathbb{R}$ and let B be Γ_j^+ or Γ_j^-. The weighted Sobolev spaces $W^{k,p}_{\beta_j}(B)$ and $V^{k,p}_{\beta_j}(B)$ denote the set of all functions v on B whose weak tangential derivatives $\partial_t^\alpha v$ exist on B for $|\alpha| \le k$ such that

$$\|v\|_{W^{k,p}_{\beta_j}(B)} := \left(\sum_{|\alpha|\le k} \|r_j^{\beta_j} \partial_t^\alpha v\|^p_{L^p(B)} \right)^{1/p} < \infty,$$
$$\|v\|_{V^{k,p}_{\beta_j}(B)} := \left(\sum_{|\alpha|\le k} \|r_j^{\beta_j-k+|\alpha|} \partial_t^\alpha v\|^p_{L^p(B)} \right)^{1/p} < \infty$$

for $1 \le p < \infty$, and

$$\|v\|_{W^{k,\infty}_{\beta_j}(B)} := \sum_{|\alpha|\le k} \|r_j^{\beta_j} \partial_t^\alpha v\|_{L^\infty(B)} < \infty,$$
$$\|v\|_{V^{k,\infty}_{\beta_j}(B)} := \sum_{|\alpha|\le k} \|r_j^{\beta_j-k+|\alpha|} \partial_t^\alpha v\|_{L^\infty(B)} < \infty$$

in case of $p = \infty$, respectively. The seminorms

$$|\cdot|_{W^{k,p}_{\beta_j}(B)} \text{ and } |\cdot|_{V^{k,p}_{\beta_j}(B)}$$

are defined analogously to the classical Sobolev seminorms by setting $|\alpha| = k$ in the definition of the corresponding norms.

Definition 2.24 (Weighted Sobolev spaces on the boundary part Γ_j)**.** Let $j \in \{1, \dots, m\}$ and let $1 \le p \le \infty$, $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0$ and $\vec{\beta} = (\beta_{j-1}, \beta_j)^T \in \mathbb{R}^2$ with the modification $j - 1 = m$ if $j = 1$. The weighted Sobolev spaces $W^{k,p}_{\vec{\beta}}(\Gamma_j)$ and $V^{k,p}_{\vec{\beta}}(\Gamma_j)$ consist of all functions v on Γ_j whose weak tangential derivatives $\partial_t^\alpha v$ exist on Γ_j for $|\alpha| \le k$ such that

$$\|v\|_{W^{k,p}_{\vec{\beta}}(\Gamma_j)} := \|v\|_{W^{k,p}(\Gamma_j \cap \Gamma^0)} + \|v\|_{W^{k,p}_{\beta_{j-1}}(\Gamma^-_{j-1})} + \|v\|_{W^{k,p}_{\beta_j}(\Gamma^+_j)} < \infty,$$
$$\|v\|_{V^{k,p}_{\vec{\beta}}(\Gamma_j)} := \|v\|_{W^{k,p}(\Gamma_j \cap \Gamma^0)} + \|v\|_{V^{k,p}_{\beta_{j-1}}(\Gamma^-_{j-1})} + \|v\|_{V^{k,p}_{\beta_j}(\Gamma^+_j)} < \infty,$$

respectively. The seminorms

$$|\cdot|_{W^{k,p}_{\vec{\beta}}(\Gamma_j)} \text{ and } |\cdot|_{V^{k,p}_{\vec{\beta}}(\Gamma_j)}$$

are defined by setting $|\alpha| = k$ in the corresponding norms.

Definition 2.25 (Weighted Sobolev spaces on the boundary Γ)**.** Let $1 \le p \le \infty$, $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0$ and $\vec{\beta} = (\beta_1, \dots, \beta_m)^T \in \mathbb{R}^m$. The weighted Sobolev spaces $W^{k,p}_{\vec{\beta}}(\Gamma)$ and $V^{k,p}_{\vec{\beta}}(\Gamma)$ denote the set of all functions v on Γ whose weak tangential derivatives $\partial_t^\alpha v$ exist on Γ_j for $|\alpha| \le k$ and $j = 1, \dots, m$ such that

$$\|v\|_{W^{k,p}_{\vec{\beta}}(\Gamma)} := \|v\|_{W^{k,p}(\Gamma^0)} + \sum_{j=1}^{m} \|v\|_{W^{k,p}_{\beta_j}(\Gamma^\pm_j)},$$
$$\|v\|_{V^{k,p}_{\vec{\beta}}(\Gamma)} := \|v\|_{W^{k,p}(\Gamma^0)} + \sum_{j=1}^{m} \|v\|_{V^{k,p}_{\beta_j}(\Gamma^\pm_j)}$$

is finite, respectively. The weighted parts in the norms are defined by

$$\|v\|_{W^{k,p}_{\beta_j}(\Gamma^\pm_j)} := \left(\|v\|^p_{W^{k,p}_{\beta_j}(\Gamma^+_j)} + \|v\|^p_{W^{k,p}_{\beta_j}(\Gamma^-_j)} \right)^{1/p},$$
$$\|v\|_{V^{k,p}_{\beta_j}(\Gamma^\pm_j)} := \left(\|v\|^p_{V^{k,p}_{\beta_j}(\Gamma^+_j)} + \|v\|^p_{V^{k,p}_{\beta_j}(\Gamma^-_j)} \right)^{1/p},$$

respectively. The seminorms

$$|\cdot|_{W^{k,p}_{\vec{\beta}}(\Gamma)} \text{ and } |\cdot|_{V^{k,p}_{\vec{\beta}}(\Gamma)}$$

are defined as in Definition 2.24.

Next, we define the trace spaces of the introduced weighted Sobolev spaces. For details we refer to e.g. [80], [81] or [74].

Definition 2.26 (Trace spaces of weighted Sobolev spaces)**.** Let $1 \le p < \infty$ and $k \in \mathbb{N}$. Furthermore, let $\mathring{W}^{k,p}_{\vec{\beta}}(\Omega)$ and $\mathring{V}^{k,p}_{\vec{\beta}}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}_{\vec{\beta}}(\Omega)}$ and $\|\cdot\|_{V^{k,p}_{\vec{\beta}}(\Omega)}$, respectively. The trace spaces $W^{k-1/p,p}_{\vec{\beta}}(\Gamma)$ of $W^{k,p}_{\vec{\beta}}(\Omega)$ and $V^{k-1/p,p}_{\vec{\beta}}(\Gamma)$ of $V^{k,p}_{\vec{\beta}}(\Omega)$ are the quotient spaces

$$W^{k-1/p,p}_{\vec{\beta}}(\Gamma) := W^{k,p}_{\vec{\beta}}(\Omega)/\mathring{W}^{k,p}_{\vec{\beta}}(\Omega)$$

and

$$V^{k-1/p,p}_{\vec{\beta}}(\Gamma) := V^{k,p}_{\vec{\beta}}(\Omega)/\mathring{V}^{k,p}_{\vec{\beta}}(\Omega),$$

respectively. These spaces are endowed with the norms

$$\|v\|_{W^{k-1/p,p}_{\vec{\beta}}(\Gamma)} := \inf\left\{\|u\|_{W^{k,p}_{\vec{\beta}}(\Omega)} : u \in W^{k,p}_{\vec{\beta}}(\Omega), v-u \in \mathring{W}^{k,p}_{\vec{\beta}}(\Omega)\right\}$$

and

$$\|v\|_{V^{k-1/p,p}_{\vec{\beta}}(\Gamma)} := \inf\left\{\|u\|_{V^{k,p}_{\vec{\beta}}(\Omega)} : u \in V^{k,p}_{\vec{\beta}}(\Omega), v-u \in \mathring{V}^{k,p}_{\vec{\beta}}(\Omega)\right\}.$$

Finally, we introduce the weighted Hölder space $N^{k,\sigma}_{\vec{\beta}}(\Omega)$ and its trace space $N^{k,\sigma}_{\vec{\beta}}(\Gamma)$.

Definition 2.27 ($N^{k,\sigma}_{\vec{\beta}}(\Omega)$- and $N^{k,\sigma}_{\vec{\beta}}(\Gamma)$-spaces)**.** Let $k \in \mathbb{N}_0$, $\sigma \in (0,1)$, $\vec{\beta} = (\beta_1, \dots, \beta_m)^T \in \mathbb{R}^m$ and let $\alpha \in \mathbb{N}_0^2$ be a multi-index. Furthermore, let $\mathcal{C}$ denote the set of all corner points. The space $N^{k,\sigma}_{\vec{\beta}}(\Omega)$ consists of all k times continuously differentiable functions in $\bar{\Omega}\backslash\mathcal{C}$ such that the norm

$$\|v\|_{N^{k,\sigma}_{\vec{\beta}}(\Omega)} := \|v\|_{C^{k,\sigma}(\bar{\Omega}^0)} + \sum_{j=1}^{m} \|v\|_{N^{k,\sigma}_{\beta_j}(\Omega_{R_j})} \tag{2.3}$$

is finite, where

$$\begin{aligned}\|v\|_{N^{k,\sigma}_{\beta_j}(\Omega_{R_j})} := &\sum_{|\alpha|\le k} \|r_j^{\beta_j-\sigma-k+|\alpha|} D^\alpha v\|_{C^0(\bar{\Omega}_{R_j})} \\ &+ \sum_{|\alpha|=k} \sup_{x_1,x_2\in\Omega_{R_j}} \frac{\left|r_j(x_1)^{\beta_j}(D^\alpha v)(x_1) - r_j(x_2)^{\beta_j}(D^\alpha v)(x_2)\right|}{|x_1-x_2|^\sigma}.\end{aligned}$$

The space $N^{k,\sigma}_{\vec{\beta}}(\Gamma)$ denotes the trace space of $N^{k,\sigma}_{\vec{\beta}}(\Omega)$ and is given by

$$N^{k,\sigma}_{\vec{\beta}}(\Gamma) := \{v|_{\Gamma\backslash\mathcal{C}} : v \in N^{k,\sigma}_{\vec{\beta}}(\Omega)\}.$$

A norm in that space can be defined analogously to (2.3), cf. [79, Section 2.7].

2.3 Properties of weighted function spaces

In this section we state selected properties of the weighted spaces which we introduced in the previous section. We start with embeddings for weighted Sobolev spaces.

Lemma 2.28. *Let $j \in \{1, \ldots, m\}$ and let $\mathcal{G}$ be Ω_{R_j}, Γ_j^+ or Γ_j^-. Furthermore, let n be the dimension of $\mathcal{G}$ and let l, k be nonnegative integers. Then the following three assertions hold:*

(i) Let $\beta_j' > -n/p$, $\beta_j - \beta_j' \leq k$ and $1 \leq p < \infty$. Then the continuous embedding $W_{\beta_j}^{l+k,p}(\mathcal{G}) \hookrightarrow W_{\beta_j'}^{l,p}(\mathcal{G})$ holds.

(ii) Let $n/q - n/p > \beta_j - \beta_j'$ and $1 \leq q < p \leq \infty$. Then the continuous embedding $W_{\beta_j}^{l,p}(\mathcal{G}) \hookrightarrow W_{\beta_j'}^{l,q}(\mathcal{G})$ is valid.

(iii) Let $\beta_j' > -n/p$, $\beta_j - \beta_j' < 1$ and $1 \leq p < \infty$. Then the compact embedding $W_{\beta_j}^{l+1,p}(\mathcal{G}) \overset{c}{\hookrightarrow} W_{\beta_j'}^{l,p}(\mathcal{G})$ holds.

Proof. (i) Let $\gamma_j := \beta_j' + k$. By Hardy's inequality applied k times and Theorem 2.7 one obtains for $\beta_j' > -n/p$ that

$$W_{\gamma_j}^{l+k,p}(\mathcal{G}) \hookrightarrow W_{\beta_j'}^{l,p}(\mathcal{G}),$$

cf. Lemma 7.1.5 in [71] for the two dimensional case with $p = 2$, and [74, (0.35)] for general p but slightly different notation. We also mention [56, 57, 58]. In these papers similar embeddings are proven for weighted spaces, where the weight function is defined as the distance to the boundary. Now, the first assertion follows immediately since

$$W_{\beta_j}^{l+k,p}(\mathcal{G}) \hookrightarrow W_{\gamma_j}^{l+k,p}(\mathcal{G})$$

for $\beta_j \leq \gamma_j$ which is equivalent to $\beta_j - \beta_j' \leq k$.

(ii) This is a consequence of the Hölder inequality.

(iii) For three space dimensions this is proven in Lemma 8.1.2 in [79]. In one and two space dimensions it can be proven analogously using the continuous embedding of (i). □

The following lemma can directly be deduced from Lemma 2.28 and Theorem 2.7.

Lemma 2.29. *Let $\mathcal{G}$ be the domain Ω, its boundary Γ or a boundary part Γ_j, let n be the dimension of $\mathcal{G}$ and let $I_{\mathcal{G}}$ denote the index set of corners corresponding to $\mathcal{G}$. Furthermore, let l, k be nonnegative integers. Then the following three assertions hold:*

(i) Let $\beta_j' > -n/p$ and $\beta_j - \beta_j' \leq k$ for $j \in I_{\mathcal{G}}$ and $1 \leq p < \infty$. Then there is the continuous embedding $W_{\vec{\beta}}^{l+k,p}(\mathcal{G}) \hookrightarrow W_{\vec{\beta}'}^{l,p}(\mathcal{G})$.

(ii) Let $n/q - n/p > \beta_j - \beta_j'$ for $j \in I_{\mathcal{G}}$ and $1 \leq q < p \leq \infty$. Then the continuous embedding $W_{\vec{\beta}}^{l,p}(\mathcal{G}) \hookrightarrow W_{\vec{\beta}'}^{l,q}(\mathcal{G})$ holds.

(iii) *Let* $\beta'_j > -n/p$ *and* $\beta_j - \beta'_j < 1$ *for* $j \in I_\mathcal{G}$ *and* $1 \le p < \infty$. *Then the compact embedding* $W^{l+1,p}_{\vec{\beta}}(\mathcal{G}) \overset{c}{\hookrightarrow} W^{l,p}_{\vec{\beta}'}(\mathcal{G})$ *is valid.*

Next, we state embeddings in weighted V-spaces. The assertions can either be proven analogously to those of Lemma 2.29 or simply hold due to the definition of these spaces, cf. [79, Sections 2.1.2 and 4.1.4].

Lemma 2.30. *Let* $\mathcal{G}$ *be the domain* Ω, *its boundary* Γ *or a boundary part* Γ_j, *let* n *be the dimension of* $\mathcal{G}$ *and let* $I_\mathcal{G}$ *denote the index set of corners corresponding to* $\mathcal{G}$. *Furthermore, let* l, k *be nonnegative integers. Then the following three assertions are valid:*

(i) *Let* $\beta_j - \beta'_j \le k$ *for* $j \in I_\mathcal{G}$ *and* $1 \le p \le \infty$. *Then the continuous embedding* $V^{l+k,p}_{\vec{\beta}}(\mathcal{G}) \hookrightarrow V^{l,p}_{\vec{\beta}'}(\mathcal{G})$ *holds.*

(ii) *Let* $n/q - n/p > \beta_j - \beta'_j$ *for* $j \in I_\mathcal{G}$ *and* $1 \le q < p \le \infty$. *Then the continuous embedding* $V^{l,p}_{\vec{\beta}}(\mathcal{G}) \hookrightarrow V^{l,q}_{\vec{\beta}'}(\mathcal{G})$ *is valid.*

(iii) *Let* $\beta_j - \beta'_j < 1$ *for* $j \in I_\mathcal{G}$ *and* $1 \le p < \infty$. *Then the compact embedding* $V^{l+1,p}_{\vec{\beta}}(\mathcal{G}) \overset{c}{\hookrightarrow} V^{l,p}_{\vec{\beta}'}(\mathcal{G})$ *holds.*

Based on Lemma 2.29 we can show the following norm equivalence, which will be essential for the derivation of interpolation error estimates on graded triangulations in Section 3.2.2.

Lemma 2.31. *Let* $\mathcal{G}$ *be the domain* Ω, *its boundary* Γ *or a boundary part* Γ_j, *let* n *be the dimension of* $\mathcal{G}$ *and let* $I_\mathcal{G}$ *denote the index set of corners corresponding to* $\mathcal{G}$. *Furthermore, let* $q \in [1, \infty)$, $-n/q < \beta_j < n - n/q + 1$ *for* $j \in I_\mathcal{G}$, $k \ge 0$ *and* $v \in W^{k+1,q}_{\vec{\beta}}(\mathcal{G})$. *Then the norm equivalence*

$$\|v\|_{W^{k+1,q}_{\vec{\beta}}(\mathcal{G})} \sim |v|_{W^{k+1,q}_{\vec{\beta}}(\mathcal{G})} + \sum_{|\alpha| \le k} \left| \int_\mathcal{G} D^\alpha v \right| \tag{2.4}$$

is valid.

Proof. This assertion has already been proven in Lemma 2.2 of [11], where the authors assume that $1 - 2/q < \beta_j \le 1$. Let $\vec{1} = (1, \ldots, 1) \in \mathbb{R}^l$ with $l = \# I_\mathcal{G}$. According to Lemma 2.29 one has

$$W^{k+1,q}_{\vec{\beta}}(\mathcal{G}) \hookrightarrow W^{k+1,1}_{\vec{1}}(\mathcal{G}) \hookrightarrow W^{k,1}(\mathcal{G}) \text{ and } W^{k+1,q}_{\vec{\beta}}(\mathcal{G}) \overset{c}{\hookrightarrow} W^{k,q}_{\vec{\beta}}(\mathcal{G}) \tag{2.5}$$

for $-n/q < \beta_j < n - n/q + 1$. These two embeddings are essential to prove the norm equivalence (2.4). In fact, tracing through the proof of Lemma 2.2 in [11] reveals that the condition $1 - 2/q < \beta_j \le 1$ can simply be replaced by $-n/q < \beta_j < n - n/q + 1$ by means of (2.5). □

Note that the classical Sobolev spaces $W^{k,p}(\Omega)$ are included in the weighted Sobolev spaces $W^{k,p}_{\vec{\beta}}(\Omega)$ by setting $\beta_j = 0$ for $j = 1, \ldots, m$, whereas they do not belong automatically to the scale of the weighted spaces $V^{k,p}_{\vec{\beta}}(\Omega)$. However, there is a relation between $W^{k,p}_{\vec{\beta}}(\Omega)$ and $V^{k,p}_{\vec{\beta}}(\Omega)$.

Lemma 2.32. *Let η_j, $j = 1, \dots, m$, be infinitely differentiable cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/64}$ and $\operatorname{supp} \eta_j \subset \Omega_{R_j}$. Then the following two assertions hold:*

(i) If $\beta_j \leq -1$ or $\beta_j > k-1$ for $j = 1, \dots, m$ with $k \in \mathbb{N}_0$. Then the spaces $W^{k,2}_{\vec{\beta}}(\Omega)$ and $V^{k,2}_{\vec{\beta}}(\Omega)$ coincide and the norms in $W^{k,2}_{\vec{\beta}}(\Omega)$ and $V^{k,2}_{\vec{\beta}}(\Omega)$ are equivalent.

(ii) Suppose that $\vec{\beta}$ satisfies the condition $s_j - 1 < \beta_j < s_j$ for $j = 1, \dots, m$ with $s_j \in \{0, 1, \dots, k-1\}$ and $k \in \mathbb{N}$. Then one has

$$W^{k,2}_{\vec{\beta}}(\Omega) = V^{k,2}_{\vec{\beta}}(\Omega) \oplus \eta_1 \mathcal{P}_{k-s_1-1}(\Omega) \oplus \cdots \oplus \eta_m \mathcal{P}_{k-s_m-1}(\Omega),$$

where $\mathcal{P}_{k-s_j-1}(\Omega)$ denotes the set of polynomials on Ω with degree not greater than $k - s_j - 1 \geq 0$ and $\mathcal{P}_{k-s_j-1}(\Omega) = \{0\}$ if $k - s_j - 1 < 0$. In particular, for any $v \in W^{k,2}_{\vec{\beta}}(\Omega)$ one can write $v = v_s + \sum_{j=1}^m \eta_j p_{k-s_j-1}(v)$ with $v_s \in V^{k,2}_{\vec{\beta}}(\Omega)$ and $p_{k-s_j-1}(v)$ being the projection of v into $\mathcal{P}_{k-s_j-1}(\Omega)$. Moreover, the norm equivalence

$$\|v\|_{W^{k,2}_{\vec{\beta}}(\Omega)} \sim \|v_s\|_{V^{k,2}_{\vec{\beta}}(\Omega)} + \sum_{j=1}^m \left(\sum_{|\alpha| \leq k-s_j-1} |(D^\alpha v)(x^{(j)})| \right) \tag{2.6}$$

is valid. In addition, $v \in V^{k,2}_{\vec{\beta}}(\Omega)$ if and only if $\sum_{j=1}^m \left(\sum_{|\alpha| \leq k-s_j-1} |(D^\alpha v)(x^{(j)})| \right) = 0$.

Proof. To show this lemma one can follow the lines of the proof Theorem 7.1.1 of [71], which represents this relation for domains with only one corner. The extension to general polygonal domains is obvious, cf. [71, p. 273]. □

Remark 2.33. The more general result for the spaces $W^{k,p}_{\vec{\beta}}(\Omega)$ and $V^{k,p}_{\vec{\beta}}(\Omega)$ with $p \in (1, \infty)$ can be found in e.g. [81, Theorem 2.1].

We also recall Lemma 7.1.6 of [71], again extended to general polygonal domains. It gives us an estimate for $|(D^\alpha v)(x^{(j)})|$ in (2.6) with $|\alpha| = k - s_j - 1$.

Lemma 2.34. *Suppose that $\vec{\beta}$ satisfies the condition $s_j - 1 < \beta_j < s_j$ for $j = 1, \dots, m$ with $s_j \in \{0, 1, \dots, k-1\}$ and $k \in \mathbb{N}$. Then for all $v \in W^{k,2}_{\vec{\beta}}(\Omega)$ there is the estimate*

$$\sum_{j=1}^m \left(\sum_{|\alpha| = k-s_j-1} |(D^\alpha v)(x^{(j)})| \right) \leq \epsilon \|v\|_{W^{k,2}_{\vec{\beta}}(\Omega)} + c_\epsilon \|v\|_{W^{k-1,2}_{\vec{\beta}}(\Omega)},$$

where ϵ is an arbitrary constant with $0 < \epsilon < 1$ and the constant $c_\epsilon > 0$ depends only on ϵ.

A result comparable to that of Lemma 2.32 also holds for the trace spaces $W^{k-1/2,2}_{\vec{\beta}}(\Gamma)$ and $V^{k-1/2,2}_{\vec{\beta}}(\Gamma)$, cf. [71, Theorem 7.1.2 and Theorem 7.1.3]. However, we will only need the analogue of Lemma 2.32 (i).

Lemma 2.35. *Suppose that $\vec{\beta}$ satisfies the condition $\beta_j \le -1$ or $\beta_j > k-1$ for $j = 1, \dots, m$ with $k \in \mathbb{N}$. Then the trace spaces $W^{k-1/2,2}_{\vec{\beta}}(\Gamma)$ and $V^{k-1/2,2}_{\vec{\beta}}(\Gamma)$ coincide and the norms in $W^{k-1/2,2}_{\vec{\beta}}(\Gamma)$ and $V^{k-1/2,2}_{\vec{\beta}}(\Gamma)$ are equivalent.*

Proof. This is a direct consequence of Lemma 2.32 (i) due to the definition of the trace spaces. □

Next, we discuss equivalent norms and embeddings for the introduced weighted Hölder spaces.

Lemma 2.36. *There is the norm equivalence*

$$\|v\|_{N^{k,\sigma}_{\beta_j}(\Omega_{R_j})} \sim \sum_{|\alpha|\le k} \|r_j^{\beta_j-\sigma-k+|\alpha|} D^\alpha v\|_{C^0(\bar{\Omega}_{R_j})} + \sum_{|\alpha|=k} \sup_{\substack{x_1,x_2\in\Omega_{R_j} \\ |x_1-x_2|\le \frac{1}{2} r_j(x_1)}} r_j(x_1)^{\beta_j} \frac{|(D^\alpha v)(x_1) - (D^\alpha v)(x_2)|}{|x_1-x_2|^\sigma}.$$

Proof. For the proof we refer to Section 1.1 of [101]. There, a similar norm equivalence is proven for the spaces $V^{k,p}_{\vec{\beta}}(\Omega)$. The same techniques can be used in the present case, cf. Section 5 of [81]. □

Lemma 2.37. *Let $k+\sigma > k'+\sigma'$ and $k+\sigma-\beta_j = k'+\sigma'-\beta'_j$ for $j = 1, \dots, m$. Then the continuous embeddings $N^{k,\sigma}_{\vec{\beta}}(\Omega) \hookrightarrow N^{k',\sigma'}_{\vec{\beta}'}(\Omega)$ and $N^{k,\sigma}_{\vec{\beta}}(\Gamma) \hookrightarrow N^{k',\sigma'}_{\vec{\beta}'}(\Gamma)$ are valid.*

Proof. The proof is based on the norm equivalence of Lemma 2.36 and the mean value theorem, cf. Lemma 2.7.1 of [79]. □

Lemma 2.38. *Suppose that the conditions $k - 2/p > l + \sigma$ and $\delta_j - l - \sigma + k - \beta_j - 2/p \ge 0$, $j = 1, \dots, m$, are fulfilled. Then the continuous embedding $V^{k,p}_{\vec{\beta}}(\Omega) \hookrightarrow N^{l,\sigma}_{\vec{\delta}}(\Omega)$ holds.*

Proof. For three dimensional domains Ω such a result is proven in Lemma 3.6.2 in [79]. In two space dimensions this can be proven analogously. The proof is based on the classical Sobolev embedding theorem together with a transformation to some reference domain. □

Finally, let us end this section with some selected properties for weighted Sobolev spaces, which can be deduced by the previous results. These properties will simplify the demonstrations in the sequel.

Lemma 2.39. *The following assertions hold:*

(i) Let η_j, $j = 1, \dots, m$, be infinitely differentiable cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/64}$ and $\operatorname{supp} \eta_j \subset \Omega_{R_j}$. Then the estimate

$$\|\eta_j r_j^{\beta_j} v\|_{W^{1/2,2}(\Gamma)} \le c\|\eta_j v\|_{V^{1/2,2}_{\vec{\beta}}(\Gamma)}$$

is valid for $j = 1, \dots, m$.

(ii) Let $0 \le \beta_j < 1$ for $j = 1,\dots,m$. Then there are parameters $r > 1$ and $s > 1$ such that the embeddings

$$W^{0,2}_{\vec\beta}(\Omega) \hookrightarrow L^r(\Omega) \quad \textit{and} \quad W^{1/2,2}_{\vec\beta}(\Gamma) \hookrightarrow L^s(\Gamma)$$

hold.

(iii) Let $k \in \mathbb{N}_0$ and $\sigma \in (0,1)$. Furthermore, let $\gamma_j = \delta_j - \sigma$ and $1+\beta_j > \gamma_j$ for $j = 1,\dots,m$. Then there are the embeddings

$$N^{k,\sigma}_{\vec\delta}(\Omega) \hookrightarrow W^{k,2}_{\vec\beta}(\Omega) \quad \textit{and} \quad N^{k+1,\sigma}_{\vec\delta}(\Gamma) \hookrightarrow W^{k+1/2,2}_{\vec\beta}(\Gamma).$$

Proof. (i) This is due to the definition of the trace spaces, the estimate

$$\|\eta_j r_j^{\beta_j} v\|_{W^{1,2}(\Omega)} \le c\|\eta_j v\|_{V^{1,2}_{\vec\beta}(\Omega)},$$

which can be deduced by straightforward calculations, and Corollary 2.14.

(ii) For the first embedding we notice that there exists a parameter r with $1 < r < 2/(\beta_j + 1)$ if $\beta_j < 1$ for $j = 1,\dots,m$. Thus, Lemma 2.29 implies the first assertion. Now, we prove the second embedding. We suppose that v is a function in $W^{1/2,2}_{\vec\beta}(\Gamma)$ with $0 \le \beta_j < 1$ for $j = 1,\dots,m$. Next, let η_j be the cut-off functions of the first part of this lemma and $\eta_0 := 1 - \sum_{j=1}^m \eta_j$. Furthermore, let $I_0 \subseteq \{1,\dots,m\}$ be the index set, where $\beta_j = 0$, and $I_1 \subseteq \{1,\dots,m\}$ the index set, where $0 < \beta_j < 1$. Then we can conclude according to the definition of the trace spaces, Corollary 2.14 and Lemma 2.35 that

$$\begin{aligned}\|v\|_{W^{1/2,2}_{\vec\beta}(\Gamma)} &\sim \|\eta_0 v\|_{W^{1/2,2}(\Gamma)} + \sum_{j\in I_0}\|\eta_j v\|_{W^{1/2,2}(\Gamma)} + \sum_{j\in I_1}\|\eta_j v\|_{W^{1/2,2}_{\vec\beta}(\Gamma)}\\ &\sim \|\eta_0 v\|_{W^{1/2,2}(\Gamma)} + \sum_{j\in I_0}\|\eta_j v\|_{W^{1/2,2}(\Gamma)} + \sum_{j\in I_1}\|\eta_j v\|_{V^{1/2,2}_{\vec\beta}(\Gamma)}.\end{aligned} \tag{2.7}$$

For $j \in I_1$ let us choose the parameter p such that $1/(1-\beta_j) < p < \infty$, which is possible since $\beta_j < 1$. Then there exists a parameter s with $1 < s < \min(1/(\beta_j + 1/p), p)$ since the condition on p implies $\beta_j + 1/p < 1$ and $p > 1$. Based on this we can deduce from Lemma 2.28, Theorem 2.7 and the result of (i) that

$$\|\eta_j v\|_{L^s(\Gamma_j^\pm)} \le c\|\eta_j v\|_{W^{0,p}_{\beta_j}(\Gamma_j^\pm)} = c\|\eta_j r_j^{\beta_j} v\|_{L^p(\Gamma)} \le c\|\eta_j r_j^{\beta_j} v\|_{W^{1/2,2}(\Gamma)} \le c\|\eta_j v\|_{V^{1/2,2}_{\vec\beta}(\Gamma)}. \tag{2.8}$$

For $j \in I_0$ we observe that

$$\|\eta_j v\|_{L^s(\Gamma)} \le c\|\eta_j v\|_{W^{1/2,2}(\Gamma)} \tag{2.9}$$

due to Theorem 2.7. Now, the second embedding of the assertion follows from (2.7)–(2.9) and Theorem 2.7 applied to $\|\eta_0 v\|_{W^{1/2,2}(\Gamma)}$.

(iii) Using the Hölder inequality as for part 2 of Lemma 2.29, one can conclude

$$N^{k,\sigma}_{\vec\delta}(\Omega) \hookrightarrow V^{k,2}_{\vec\beta}(\Omega) \hookrightarrow W^{k,2}_{\vec\beta}(\Omega),$$

where the embedding in the last step is trivial. The second assertion can be deduced in the same manner having regard to the definition of the trace spaces. □

CHAPTER 3

Elliptic boundary value problems

In this chapter we analyze linear as well as semilinear elliptic boundary value problems with Neumann boundary data in polygonal domains. It consists of two parts. In Section 3.1, the first part, we focus on regularity results in classical as well as in weighted Sobolev spaces. The error analysis for a finite element discretization of both problems by linear finite elements can be found in the second part, Section 3.2.

For the purpose of a short notation we set $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)^T = (\pi/\omega_1, \ldots, \pi/\omega_m)^T$ and $\vec{a} = (a, \ldots, a)^T \in \mathbb{R}^m$ for any real number a, e.g. $\vec{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$. Furthermore, all inequalities containing vectorial parameters should be understood component-by-component in the sequel. Let us also remark that the constant c will denote a generic positive constant, which may take different values at each occurrence. Moreover, in this chapter we are going to track the dependencies of this constant on the data of the elliptic boundary value problems. This is due to the fact, that the data will also depend on discrete functions, when discussing semilinear elliptic Neumann boundary control problems in Section 4.4. By this approach we will be able to ensure that the constants are independent of the mesh parameter.

3.1 Regularity results in classical and weighted spaces

As already announced in the prefix of this chapter we derive regularity results for linear as well as for semilinear elliptic boundary value problems with Neumann boundary data in polygonal domains. In the first part, in Section 3.1.1, we focus on regularity results for linear problems, whereas the results for semilinear problems are proven in Section 3.1.2. More precisely, employing known results from the literature, cf. [67, 102, 48], we start in Section 3.1.1 with regularity results in Sobolev Slobodetskij spaces for the weak solution of linear problems, which hold independently of the interior angles. Furthermore, we derive regularity results in weighted $W^{2,2}$- and $W^{2,\infty}$-spaces, which are mainly based on the results of [70, 80, 81, 90, 71, 72, 79, 54]. Afterwards, in Section 3.1.2, we transfer all these results to the weak solution of semilinear elliptic boundary value problems by employing the corresponding results for the linear problem and the assumptions on the nonlinearity, especially the monotonicity and the Lipschitz continuity. Furthermore, in the final part of each subsection, we derive for each problem Lipschitz estimates, which will frequently be used in Section 4.4 for the discretization error analysis of semilinear Neumann boundary control problems.

3.1.1 Linear elliptic problems

This section is devoted to solvability and regularity results for the boundary value problem

$$\begin{aligned} -\Delta y + \alpha y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.1}$$

where the domain Ω is a polygonal domain according to Definition 2.17 with m corner points and boundary $\Gamma = \bigcup_{j=1}^{m} \bar{\Gamma}_j$. Depending on the desired regularity of the solution y, we require that one of the following two assumptions for the function α holds.

Assumption 3.1. Let m, M be constants greater than zero and let E_Ω be a subset of Ω with $|E_\Omega| > 0$.

(A1) The function $\alpha \in L^\infty(\Omega)$ fulfills $\alpha(x) \geq 0$ for a.a. $x \in \Omega$, $\alpha(x) \geq m$ for a.a. $x \in E_\Omega$ and $\|\alpha\|_{L^\infty(\Omega)} \leq M$.

(A2) The function α belongs to $C^{0,\sigma}(\bar{\Omega})$ with $\sigma \in (0, 1]$ and $\|\alpha\|_{C^{0,\sigma}(\Omega)} \leq M$ and satisfies $\alpha(x) \geq 0$ for all $x \in \Omega$ and $\alpha(x) \geq m$ for all $x \in E_\Omega$.

As we will see later in this section the precise regularity assumptions for f and g also depend on the desired regularity of y. For the moment we only assume the regularity that allows us to introduce the concept of weak solutions. Based on this, we will explain the difficulties with polygonal domains and how we will proceed. Note, in the sequel we will denote by V^* the dual space of some space V.

Definition 3.2. Let $f \in H^1(\Omega)^*$ and $g \in H^{1/2}(\Gamma)^*$. Furthermore, let Assumption 3.1 (A1) be fulfilled. Then a weak solution of (3.1) is an element $y \in H^1(\Omega)$ that satisfies

$$a(y, v) = \int_\Omega f v + \int_\Gamma g v \quad \forall v \in H^1(\Omega), \tag{3.2}$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(y, v) := \int_\Omega \left(\nabla y \cdot \nabla v + \alpha y v \right). \tag{3.3}$$

Let us remark that the existence and uniqueness of such a solution can be deduced by the Lax-Milgram Theorem for general Lipschitz domains, cf. Lemma 3.4. If the boundary of the domain is smooth enough, or more precisely the boundary Γ of the domain Ω is of class $C^{k+1,1}$ with some $k \geq 0$, then one can apply shift theorems to deduce higher regularity of the weak solution y. By this we obtain $y \in W^{k+2,2}(\Omega)$ for $f \in W^{k,2}(\Omega)$ and $g \in W^{k+1/2,2}(\Gamma)$, cf. [54, Theorem 2.4.2.7 and Theorem 2.5.1.1]. In polygonal domains this statement fails in general due to the appearance of singular terms in the solution, which are caused by the corners. In fact, let $f \in L^2(\Omega)$ and $g \in W^{1/2,2}(\Gamma)$ and let η_j, $j = 1, \dots, m$, be infinitely differentiable cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/64}$ and $\operatorname{supp} \eta_j \subset \Omega_{R_j}$. Then one can show that the solution in polygonal domains has the asymptotics

$$y = \sum_{j=1}^{m} \eta_j c_{0,j} + \sum_{j:\omega_j > \pi} \eta_j c_{1,j} r_j^{\lambda_j} \cos(\lambda_j \varphi_j) + y_{reg},$$

where $c_{0,j}$ and $c_{1,j}$ are some constants, $\lambda_j = \pi/\omega_j$ and the function y_{reg} belongs to $H^2(\Omega)$, cf. [90, Chapter 2]. It is easy to check, that the functions $r_j^{\lambda_j}$ does not belong to $H^2(\Omega)$ if the associated angle ω_j is greater than π. For more general data $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ with $\max(0, 1 - \lambda_j) < \beta_j < 1$ there even does not exist the regular part $y_{reg} \in H^2(\Omega)$. In this case the asymptotic representation reduces to

$$y = \sum_{j=1}^{m} \eta_j c_{0,j} + y_{sing}, \tag{3.4}$$

where y_{sing} belongs to $V^{2,2}_{\vec{\beta}}(\Omega)$ but not to $H^2(\Omega)$ in general, see again [90, Chapter 2]. At this point we also want to emphasize that, in contrast to the Dirichlet problem, the solution y of (3.1) does not belong to $V^{2,2}_{\vec{\beta}}(\Omega)$ in general for $\max(0, 1 - \lambda_j) < \beta_j < 1$. This can be explained by the fact, that the terms $\eta_j c_{0,j}$ in the asymptotic representation (3.4) are not contained in the space $V^{2,2}_{\vec{\beta}}(\Omega)$ for $\max(0, 1 - \lambda_j) < \beta_j < 1$, cf. Lemma 2.32. Instead, based on such representations, one can prove regularity results in the weighted Sobolev spaces $W^{2,2}_{\vec{\beta}}(\Omega)$ and also regularity results in the weighted spaces $W^{2,\infty}_{\vec{\gamma}}(\Omega)$ and $W^{2,\infty}_{\vec{\gamma}}(\Gamma)$, respectively, with $\max(0, 2 - \lambda_j) < \gamma_j < 2$. Of course, for the latter regularity results one has to ensure that the data admit such a solution.

Now, let us outline how we proceed in the further course of this section. First, we obtain by means of the Lax-Milgram Theorem that the solution y belongs to the space $H^1(\Omega)$. Then we show that it belongs independently of the interior angles to some Sobolev Slobodetskij space even in case of lower regularity assumptions on the data than in the demonstrations before. This kind of regularity is especially needed for the discussion of the semilinear Neumann boundary control problems in Section 4.3. Afterwards, we address the derivation of regularity results in weighted Sobolev spaces. First, we prove that the solution belongs to the space $V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)$. Having such a regular solution at hand, we show for this solution that it admits the splitting (3.4). Based on this we derive regularity results in the weighted Sobolev spaces $W^{2,2}_{\vec{\beta}}(\Omega)$. Furthermore, we show for such solutions that they also belong to the weighted spaces $W^{2,\infty}_{\vec{\gamma}}(\Omega)$ and $W^{2,\infty}_{\vec{\gamma}}(\Gamma)$, if the data belong to some weighted Hölder spaces. Finally, we derive certain Lipschitz estimates for the weak solutions of linear elliptic problems, which are required for the numerical analysis of the semilinear Neumann boundary control problems.

Let us recall the Lax-Milgram Theorem, cf. e.g. [20, Theorem 2.7.7 and Remark 2.7.11]. This allows us to deduce existence and uniqueness of a weak solution of (3.1), afterwards.

Theorem 3.3 (Lax-Milgram)**.** *Let c and c_* be constants greater than zero. Furthermore, let a Hilbert space $(V, (\cdot,\cdot))$, a continuous, coercive bilinear form $a(\cdot,\cdot)$, i.e.,*

$$\begin{aligned} |a(y,v)| &\le c\|y\|_V\|v\|_V \quad \forall y,v \in V \ \textit{(continuity)},\\ a(y,y) &\ge c_*\|y\|_V^2 \qquad \forall y \in V \ \textit{(coercivity)}, \end{aligned}$$

and a continuous linear functional F belonging to the dual space V^* of V be given. Then there exists a unique $y \in V$ such that

$$a(y,v) = F(v) \quad \forall v \in V.$$

Furthermore, there is the estimate

$$\|y\|_V \le \frac{1}{c_*}\|F\|_{V^*}, \tag{3.5}$$

where c_* denotes the coercivity constant.

Lemma 3.4. *Suppose that Assumption 3.1 (A1) holds. Then problem* (3.1) *has a unique weak solution $y \in H^1(\Omega)$ for*

(i) $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$. Furthermore, there exists a positive constant $c = c(E_\Omega, m)$, independent of f, g and α, such that

$$\|y\|_{H^1(\Omega)} \le c\left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right).$$

(ii) $f \in W^{0,2}_{\vec\beta}(\Omega)$ and $g \in W^{1/2,2}_{\vec\beta}(\Gamma)$ with $0 \le \beta_j < 1$ for $j = 1,\dots,m$. Moreover, there holds the estimate

$$\|y\|_{H^1(\Omega)} \le c\left(\|f\|_{W^{0,2}_{\vec\beta}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\beta}(\Gamma)}\right)$$

with a positive constant $c = c(E_\Omega, m)$ independent of f, g and α.

Proof. The proof relies on the Lax-Milgram Theorem stated above. For its application we have to show that the bilinear form is continuous and coercive on $H^1(\Omega)$. We begin with proving the continuity of the bilinear form. There holds

$$|a(y,v)| = \left|\int_\Omega \nabla y\cdot\nabla v + \int_\Omega \alpha y v\right| \le (1+M)\|y\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \tag{3.6}$$

where we used the Cauchy-Schwarz inequality, the boundedness of α in $L^\infty(\Omega)$ with norm M and the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. When showing the coercivity of the bilinear form we have to take care, that the coercivity constant does not depend on α since the estimate (3.5) depends on that constant. For that reason we first employ the assumption $\alpha(x) \ge m$ for a.a. $x \in E_\Omega$. Afterwards we apply the Cauchy-Schwarz inequality and the Poincaré inequality. This yields

$$\begin{aligned} a(y,y) &= \int_\Omega |\nabla y|^2 + \int_\Omega \alpha y^2 \ge \int_\Omega |\nabla y|^2 + m\int_{E_\Omega} y^2 \ge \int_\Omega |\nabla y|^2 + \frac{m}{|E_\Omega|}\left(\int_{E_\Omega} y\right)^2\\ &\ge \min(1, \frac{m}{|E_\Omega|})\left(\int_\Omega |\nabla y|^2 + \left(\int_{E_\Omega} y\right)^2\right) \ge \frac{1}{c_{E_\Omega}}\min(1,\frac{m}{|E_\Omega|})\|y\|^2_{H^1(\Omega)}, \end{aligned} \tag{3.7}$$

where c_{E_Ω} denotes the positive constant form the Poincaré inequality which only depends on E_Ω and Ω. Now, the unique solvability of problem (3.1) in $H^1(\Omega)$ can be deduced from the Lax-Milgram Theorem provided that the functional

$$F(v) := \int_\Omega fv + \int_\Gamma gv$$

belongs to the dual space $H^1(\Omega)^*$ of $H^1(\Omega)$. Furthermore, the estimate

$$\|y\|_{H^1(\Omega)} \leq c\|F\|_{H^1(\Omega)^*} \tag{3.8}$$

is valid with some constant c only depending on E_Ω, Ω and m. In case (i) there holds

$$\begin{aligned} |F(v)| = \left|\int_\Omega fv + \int_\Gamma gv\right| &\leq \|f\|_{L^r(\Omega)}\|v\|_{L^{r/(r-1)}(\Omega)} + \|g\|_{L^s(\Gamma)}\|v\|_{L^{s/(s-1)}(\Gamma)} \\ &\leq c\left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right)\|v\|_{H^1(\Omega)}, \end{aligned} \tag{3.9}$$

where we used the Hölder inequality, the embedding $H^1(\Omega) \hookrightarrow L^{r/(r-1)}(\Omega)$ and Theorem 2.8. In case (ii), we use (3.9) and Lemma 2.39. By this we can show that the functional F also admits the estimate

$$|F(v)| \leq c\left(\|f\|_{W^{0,2}_{\vec\beta}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\beta}(\Gamma)}\right)\|v\|_{H^1(\Omega)} \tag{3.10}$$

for arbitrary $f \in W^{0,2}_{\vec\beta}(\Omega)$ and $g \in W^{1/2,2}_{\vec\beta}(\Gamma)$ with β_j satisfying $0 \leq \beta_j < 1$ for $j = 1, \ldots, m$. Thus, in both cases, the linear functional F belongs to $H^1(\Omega)^*$ and the Lax-Milgram Theorem can be applied to deduce the unique solvability. Finally, the estimates of the assertion can be obtained from (3.8), (3.9) and (3.10). □

Next, we are going to show regularity higher than $H^1(\Omega)$ for $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$. It is based on the following lemma, which can be deduced from [48, Theorem 9.2]. We also mention [67], [102] and [51], where similar results are proven.

Lemma 3.5. *Let $F \in H^{2-t}(\Omega)^*$ and $G \in H^{3/2-t}(\Gamma)^*$ with $t \in (1, 3/2)$ satisfy*

$$\int_\Omega F + \int_\Gamma G = 0.$$

Then the problem

$$\begin{aligned} -\Delta u &= F \quad \text{in } \Omega, \\ \partial_n u &= G \quad \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned}$$

has a unique (modulo additive constants) weak solution $u \in H^t(\Omega)$ which fulfills the a priori estimate

$$|u|_{H^t(\Omega)} \leq c\left(\|F\|_{H^{2-t}(\Omega)^*} + \|G\|_{H^{3/2-t}(\Gamma)^*}\right).$$

Corollary 3.6. *Suppose that Assumption 3.1 (A1) holds. Furthermore, let* $r \in (1,4/3)$, $s \in (1,2)$ *and* $t = \min(3-2/r, 2-1/s)$. *Then the weak solution of problem* (3.1) *belongs to* $H^t(\Omega)$ *for* $f \in L^r(\Omega)$ *and* $g \in L^s(\Gamma)$ *and satisfies*

$$\|y\|_{H^t(\Omega)} \le c\left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right)$$

with a positive constant $c = c(E_\Omega, m, M)$ *independent of* f, g *and* α.

Proof. First, let us note that the solution y belongs to $H^1(\Omega)$ according to Lemma 3.4. Furthermore, we observe that $r \in (1,4/3)$ and $s \in (1,2)$ imply $t \in (1,3/2)$. Next, we are going to show $g \in H^{3/2-t}(\Gamma)^*$, $\alpha y \in H^{2-t}(\Omega)^*$ and $f \in H^{2-t}(\Omega)^*$ for $g \in L^s(\Gamma)$, $f \in L^r(\Omega)$ and for α satisfying Assumption 3.1 (A1). We get

$$\|g\|_{H^{3/2-t}(\Gamma)^*} = \sup_{\substack{v \in H^{3/2-t}(\Gamma)\\ v \neq 0}} \frac{|\int_\Gamma gv|}{\|v\|_{H^{3/2-t}(\Gamma)}} \le c \sup_{\substack{v \in H^{3/2-t}(\Gamma)\\ v \neq 0}} \frac{\|g\|_{L^s(\Gamma)}\|v\|_{L^{1/(t-1)}(\Gamma)}}{\|v\|_{H^{3/2-t}(\Gamma)}}, \tag{3.11}$$

where we applied the Hölder inequality in the last step. One obtains from Theorem 2.7

$$\|v\|_{L^{1/(t-1)}(\Gamma)} \le c\|v\|_{H^{3/2-t}(\Gamma)}. \tag{3.12}$$

Thus, the inequalities (3.11) and (3.12) imply $g \in H^{3/2-t}(\Gamma)^*$ if $g \in L^s(\Gamma)$. In the same manner one can show for the function f

$$\|f\|_{H^{2-t}(\Omega)^*} = \sup_{\substack{v \in H^{2-t}(\Omega)\\ v \neq 0}} \frac{|\int_\Omega fv|}{\|v\|_{H^{2-t}(\Omega)}} \le c \sup_{\substack{v \in H^{2-t}(\Omega)\\ v \neq 0}} \frac{\|f\|_{L^r(\Omega)}\|v\|_{L^{2/(t-1)}(\Omega)}}{\|v\|_{H^{2-t}(\Omega)}}. \tag{3.13}$$

Furthermore, an application of Theorem 2.7 yields

$$\|v\|_{L^{2/(t-1)}(\Omega)} \le c\|v\|_{H^{2-t}(\Omega)} \tag{3.14}$$

and

$$\|\alpha y\|_{L^r(\Omega)} \le c\|y\|_{L^r(\Omega)} \le c\|y\|_{H^1(\Omega)}, \tag{3.15}$$

where the positive constant c only depends on M. Consequently, we can conclude from (3.13) and (3.14) that f belongs to $H^{2-t}(\Omega)^*$ for $f \in L^r(\Omega)$ and analogously $\alpha y \in H^{2-t}(\Omega)^*$ for $g \in L^s(\Gamma)$, $f \in L^r(\Omega)$ and for α satisfying Assumption 3.1 (A1) by employing (3.15). Next, we observe that the solution $y \in H^1(\Omega)$ of (3.1) also solves

$$\begin{aligned} -\Delta y &= f - \alpha y \quad && \text{in } \Omega,\\ \partial_n y &= g && \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned}$$

in the weak sense. Therefore, using the results of Lemma 3.5 we can conclude that the weak solution y of (3.1) fulfills

$$\begin{aligned} \|y\|_{H^t(\Omega)} \le \|y\|_{H^1(\Omega)} + |y|_{H^t(\Omega)} &\le c\left(\|y\|_{H^1(\Omega)} + \|f - \alpha y\|_{H^{2-t}(\Omega)^*} + \|g\|_{H^{3/2-t}(\Gamma)^*}\right)\\ &\le c\left(\|y\|_{H^1(\Omega)} + \|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right) \end{aligned}$$

with a constant $c = c(M)$, where we inserted the inequalities (3.11)–(3.15). Finally, the a priori estimate of Lemma 3.4 (i) proves the assertion. □

Remark 3.7. Corollary 3.6 allows us to conclude $y \in C^0(\bar{\Omega})$ according to Theorem 2.7. Moreover, we get the validity of the a priori estimate

$$\|y\|_{C^0(\bar{\Omega})} \leq c\left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right) \tag{3.16}$$

with some arbitrary $r, s > 1$ and a constant $c = c(E_\Omega, m, M) > 0$ independent of f, g and α.

Remark 3.8. One can also show $y \in H^{3/2}(\Omega)$ for $f \in L^p(\Omega)$ with $p > 4/3$ and $g \in L^2(\Gamma)$, see Corollary 1.15 of [42]. Since our error analysis for the finite element method in Section 3.2 is mainly based on regularity results in weighted Sobolev spaces, this regularity result is not included in Corollary 3.6. In fact, we will only need $y \in H^t(\Omega)$ with some $t > 1$ to derive some rate of convergence for the finite element method, which is needed in Section 4.4 to ensure the convergence of the different discretization strategies applied to semilinear elliptic Neumann boundary control problems.

In the next step we are going to show that the solution y of (3.1) also belongs to the weighted Sobolev space $V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)$ with some arbitrary $\vec{\epsilon} > \vec{0}$. Comparable results can be found in e.g. [70, Section 5.5], [90, Section 2.4], [71, Section 6.3] and [36, Section 3]. The proof is based on local regularity results in classical Sobolev spaces for domains with smooth boundary and a dyadic partitioning of the neighborhood of every corner. Since the proof is essential and a similar technique will occur in the proof of the finite element error estimates on the boundary in Section 3.2.3, we state it here. An illustration of a comparable partitioning of the domain can be found on page 66.

Lemma 3.9. *Let Assumption 3.1 (A1) and the requirements of Lemma 3.4 (ii) be fulfilled. Then the unique weak solution $y \in H^1(\Omega)$ of* (3.1) *belongs to $V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)$ with some arbitrary $\vec{\epsilon} > \vec{0}$ and satisfies the estimate*

$$\|y\|_{V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} \leq c\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}\right) \tag{3.17}$$

with a positive constant $c = c(E_\Omega, m, M)$ independent of f, g and α.

Proof. Let η_j, $j = 1, \ldots, m$, be infinitely differentiable cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/32}$, $\operatorname{supp} \eta_j \subset \Omega_{R_j/2}$ and $\partial_n \eta_j \equiv 0$ with $\|\eta_j\|_{W^{k,\infty}(\Omega)} \leq c$ for $k \in \mathbb{N}_0$. Furthermore, we set $\eta_0 := 1 - \sum_{j=1}^m \eta_j$. Next, we define $\tilde{y}_j = \eta_j y$ for $j = 0, \ldots, m$, where y is the unique weak solution of (3.1) according to Lemma 3.4 (ii). For $j = 1, \ldots, m$ the functions $\tilde{y}_j$ satisfy

$$\begin{aligned} -\Delta \tilde{y}_j &= \eta_j f - \eta_j \alpha y - \Delta \eta_j y - 2\nabla \eta_j \cdot \nabla y =: F_j && \text{in } \Omega, \\ \partial_n \tilde{y}_j &= \eta_j g + \partial_n \eta_j y = \eta_j g =: G_j && \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned}$$

in the weak sense. Due to the properties of the cut-off functions η_j, $\|\alpha\|_{L^\infty(\Omega)} \leq M$ and $\|r_j^{1+\epsilon}\|_{L^\infty(\Omega_{R_j})} \leq c$ we can show that there is a constant $c = c(M)$ such that

$$\|F_j\|_{V^{0,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} \leq c\left(\|f\|_{V^{0,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} + \|y\|_{H^1(\Omega)}\right). \tag{3.18}$$

In the same manner we get the functions G_j bounded by

$$\|G_j\|_{V^{1/2,2}_{\vec{1}+\vec{\epsilon}}(\Gamma)} \leq c\|g\|_{V^{1/2,2}_{\vec{1}+\vec{\epsilon}}(\Gamma)}. \tag{3.19}$$

Next, we introduce the subsets

$$\Omega_{R_j,i} = \{x \in \Omega_{R_j} : d_{i+1}R_j < r_j(x) < d_i R_j\}$$

and the extended subsets

$$\Omega'_{R_j,i} = \{x \in \Omega_{R_j} : d_{i+2}R_j < r_j(x) < d_{i-1} R_j\}$$

with $d_i = 2^{-i}$ for $i \in \mathbb{N}$. The boundaries of $\Omega_{R_j,i}$ and $\Omega'_{R_j,i}$ are denoted by $\partial\Omega_{R_j,i}$ and $\partial\Omega'_{R_j,i}$, respectively. Note that

$$d_i \sim r_j(x) \text{ for } x \in \bar{\Omega}'_{R_j,i}, \tag{3.20}$$

$$4d_{i+2} = 2d_{i+1} = d_i = d_{i-1}/2, \tag{3.21}$$

$$|\Omega'_{R_j,i}| \sim |\Omega_{R_j,i}| \sim d_i^2, \tag{3.22}$$

$$|\partial\Omega'_{R_j,i} \cap \Gamma_j^{\pm}| \sim |\partial\Omega_{R_j,i} \cap \Gamma_j^{\pm}| \sim d_i. \tag{3.23}$$

By using the coordinate transformation $\hat{x} \mapsto x = d_{i-1}(\hat{x} - x^{(j)}) + x^{(j)}$ we can map the sets $\Omega_{R_j,1}$ and $\Omega'_{R_j,1}$ to the sets $\Omega_{R_j,i}$ and $\Omega'_{R_j,i}$, respectively. Furthermore, we can observe that the functions $\tilde{y}_{j,i}(\hat{x}) := \tilde{y}_j(d_{i-1}(\hat{x} - x^{(j)}) + x^{(j)})$ satisfy

$$\begin{aligned} -\left(\frac{\partial^2}{\partial\hat{x}_1^2} + \frac{\partial^2}{\partial\hat{x}_2^2}\right)\tilde{y}_{j,i}(\hat{x}) &= d_{i-1}^2 F_{j,i}(\hat{x}) \quad \text{in } \Omega'_{R_j,1}, \\ \partial_n \tilde{y}_{j,i}(\hat{x}) &= d_{i-1} G_{j,i}(\hat{x}) \quad \text{on } \partial\Omega'_{R_j,1} \cap \Gamma_j^{\pm} \end{aligned}$$

in the weak sense with

$$F_{j,i}(\hat{x}) := F_j(d_{i-1}(\hat{x} - x^{(j)}) + x^{(j)}) \quad \text{and} \quad G_{j,i}(\hat{x}) := G_j(d_{i-1}(\hat{x} - x^{(j)}) + x^{(j)}).$$

Applying local estimates in classical Sobolev spaces for smooth domains from e.g. [90, Theorem 1.1.5 and Proposition 2.1.3], [71, Section 3.2.3 and Section 4.3] or [88, Chapter 6] yields

$$\|\tilde{y}_{j,i}\|_{H^2(\Omega_{R_j,1})} \leq c\left(d_{i-1}^2\|F_{j,i}\|_{L^2(\Omega'_{R_j,1})} + d_{i-1}\|G_{j,i}\|_{H^{1/2}(\partial\Omega'_{R_j,1}\cap\Gamma_j^{\pm})} + \|\tilde{y}_{j,i}\|_{H^1(\Omega'_{R_j,1})}\right). \tag{3.24}$$

By using property (3.20), [34, Theorem 15.1] together with property (3.22), the a priori estimate (3.24) and property (3.21) we can continue with

$$\begin{aligned} |\tilde{y}_j|_{V^{2,2}_{1+\epsilon}(\Omega_{R_j,i})} &\leq c d_i^{1+\epsilon}|\tilde{y}_j|_{H^2(\Omega_{R_j,j})} \leq c d_i^{\epsilon}|\tilde{y}_{j,i}|_{H^2(\Omega_{R_j,1})} \\ &\leq c d_i^{\epsilon}\left(d_{i-1}^2\|F_{j,i}\|_{L^2(\Omega'_{R_j,1})} + d_{i-1}\|G_{j,i}\|_{H^{1/2}(\partial\Omega'_{R_j,1}\cap\Gamma_j^{\pm})} + \|\tilde{y}_{j,i}\|_{H^1(\Omega'_{R_j,1})}\right) \\ &\leq c\left(d_i^{2+\epsilon}\|F_{j,i}\|_{L^2(\Omega'_{R_j,1})} + d_i^{1+\epsilon}\|G_{j,i}\|_{H^{1/2}(\partial\Omega'_{R_j,1}\cap\Gamma_j^{\pm})} + d_i^{\epsilon}\|\tilde{y}_{j,i}\|_{H^1(\Omega'_{R_j,1})}\right). \end{aligned} \tag{3.25}$$

For the first and third term in the previous estimate we can conclude by means of [34, Theorem 15.1], (3.22) and (3.20)

$$\begin{aligned} d_i^{2+\epsilon}\|F_{j,i}\|_{L^2(\Omega'_{R_j,1})} + d_i^{\epsilon}\|\tilde{y}_{j,i}\|_{H^1(\Omega'_{R_j,1})} \\ &\leq c\left(d_i^{1+\epsilon}\|F_j\|_{L^2(\Omega'_{R_j,i})} + d_i^{\epsilon-1}\|\tilde{y}_j\|_{L^2(\Omega'_{R_j,i})} + d_i^{\epsilon}|\tilde{y}_j|_{H^1(\Omega'_{R_j,i})}\right) \\ &\leq c\left(\|r_j^{1+\epsilon}F_j\|_{L^2(\Omega'_{R_j,i})} + \|r_j^{\epsilon}\tilde{y}_j\|_{H^1(\Omega'_{R_j,i})}\right) && (3.26) \\ &\leq c\left(\|F_j\|_{V^{0,2}_{1+\epsilon}(\Omega'_{R_j,i})} + \|\tilde{y}_j\|_{V^{1,2}_{\epsilon}(\Omega'_{R_j,i})}\right). && (3.27) \end{aligned}$$

Arguing as in Example 3 of [46] and as in the proof of Lemma 1.1 of [101] we obtain for the second term in (3.25) by employing the properties (3.20)–(3.23)

$$\begin{aligned} d_i^{1+\epsilon}\|G_{j,i}\|_{H^{1/2}(\partial\Omega'_{R_j,1}\cap\Gamma_j^{\pm})} &\leq c\left(d_i^{1/2+\epsilon}\|G_j\|_{L^2(\partial\Omega'_{R_j,i}\cap\Gamma_j^{\pm})} + d_i^{1+\epsilon}|G_j|_{H^{1/2}(\partial\Omega'_{R_j,i}\cap\Gamma_j^{\pm})}\right) \\ &\leq c\left(\|r_j^{1/2+\epsilon}G_j\|_{L^2(\partial\Omega'_{R_j,i}\cap\Gamma_j^{\pm})} + |r_j^{1+\epsilon}G_j|_{H^{1/2}(\partial\Omega'_{R_j,i}\cap\Gamma_j^{\pm})}\right), \quad (3.28) \end{aligned}$$

see also the proof of Lemma 4.5 of [6]. Based on the estimates (3.25)–(3.28) we can conclude according to the norm equivalences [71, Lemma 6.1.1] and [71, Lemma 6.1.2], see also Section 4.5.3 of [90], that

$$\begin{aligned} \|\tilde{y}_j\|_{V^{2,2}_{1+\epsilon}(\Omega_{R_j/2})} &\leq c\left(\sum_{i=1}^{\infty}\left(\|\tilde{y}_j\|^2_{V^{1,2}_{\epsilon}(\Omega_{R_j,i})} + |\tilde{y}_j|^2_{V^{2,2}_{1+\epsilon}(\Omega_{R_j,i})}\right)\right)^{1/2} \\ &\leq c\left(\sum_{i=1}^{\infty}\left(\|F_j\|^2_{V^{0,2}_{1+\epsilon}(\Omega'_{R_j,i})} + \|r_j^{1/2+\epsilon}G_j\|^2_{L^2(\partial\Omega'_{R_j,i}\cap\Gamma_j^{\pm})}\right.\right. \\ &\qquad \left.\left. + |r_j^{1+\epsilon}G_j|^2_{H^{1/2}(\partial\Omega'_{R_j,i}\cap\Gamma_j^{\pm})} + \|\tilde{y}_j\|^2_{V^{1,2}_{\epsilon}(\Omega'_{R_j,i})}\right)\right)^{1/2} \\ &\leq c\left(\|F_j\|_{V^{0,2}_{1+\epsilon}(\Omega_{R_j})} + \|r_j^{1/2+\epsilon}G_j\|_{L^2(\Gamma_j^{\pm})} + |r_j^{1+\epsilon}G_j|_{H^{1/2}(\Gamma_j^{\pm})} + \|\tilde{y}_j\|_{V^{1,2}_{\epsilon}(\Omega_{R_j})}\right) \\ &\leq c\left(\|F_j\|_{V^{0,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} + \|G_j\|_{V^{1/2,2}_{\vec{1}+\vec{\epsilon}}(\Gamma)} + \|\tilde{y}_j\|_{H^1(\Omega_{R_j})}\right) \\ &\leq c\left(\|f\|_{V^{0,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} + \|g\|_{V^{1/2,2}_{\vec{1}+\vec{\epsilon}}(\Gamma)} + \|y\|_{H^1(\Omega)}\right), \quad (3.29) \end{aligned}$$

where we inserted (3.18) and (3.19) and used the embedding $H^1(\Omega_{R_j}) \hookrightarrow V^{1,2}_{\vec{\epsilon}}(\Omega_{R_j})$, which is valid for arbitrary $\epsilon > 0$ with respect to the Hardy inequality, and the boundedness of $D^\alpha\eta_j$ ($|\alpha| \geq 0$). It remains to estimate $\|\tilde{y}_0\|_{H^2(\Omega)}$. Using again local estimates in classical Sobolev spaces for smooth domains one obtains analogously

$$\|\tilde{y}_0\|_{H^2(\Omega)} \leq c\left(\|f\|_{L^2(\Omega^0)} + \|g\|_{H^{1/2}(\Gamma^0)} + \|y\|_{H^1(\Omega^0)}\right). \quad (3.30)$$

Collecting the estimates (3.29) for $j = 1, \dots, m$ and the estimate (3.30) yields

$$\|y\|_{V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} \le c \left(\|\tilde{y}_0\|_{H^2(\Omega)} + \sum_{j=1}^{m} \|\tilde{y}_j\|_{V^{2,2}_{1+\epsilon}(\Omega_{R_j/2})} \right) \le c \left(\|f\|_{V^{0,2}_{\vec{1}+\vec{\epsilon}}(\Omega)} + \|g\|_{V^{1/2,2}_{\vec{1}+\vec{\epsilon}}(\Gamma)} + \|y\|_{H^1(\Omega)} \right)$$
$$\le c \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} + \|y\|_{H^1(\Omega)} \right),$$

where we used in the last step Lemma 2.32 and Lemma 2.35, together with $\|r_j^{1+\epsilon}\|_{L^\infty(\Omega_{R_j})} \le c\|r_j^{\beta_j}\|_{L^\infty(\Omega_{R_j})}$. Finally, Lemma 3.4 (ii) yields the assertion. □

The next lemma is devoted to the asymptotics of a solution of (3.1) in the neighborhood of every corner. In case of bounded domains with only one corner it is proven in [90, Section 2.4]. The extension to general polygonal domains is obvious. Comparable results, also for more general elliptic operators, are given in e.g. [70, Section 3], [90, Section 4.2] or [71, Section 6.4].

Lemma 3.10. *Let $\lambda_j = \pi/\omega_j$ and let $u \in V^{l+2,2}_{\vec{\gamma}}(\Omega)$ with $l \in \mathbb{N}_0$ and $\gamma_j - l - 1 \in (0, \lambda_j)$, $j = 1, \dots, m$, be a solution of*

$$-\Delta u = F \quad \text{in } \Omega,$$
$$\partial_n u = G \quad \text{on } \Gamma_j, \quad j = 1, \dots, m,$$

where $F \in V^{l,2}_{\vec{\beta}}(\Omega)$ and $G \in V^{l+1/2,2}_{\vec{\beta}}(\Gamma)$ with $l + 1 - \beta_j \in (\lambda_j k, \lambda_j(k+1))$, $j = 1, \dots, m$, and $k \in \mathbb{N}_0$. Furthermore, let η_j, $j = 1, \dots, m$, be infinitely differentiable cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/64}$ and $\operatorname{supp} \eta_j \subset \Omega_{R_j}$. Then the solution u has the asymptotic representation

$$u = \sum_{j=1}^{m} \left\{ \eta_j \left[c_{01,j} \ln r_j + c_{0,j} + \sum_{i=1}^{k} c_{i,j} r_j^{i\lambda_j} \cos(i\lambda_j \varphi_j) \right] \right\} + w, \tag{3.31}$$

where $c_{01,j}$ and $c_{0,j}, \dots, c_{k,j}$ are constants (depending on F and G and the choice of the non-unique solution u). Especially, for $k = 0$ the asymptotic representation (3.31) *reduces to*

$$u = \sum_{j=1}^{m} \{\eta_j [c_{01,j} \ln r_j + c_{0,j}]\} + w.$$

Moreover, the function w belongs to $V^{l+2,2}_{\vec{\beta}}(\Omega)$ and satisfies the estimate

$$\|w\|_{V^{l+2,2}_{\vec{\beta}}(\Omega)} \le c \left(\|F\|_{V^{l,2}_{\vec{\beta}}(\Omega)} + \|G\|_{V^{l+1/2,2}_{\vec{\beta}}(\Gamma)} + \|u\|_{V^{l+2,2}_{\vec{\gamma}}(\Omega)} \right). \tag{3.32}$$

Based on the previous lemma we can derive regularity results in the weighted spaces $W^{2,2}_{\vec{\beta}}(\Omega)$. Comparable results can be found in e.g. [111], [81], [90, Section 4.5], [71, Section 7], [89] , [79] or [36].

Lemma 3.11. *Suppose that Assumption 3.1 (A1) is fulfilled. Furthermore, let* $\lambda_j = \pi/\omega_j$ *and let* β_j *satisfy the condition*

$$1 > \beta_j > \max(0, 1-\lambda_j) \quad \textit{or} \quad \beta_j = 0 \textit{ and } 1-\lambda_j < 0 \tag{3.33}$$

for $j = 1, \dots, m$. *Then problem* (3.1) *has a unique weak solution* $y \in W^{2,2}_{\vec{\beta}}(\Omega)$ *for every* $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ *and* $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$, *and the a priori estimate*

$$\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le c\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}\right) \tag{3.34}$$

is valid with a positive constant $c = c(E_\Omega, m, M)$ *independent of* f, g *and* α.

Proof. We distinguish between the two cases in (3.33). For $1 > \beta_j > \max(0, 1-\lambda_j)$ we can deduce from Lemma 3.9 that there exists a unique weak solution y of (3.1) in $H^1(\Omega) \cap V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)$ for some arbitrary $\vec{\epsilon} > 0$. In the following we choose $\vec{\epsilon}$ such that $0 < \epsilon_j < \min(1, \lambda_j)$ for $j = 1, \dots, m$. Note that the weak solution y of (3.1) also solves the problem

$$\begin{aligned} -\Delta y &= f - \alpha y =: F && \text{in } \Omega, \\ \partial_n y &= g && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

in the weak sense. By employing $\|\alpha\|_{L^\infty(\Omega)} \le M$ and the embeddings $V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega) \hookrightarrow V^{2,2}_{\vec{2}+\vec{\beta}}(\Omega) \hookrightarrow V^{0,2}_{\vec{\beta}}(\Omega)$ we observe that there is a constant $c = c(M)$ such that

$$\begin{aligned} \|F\|_{V^{0,2}_{\vec{\beta}}(\Omega)} &\le c\left(\|f\|_{V^{0,2}_{\vec{\beta}}(\Omega)} + \|y\|_{V^{0,2}_{\vec{\beta}}(\Omega)}\right) \le c\left(\|f\|_{V^{0,2}_{\vec{\beta}}(\Omega)} + \|y\|_{V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)}\right) \\ &\le c\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}\right), \end{aligned} \tag{3.35}$$

where we inserted the a priori estimate of Lemma 3.9 and used that the norm in $W^{0,2}_{\vec{\beta}}(\Omega)$ and $V^{0,2}_{\vec{\beta}}(\Omega)$ are equal. Thus, F belongs to $V^{0,2}_{\vec{\beta}}(\Omega)$. Now, according to Lemma 3.10 the solution y has the asymptotic representation

$$y = \sum_{j=1}^{m} \eta_j \left[c_{01,j} \ln r_j + c_{0,j}\right] + w, \tag{3.36}$$

where w belongs to $V^{2,2}_{\vec{\beta}}(\Omega)$ with $\max(0, 1-\lambda_j) < \beta_j < 1$. The constants $c_{01,j}$ are equal to zero in addition since $\ln r_j \notin H^1(\Omega)$ but $y \in H^1(\Omega)$. According to part (ii) of Lemma 2.32 the function w also vanishes at the corners since $0 < \beta_j < 1$ for $j = 1, \dots, m$. Therefore, the asymptotic representation (3.36) can be written as

$$y = \sum_{j=1}^{m} \eta_j y(x^{(j)}) + w. \tag{3.37}$$

With respect to part (ii) of Lemma 2.32 we can conclude $y \in W^{2,2}_{\vec{\beta}}(\Omega)$. Next, we are going to prove the validity of the a priori estimate (3.34). According to (3.32), (3.35), Lemma 2.35 and Lemma 3.9 there is the estimate

$$\|w\|_{V^{2,2}_{\vec{\beta}}(\Omega)} \le c\left(\|F\|_{V^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{V^{1/2,2}_{\vec{\beta}}(\Gamma)} + \|y\|_{V^{2,2}_{\vec{1}+\vec{\epsilon}}(\Omega)}\right) \le c\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}\right). \tag{3.38}$$

From (3.37), part (ii) of Lemma 2.32, (3.38) and Lemma 2.34 we can deduce

$$\begin{aligned}\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} &\le c\left(\|w\|_{V^{2,2}_{\vec{\beta}}(\Omega)} + \sum_{j=1}^{m}\left|y(x^{(j)})\right|\right)\\ &\le c\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} + \varepsilon\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + c_\varepsilon\|y\|_{W^{1,2}_{\vec{\beta}}(\Omega)}\right)\end{aligned}$$

with an arbitrary constant $\varepsilon \in (0,1)$ and a constant c_ε only depending on ε. If we choose ε such that $\varepsilon c < 1$ we obtain

$$\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le \frac{c}{1-\varepsilon c}\left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} + c_\varepsilon\|y\|_{W^{1,2}_{\vec{\beta}}(\Omega)}\right). \tag{3.39}$$

Now, the a priori estimate (3.34) follows from (3.39) and Lemma 3.4 (ii) with respect to the embedding $H^1(\Omega) \hookrightarrow W^{1,2}_{\vec{\beta}}(\Omega)$, cf. Lemma 2.29.

In case that $\beta_j = 0$ and $1 - \lambda_j < 0$ we can deduce the unique solvability in $H^2(\Omega)$ from Lemma 3.4 (ii) and [54, Corollary 4.4.3.8]. The a priori estimate holds in that case according to [54, Theorem 4.3.1.4] and the a priori estimate of Lemma 3.4 (ii). □

Next, we would like to get regularity results in $W^{2,\infty}_{\vec{\gamma}}(\Omega)$ and $W^{2,\infty}_{\vec{\gamma}}(\Gamma)$. To the best of our knowledge there is no reference where this is done directly. Instead, we use regularity results in weighted Hölder spaces for that purpose. The following Lemma represents parts of Theorem 1.4.5 of [72], which has been adapted to our setting, compare also [79, Section 2.7-2.8], [81] and [89].

Lemma 3.12. *Let $\lambda_j = \pi/\omega_j$ and $\sigma \in (0,1)$. Furthermore, let $u \in V^{2,2}_{\vec{\beta}}(\Omega)$ with β_j satisfying (3.33) be a solution of*

$$\begin{aligned}-\Delta u &= F \quad \textit{in } \Omega\\ \partial_n u &= G \quad \textit{on } \Gamma_j, \quad j = 1,\dots,m,\end{aligned}$$

where $F \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ and $G \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$. If $0 < 2 + \sigma - \delta_j < \lambda_j$ for $j = 1,\dots,m$, then u belongs to $N^{2,\sigma}_{\vec{\delta}}(\Omega)$ and the a priori estimate

$$\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} \le c\left(\|F\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|G\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|u\|_{L^1(\Omega)}\right) \tag{3.40}$$

is valid.

Lemma 3.13. *Let Assumption 3.1 (A2) be fulfilled. Moreover, let* $\lambda_j = \pi/\omega_j$ *and* $\sigma \in (0,1)$. *If for each* $j \in \{1,\dots,m\}$ *one of the following two conditions*

(i) $2 > \gamma_j > \max(0, 2-\lambda_j)$ *and* $\delta_j = \gamma_j + \sigma$

(ii) $\gamma_j = 0,\ 2-\lambda_j < 0$ *and* $\delta_j = \sigma$

is fulfilled, then for every $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ *the unique weak solution* y *of problem* (3.1) *fulfills the a priori estimate*

$$\begin{aligned}\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} &\le c\left(\|y\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^m \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})}\right)\\ &\le c\left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}\right)\end{aligned}$$

with a positive constant $c = c(E_\Omega, m, M)$ *independent of* f, g *and* α.

Proof. First, we notice that it is possible to choose parameters β_j such that $\max(0, \gamma_j - 1) < \beta_j < 1$ for $j = 1,\dots,m$, which implies $\max(0, 1-\lambda_j) < \beta_j < 1$. Then we obtain from Lemma 2.39 for this choice of the parameters that

$$N^{0,\sigma}_{\vec{\delta}}(\Omega) \hookrightarrow W^{0,2}_{\vec{\beta}}(\Omega) \quad \text{and} \quad N^{1,\sigma}_{\vec{\delta}}(\Gamma) \hookrightarrow W^{1/2,2}_{\vec{\beta}}(\Gamma). \tag{3.41}$$

Furthermore, we get from Lemma 3.11 that the solution y of (3.1) belongs to $W^{2,2}_{\vec{\beta}}(\Omega)$. Next, we would like to apply Lemma 3.12, but $y \notin V^{2,2}_{\vec{\beta}}(\Omega)$. Instead, we first use Lemma 2.32 (compare also (3.37)). This yields the splitting

$$u = y - \sum_{j=1}^m \eta_j y(x^{(j)}) \in V^{2,2}_{\vec{\beta}}(\Omega), \tag{3.42}$$

where η_j denote the cut-off functions introduced in Lemma 2.32. Furthermore, we know that u solves

$$\begin{aligned}-\Delta u &= f - \alpha u - \alpha \sum_{j=1}^m \eta_j y(x^{(j)}) + \sum_{j=1}^m y(x^{(j)})\Delta\eta_j =: F && \text{in } \Omega,\\ \partial_n u &= g - \sum_{j=1}^m y(x^{(j)})\partial_n \eta_j =: G && \text{on } \Gamma_k,\ k = 1,\dots,m.\end{aligned}$$

Next we are going to show

$$F \in N^{0,\sigma}_{\vec{\delta}}(\Omega) \text{ and } G \in N^{1,\sigma}_{\vec{\delta}}(\Gamma). \tag{3.43}$$

One can conclude for the function αu that

$$\begin{aligned}
&\|\alpha u\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} = \|\alpha u\|_{C^{0,\sigma}(\bar{\Omega}_0)} \\
&+ c\sum_{j=1}^{m}\left(\|r_j^{\delta_j-\sigma}\alpha u\|_{C^0(\bar{\Omega}_{R_j})} + \sup_{x_1,x_2\in\Omega_{R_j}}\frac{\left|r_j(x_1)^{\delta_j}(\alpha u)(x_1) - r_j(x_2)^{\delta_j}(\alpha u)(x_2)\right|}{|x_1-x_2|^\sigma}\right) \\
&\leq c\left(\|\alpha\|_{C^{0,\sigma}(\bar{\Omega}_0)}\|u\|_{C^{0,\sigma}(\bar{\Omega}_0)} + \sum_{j=1}^{m}\left(\|\alpha\|_{C^0(\bar{\Omega}_{R_j})}\|r_j^{\delta_j-\sigma}u\|_{C^0(\bar{\Omega}_{R_j})}\right.\right. \\
&\left.\left.+ \sup_{x_1,x_2\in\Omega_{R_j}}\left(|\alpha(x_1)|\frac{\left|r_j(x_1)^{\delta_j}u(x_1) - r_j(x_2)^{\delta_j}u(x_2)\right|}{|x_1-x_2|^\sigma} + \left|r_j(x_2)^{\delta_j}u(x_2)\right|\frac{|\alpha(x_1)-\alpha(x_2)|}{|x_1-x_2|^\sigma}\right)\right)\right) \\
&\leq c\|\alpha\|_{C^{0,\sigma}(\bar{\Omega})}\|u\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} \leq c\|u\|_{V^{2,2}_{\vec{\beta}}(\Omega)}, \qquad (3.44)
\end{aligned}$$

where we used $\|\alpha\|_{C^{0,\sigma}(\bar{\Omega})} \leq M$ in the last step. Thus, the constant $c = c(M)$ is independent of α. In order to verify that u belongs to $N^{0,\sigma}_{\vec{\delta}}(\Omega)$ we used the embedding $V^{2,2}_{\vec{\beta}}(\Omega) \hookrightarrow N^{0,\sigma}_{\vec{\delta}}(\Omega)$ according to Lemma 2.38. This embedding holds since $\delta_j - \sigma = \gamma_j \geq 0 > \beta_j - 1$ by assumption. As a consequence there holds

$$\|\alpha u\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} \leq c\|u\|_{V^{2,2}_{\vec{\beta}}(\Omega)}. \qquad (3.45)$$

We obtain analogously to (3.44)

$$\|\alpha\sum_{j=1}^{m}\eta_j y(x^{(j)})\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} \leq c\|\alpha\|_{C^{0,\sigma}(\bar{\Omega})}\sum_{j=1}^{m}\left|y(x^{(j)})\right|\|\eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} \leq c\sum_{j=1}^{m}\left|y(x^{(j)})\right|\|\eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}, \qquad (3.46)$$

where the constant $c = c(M)$ is again independent of α. With respect to Lemma 2.36 we can conclude for $\delta_j \geq \sigma$

$$\begin{aligned}
\|\eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} &\leq c\left(\|r_j^{\delta_j-\sigma}\eta_j\|_{C^0(\bar{\Omega}_{R_j})} + \sup_{\substack{x_1,x_2\in\Omega_{R_j}\\ |x_1-x_2|\leq\frac{1}{2}r_j(x_1)}} r_j(x_1)^{\delta_j}\frac{|\eta_j(x_1)-\eta_j(x_2)|}{|x_1-x_2|^\sigma}\right) \\
&\leq c\|\eta_j\|_{C^{0,\sigma}(\bar{\Omega})} \leq c. \qquad (3.47)
\end{aligned}$$

Thus, the regularity of u stated in (3.42) and the inequalities (3.45), (3.46) and (3.47) yield (3.43). Note that the functions $\Delta\eta_j$ and $\partial_n\eta_j$ belong trivially to the corresponding weighted Hölder spaces since they even vanish in the neighborhood of every corner. Now we are able to apply Lemma 3.12 to u. By this we obtain that u belongs to $N^{2,\sigma}_{\vec{\delta}}(\Omega)$ and fulfills the a priori estimate

$$\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} \leq c\left(\|F\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|G\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|u\|_{L^1(\Omega)}\right) \qquad (3.48)$$

if $\gamma_j = \delta_j - \sigma \geq 0$ and $0 < 2 - \gamma_j < \lambda_j$ for $j = 1,\ldots,m$. Based on this we now derive the a priori estimate for y. There holds

$$\begin{aligned}
\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} &\le c\left(\|y\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^m \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})}\right)\\
&\le c\left(\|u\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^m \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha u\|_{C^0(\bar{\Omega}_{R_j})}\right.\\
&\left.+\sum_{j=1}^m \left|y(x^{(j)})\right| \left[\|\eta_j\|_{C^2(\bar{\Omega}^0)} + \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha \eta_j\|_{C^0(\bar{\Omega}_{R_j})}\right]\right)\\
&\le c\left(\|u\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^m \sum_{|\alpha|\le 2} \|r_j^{\gamma_j} D^\alpha u\|_{C^0(\bar{\Omega}_{R_j})} + \sum_{j=1}^m \left|y(x^{(j)})\right|\right),
\end{aligned}$$

where we inserted (3.42) and used that the functions $r_j^{\gamma_j}$ (for $\gamma_j \ge 0$) and $|D^\alpha \eta_j|$ are bounded by a constant. Since $\gamma_j = \delta_j - \sigma$ and $2 - |\alpha| \ge 0$ for $|\alpha| \le 2$ we can conclude

$$\begin{aligned}
\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} &\le c\left(\|u\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^m \sum_{|\alpha|\le 2} \|r_j^{\delta_j-\sigma-2+|\alpha|} D^\alpha u\|_{C^0(\bar{\Omega}_{R_j})} + \sum_{j=1}^m \left|y(x^{(j)})\right|\right)\\
&\le c\left(\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} + \sum_{j=1}^m \left|y(x^{(j)})\right|\right). \qquad (3.49)
\end{aligned}$$

Next, we employ the a priori estimate (3.48) and insert the definitions of F and G. This yields

$$\begin{aligned}
\|u\|_{N^{2,\sigma}_{\vec{\delta}}(\Omega)} &\le c\left(\|f - \alpha u - \alpha \sum_{j=1}^m \eta_j y(x^{(j)}) + \sum_{j=1}^m y(x^{(j)}) \Delta \eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}\right.\\
&\left.+\|g - \sum_{j=1}^m y(x^{(j)}) \partial_n \eta_j\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|u\|_{L^1(\Omega)}\right)\\
&\le c\left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|\alpha u\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|\alpha \sum_{j=1}^m \eta_j y(x^{(j)})\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}\right.\\
&\left.+\|u\|_{L^1(\Omega)} + \sum_{j=1}^m \left|y(x^{(j)})\right| \left(\|\Delta \eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|\partial_n \eta_j\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}\right)\right). \qquad (3.50)
\end{aligned}$$

Combining the inequalities (3.49), (3.50), (3.45), (3.46) and (3.47) implies

$$\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} \le c\left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|u\|_{V^{2,2}_{\vec{\beta}}(\Omega)} + \sum_{j=1}^m \left|y(x^{(j)})\right|\right),$$

with a constant $c = c(M)$, where we also used the embedding $V^{2,2}_{\vec{\beta}}(\Omega) \hookrightarrow L^1(\Omega)$ according to Lemma 2.30 and the boundedness of the functions $\Delta \eta_j$ and $\partial_n \eta_j$ in $N^{0,\sigma}_{\vec{\delta}}(\Omega)$ and $N^{0,\sigma}_{\vec{\delta}}(\Gamma)$, respectively. Finally, the norm equivalence of Lemma 2.32, the a priori estimate of Lemma 3.11 and (3.41) yield the assertion. □

Corollary 3.14. *Suppose that Assumption 3.1 (A2) is fulfilled. Let* $\lambda_j = \pi/\omega_j$ *and* $\sigma \in (0,1)$. *Furthermore, let one of the following two conditions*

(i) $1 > \tau_j > \max(0, 1-\lambda_j)$ *and* $\delta_j = \tau_j + \sigma + 1$

(ii) $\tau_j = 0$, $1 - \lambda_j < 0$ *and* $\delta_j < \sigma + 1$

be satisfied for each $j \in \{1, \dots, m\}$. *Then for every* $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ *the unique weak solution* y *of problem* (3.1) *fulfills the estimate*

$$\|y\|_{W^{1,\infty}_{\vec{\tau}}(\Gamma)} \leq \left(\|y\|_{C^1(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\leq 1} \|r_j^{\tau_j} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})} \right) \leq c \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \right)$$

with a positive constant $c = c(E_\Omega, m, M)$ *independent of* f, g *and* α.

Proof. We start with the case $1 > \tau_j > \max(0, 1-\lambda_j)$. According to [79, Lemma 6.7.4] we can conclude for $\tau_j > 0$ that

$$\|y\|_{C^1(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\leq 1} \|r_j^{\tau_j} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})} \leq c \left(\|y\|_{C^2(\bar{\Omega}^0)} + \sum_{j=1}^{m} \sum_{|\alpha|\leq 2} \|r_j^{\tau_j+1} D^\alpha y\|_{C^0(\bar{\Omega}_{R_j})} \right).$$

Thus, the assertion in the first case is a direct consequence of Lemma 3.13 by setting $\gamma_j = \tau_j + 1$. Next, let us consider the second condition $\tau_j = 0$ and $1 - \lambda_j < 0$. We observe that there exist parameters γ_j and p with $\gamma_j \geq \delta_j - \sigma$, $1 > \gamma_j > \max(0, 2-\lambda_j)$ and $2/\gamma_j > p > 2$ since $\delta_j - \sigma < 1$ and $1 - \lambda_j < 0$. Thus, Theorem 2.7 and Lemma 2.29 imply

$$\|y\|_{C^1(\bar{\Omega})} \leq c\|y\|_{W^{2,p}(\Omega)} \leq c\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)}.$$

Consequently, Lemma 3.13 yields the second assertion. □

Corollary 3.15. *Suppose that Assumption 3.1 (A2) is satisfied. Furthermore, let* $\lambda_j = \pi/\omega_j$, $\sigma \in (0,1)$ *and let the conditions*

$$\frac{3}{2} > \kappa_j > \max(-1/2, 3/2 - \lambda_j) \text{ and } \delta_j < \kappa_j + \sigma + 1/2 \tag{3.51}$$

be fulfilled for each $j \in \{1, \dots, m\}$. *Then for every* $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ *the unique weak solution* y *of problem* (3.1) *satisfies the estimate*

$$\|y\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \leq c \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \right)$$

with a positive constant $c = c(E_\Omega, m, M)$ *independent of* f, g *and* α.

Proof. We notice that there are parameters γ_j with $\gamma_j \geq \delta_j - \sigma$, $2 > \gamma_j > \max(0, 2-\lambda_j)$ and $1/2 + \kappa_j > \gamma_j$ since $3/2 > \kappa_j > \max(-1/2, 3/2 - \lambda_j)$ and $1/2 + \kappa_j > \delta_j - \sigma$. Therefore, an application of Lemma 2.29 yields

$$\|y\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \leq c\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)}. \tag{3.52}$$

Thus, the assertion follows from Lemma 3.13. □

Before we close this section about regularity results in weighted Sobolev spaces for linear elliptic equations let us state some Lipschitz estimates which are needed for the numerical analysis of the semilinear boundary control problems in Section 4.4.

Lemma 3.16. *Let Assumption 3.1 (A1) be fulfilled and let some arbitrary $r > 1$ and $s > 1$ be given. Furthermore, let $y_1 \in H^1(\Omega)$ and $y_2 \in H^1(\Omega)$ be the weak solutions of (3.1) with right hand sides $f_1 \in L^r(\Omega)$ and $f_2 \in L^r(\Omega)$ and Neumann boundary data $g_1 \in L^s(\Gamma)$ and $g_2 \in L^s(\Gamma)$, respectively. Then the estimate*

$$\|y_1 - y_2\|_{L^2(\Omega)} \le c\left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right)$$

is valid with a positive constant $c = c(E_\Omega, m, M)$ independent of f_1, f_2, g_1, g_2 and α. Moreover, the estimate

$$\|y_1 - y_2\|_{H^1(\Omega)} \le c\left(\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)}\right)$$

holds with a positive constant $c = c(E_\Omega, m)$ independent of f_1, f_2, g_1, g_2 and α.

Proof. Let w be the weak solution of

$$\begin{aligned} -\Delta w + \alpha w &= y_1 - y_2 \quad \text{in } \Omega, \\ \partial_n w &= 0 \quad\quad\quad\; \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

which is continuous according to Remark 3.7. Therefore, we can conclude

$$\begin{aligned} \|y_1 - y_2\|^2_{L^2(\Omega)} &= a(y_1 - y_2, w) = \int_\Omega (f_1 - f_2)w + \int_\Gamma (g_1 - g_2)w \\ &\le \left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right)\|w\|_{C^0(\bar\Omega)}. \end{aligned}$$

The first assertion follows from (3.16). For the second one we observe that there holds according to (3.7)

$$\|y_1 - y_2\|^2_{H^1(\Omega)} \le ca(y_1 - y_2, y_1 - y_2) = c\left(\int_\Omega (f_1 - f_2)(y_1 - y_2) + \int_\Gamma (g_1 - g_2)(y_1 - y_2)\right)$$

with a positive constant $c = c(E_\Omega, m)$ independent of f_1, f_2, g_1, g_2 and α. The Hölder inequality, Theorem 2.7 and Theorem 2.8 yield

$$\begin{aligned} \|y_1 - y_2\|^2_{H^1(\Omega)} &\le c\left(\|f_1 - f_2\|_{L^r(\Omega)}\|y_1 - y_2\|_{L^{r/(r-1)}(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)}\|y_1 - y_2\|_{L^{s/(s-1)}(\Gamma)}\right) \\ &\le c\left(\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)}\right)\|y_1 - y_2\|_{H^1(\Omega)}. \end{aligned}$$

Dividing by $\|y_1 - y_2\|_{H^1(\Omega)}$ implies the second assertion. □

Lemma 3.17. *Let E_Ω be a subset of Ω with $|E_\Omega| > 0$ and let m, M_1, M_2 be constants greater than zero. Furthermore, let the functions $\alpha_i \in L^\infty(\Omega)$, $i \in \{1, 2\}$, fulfill $\alpha_i(x) \ge 0$ for a.a. $x \in \Omega$, $\alpha_i \ge m$ for a.a. $x \in E_\Omega$ and $\|\alpha_i\|_{L^\infty(\Omega)} \le M_i$. Then the weak solutions of*

$$\begin{aligned} -\Delta y_1 + \alpha_1 y_1 &= f_1 \quad \text{in } \Omega, \\ \partial_n y_1 &= g_1 \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} -\Delta y_2 + \alpha_2 y_2 &= f_2 \quad \text{in } \Omega, \\ \partial_n y_2 &= g_2 \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.54}$$

with $f_1, f_2 \in L^r(\Omega)$ *and* $g_1, g_2 \in L^s(\Gamma)$, $r, s > 1$, *satisfy the estimate*

$$\|y_1 - y_2\|_{H^1(\Omega)} \le c \Big(\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} + \|\alpha_2 - \alpha_1\|_{L^t(\Omega)} \big(\|f_1\|_{L^r(\Omega)} + \|g_1\|_{L^s(\Gamma)} \big) \Big)$$

with some arbitrary $t > 1$ *and a positive constant* $c = c(E_\Omega, m)$ *independent of* α_1, α_2, f_1, f_2, g_1 *and* g_2.

Proof. Let χ_{E_Ω} be the characteristic function of E_Ω and let us set $\tilde{\alpha}_i = \alpha_i - m\chi_{E_\Omega}$ for $i \in \{1, 2\}$. By testing the variational formulations of (3.53) and (3.54) with $y_1 - y_2$ one obtains for the difference of both

$$\begin{aligned} &\int_\Omega \nabla(y_1 - y_2)\nabla(y_1 - y_2) + m \int_{E_\Omega} (y_1 - y_2)(y_1 - y_2) \\ &\qquad = \int_\Omega (f_1 - f_2)(y_1 - y_2) + \int_\Gamma (g_1 - g_2)(y_1 - y_2) + \int_\Omega (\tilde{\alpha}_2 y_2 - \tilde{\alpha}_1 y_1)(y_1 - y_2). \end{aligned}$$

As in (3.7) we can continue with

$$\begin{aligned} \|y_1 - y_2\|^2_{H^1(\Omega)} &\le c \Big(\int_\Omega (f_1 - f_2)(y_1 - y_2) + \int_\Gamma (g_1 - g_2)(y_1 - y_2) \\ &\quad + \int_\Omega (\tilde{\alpha}_2 y_2 - \tilde{\alpha}_1 y_1)(y_1 - y_2) \Big) \\ &= c \Big(\int_\Omega (f_1 - f_2)(y_1 - y_2) + \int_\Gamma (g_1 - g_2)(y_1 - y_2) \\ &\quad + \int_\Omega \tilde{\alpha}_2 (y_2 - y_1)(y_1 - y_2) + \int_\Omega (\alpha_2 - \alpha_1) y_1 (y_1 - y_2) \Big) \end{aligned} \tag{3.55}$$

with $c = c(E_\Omega, m) > 0$. Next, we observe that the third term of (3.55) is less or equal to zero since $\tilde{\alpha}_2(x) \ge 0$ for a.a. $x \in \Omega$. Using the Hölder inequality, Theorem 2.7, Theorem 2.8 and Lemma 3.4 yields

$$\begin{aligned} \|y_1 - y_2\|^2_{H^1(\Omega)} &\le c \Big(\|f_1 - f_2\|_{L^r(\Omega)} \|y_1 - y_2\|_{L^{r/(r-1)}(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \|y_1 - y_2\|_{L^{s/(s-1)}(\Gamma)} \\ &\quad + \|\alpha_2 - \alpha_1\|_{L^t(\Omega)} \|y_1\|_{L^{2t/(t-1)}(\Omega)} \|y_1 - y_2\|_{L^{2t/(t-1)}(\Omega)} \Big) \\ &\le c \Big(\|f_1 - f_2\|_{L^r(\Omega)} \|y_1 - y_2\|_{H^1(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \|y_1 - y_2\|_{H^1(\Omega)} \\ &\quad + \|\alpha_2 - \alpha_1\|_{L^t(\Omega)} \|y_1\|_{H^1(\Omega)} \|y_1 - y_2\|_{H^1(\Omega)} \Big) \\ &\le c \Big[\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \\ &\quad + \|\alpha_2 - \alpha_1\|_{L^t(\Omega)} \big(\|f_1\|_{L^r(\Omega)} + \|g_1\|_{L^s(\Gamma)} \big) \Big] \|y_1 - y_2\|_{H^1(\Omega)} \end{aligned}$$

with a positive constant $c = c(E_\Omega, m)$ independent of α_1, α_2, f_1, f_2, g_1 and g_2. Finally, one has to divide by $\|y_1 - y_2\|_{H^1(\Omega)}$ to prove the assertion. ☐

3.1.2 Semilinear elliptic problems

This section is devoted to solvability and regularity results of the semilinear elliptic boundary value problem

$$\begin{aligned} -\Delta y + d(\cdot, y) &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned} \tag{3.56}$$

where the domain Ω is a polygonal domain according to Definition 2.17 with m corner points and boundary $\Gamma = \bigcup_{j=1}^{m} \bar{\Gamma}_j$. Throughout the remainder of this section we require for the discussion of problem (3.56) the following assumption on the nonlinearity d.

Assumption 3.18.

(A1) The function $d = d(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to $x \in \Omega$ for all fixed $y \in \mathbb{R}$, and differentiable with respect to y for almost all $x \in \Omega$. Moreover, we require

$$\frac{\partial d}{\partial y}(x, y) \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and } y \in \mathbb{R},$$

and the following Lipschitz condition: for all $M > 0$ there exists $L_{d,M} > 0$ such that d satisfies

$$|d(x, y_1) - d(x, y_2)| \leq L_{d,M} |y_1 - y_2|$$

for a.a. $x \in \Omega$ and $y_i \in \mathbb{R}$ with $|y_i| \leq M$, $i = 1, 2$.

(A2) There is a subset $E_\Omega \subset \Omega$ of positive measure and a constant $c_\Omega > 0$ such that $\frac{\partial d}{\partial y}(x, y) \geq c_\Omega$ in $E_\Omega \times \mathbb{R}$.

Again the precise regularity assumptions on $d(\cdot, 0)$, f and g depend on the desired regularity of the solution y. For the moment, we assume the regularity required for the introduction of a weak solution of (3.56).

Definition 3.19. Let Assumption 3.18 be fulfilled. Furthermore, let $d(\cdot, 0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$. Then a weak solution of (3.56) is an element $y \in H^1(\Omega) \cap C^0(\bar{\Omega})$ that satisfies

$$a(y, v) + \int_\Omega d(\cdot, y) v = \int_\Omega f v + \int_\Gamma g v \quad \forall v \in H^1(\Omega), \tag{3.57}$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(y, v) := \int_\Omega \nabla y \cdot \nabla v. \tag{3.58}$$

For the analysis of problem (3.56) it is useful to state an equivalent formulation of the variational equation (3.57). Let $\alpha \in C^\infty(\bar{\Omega})$ be defined by

$$\alpha := \eta_{E_\Omega} c_\Omega,$$

where η_{E_Ω} is an infinitely differentiable cut-off function equal to one in a proper subset of E_Ω and $\operatorname{supp} \eta_{E_\Omega} \subset E_\Omega$. The variational equation (3.57) can be reformulated as

$$\tilde{a}(y, v) + \int_\Omega \tilde{d}(\cdot, y) v = \int_\Omega (f - d(\cdot, 0)) v + \int_\Gamma g v \quad \forall v \in H^1(\Omega), \tag{3.59}$$

where $\tilde{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$\tilde{a}(y, v) := \int_\Omega (\nabla y \cdot \nabla v + \alpha y v) \tag{3.60}$$

and the function $\tilde{d}$ is given by

$$\tilde{d}(x, y) := d(x, y) - d(x, 0) - \alpha(x) y.$$

Note that the functions $\tilde{d}$ and α fulfill Assumption 3.18 (A1) and Assumption 3.1 (A2), respectively. Furthermore, we have $\tilde{d}(x, 0) = 0$ for a.a. $x \in \Omega$. Next, we show, that a weak solution of (3.56) exists and satisfies certain a priori estimates.

Lemma 3.20. *Let Assumption 3.18 be fulfilled. Then problem* (3.56) *has a unique weak solution $y \in H^1(\Omega) \cap L^\infty(\Omega)$, which is continuous, for*

(i) $d(\cdot, 0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$. Moreover, there exists a positive constant $c = c(E_\Omega, c_\Omega)$, independent of d, f and g, such that

$$\|y\|_{H^1(\Omega)} + \|y\|_{C^0(\bar{\Omega})} \le c \left(\|f - d(\cdot, 0)\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)} \right).$$

(ii) $d(\cdot, 0) \in W^{0,2}_{\vec{\beta}}(\Omega)$, $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ with $0 \le \beta_j < 1$ for $j = 1, \dots, m$. Furthermore, there is the estimate

$$\|y\|_{H^1(\Omega)} + \|y\|_{C^0(\bar{\Omega})} \le c \left(\|f - d(\cdot, 0)\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right)$$

with a positive constant $c = c(E_\Omega, c_\Omega)$ independent of d, f and g.

Proof. (i) The existence and uniqueness of the solution of (3.59) and the validity of the a priori estimate in that case can be deduced from Theorem 4.7, Theorem 4.8 and Theorem 4.10 of [107].

(ii) The assertion can be deduced from (i) having regard to Lemma 2.39. □

The following remark is essential to understand the notion in the sequel.

Remark 3.21. For the solution y of the semilinear elliptic problem (3.56) we can also show higher regularity than $H^1(\Omega)$-regularity. The proof relies on the corresponding results for linear elliptic problems and the assumptions on the nonlinearity, in particular that it is Lipschitz continuous. Since we only a assume a local Lipschitz condition, see Assumption 3.18, the Lipschitz constant $L_{d,M}$ implicitly depends on $\|y\|_{C^0(\bar{\Omega})}$. Therefore, according to Lemma 3.20, this constant depends on the data in different norms, e.g. $\|f - d(\cdot, 0)\|_{L^r(\Omega)}$ and $\|g\|_{L^s(\Gamma)}$. For that reason, we cannot separate the data from the constants in the following estimates. However, we choose a notation which shows these dependencies in detail and enables us to decide whether a constant is bounded or not: we write

$$c = c(\|f\|_X)$$

to express that the quantity c depends on the norm of the function f in X, but not on further properties of f. In particular, the quantity c is bounded if $\|f\|_X$ is bounded. This knowledge about the constants will be especially important in Section 4.4, when discussing discretization error estimates for semilinear elliptic Neumann boundary control problems, since the data in this section will also depend on discrete functions. Therefore, we will be able by this approach to ensure that the constants are independent of the mesh parameter, if the discrete functions are bounded in these norms.

Now, we turn our attention to the proof of regularity results in Sobolev Slobodetskij and weighted Sobolev spaces.

Corollary 3.22. *Let Assumption 3.18 be fulfilled. Furthermore, let $r \in (1, 4/3)$, $s \in (1, 2)$ and $t = \min(3 - 2/r, 2 - 1/s)$. Then the weak solution of problem (3.56) belongs to $H^t(\Omega)$ for $d(\cdot, 0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$. Furthermore, there is the estimate*

$$\|y\|_{H^t(\Omega)} \leq c$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot, 0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)}) > 0$.

Proof. We know from Lemma 3.20 that there exists a unique solution $y \in H^1(\Omega) \cap C^0(\bar{\Omega})$ of (3.56). This solution also solves the problem

$$\begin{aligned} -\Delta y + \alpha y &= f - d(\cdot, 0) - \tilde{d}(\cdot, y) && \text{in } \Omega, \\ \partial_n y &= g && \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned} \tag{3.61}$$

in the weak sense. Due to the local Lipschitz condition stated in Assumption 3.18 (A1) and Lemma 3.20, there exists a positive constant c depending on the Lipschitz constant $L_{d,M} = L_{d,M}(\|f - d(\cdot, 0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)})$ such that

$$\|\tilde{d}(\cdot, y)\|_{L^r(\Omega)} = \|\tilde{d}(\cdot, y) - \tilde{d}(\cdot, 0)\|_{L^r(\Omega)} \leq c\|y\|_{L^r(\Omega)} \leq c\|y\|_{H^1(\Omega)}, \tag{3.62}$$

where we additionally used $\tilde{d}(x, 0) = 0$ and the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$, cf. Theorem 2.7. Thus, we have $\tilde{d}(\cdot, y) \in L^r(\Omega)$. The regularity assertion for the solution y follows now from Corollary 3.6. The estimate of the assertion is a consequence of the a priori estimate of Corollary 3.6, (3.62) and Lemma 3.20. □

Corollary 3.23. *Suppose that Assumption 3.18 is satisfied. Then problem (3.56) has a unique weak solution $y \in W^{2,2}_{\vec{\beta}}(\Omega)$ for $d(\cdot, 0) \in W^{0,2}_{\vec{\beta}}(\Omega)$, $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ with β_j satisfying condition (3.33). Moreover, there is the estimate*

$$\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot, 0)\|_{W^{0,2}_{\vec{\beta}}(\Omega)}, \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}) > 0$.

Proof. We argue as in the proof of Corollary 3.22. The local Lipschitz condition from Assumption 3.18 (A1) and Lemma 3.20 imply the existence of a positive constant c depending on the Lipschitz constant $L_{d,M} = L_{d,M}(\|f - d(\cdot,0)\|_{W^{0,2}_{\vec\beta}(\Omega)}, \|g\|_{W^{1/2,2}_{\vec\beta}(\Gamma)})$ such that

$$\|\tilde d(\cdot,y)\|_{W^{0,2}_{\vec\beta}(\Omega)} = \|\tilde d(\cdot,y) - \tilde d(\cdot,0)\|_{W^{0,2}_{\vec\beta}(\Omega)} \le c\|y\|_{W^{0,2}_{\vec\beta}(\Omega)} \le c\|y\|_{H^1(\Omega)}, \tag{3.63}$$

where we employed the embedding $H^1(\Omega) \hookrightarrow W^{0,2}_{\vec\beta}(\Omega)$ according to Lemma 2.29. Therefore, we can conclude $\tilde d(\cdot,y) \in W^{0,2}_{\vec\beta}(\Omega)$. The regularity assertion can now be deduced from Lemma 3.11. The estimate, stated in the assertion, follows from the a priori estimate of Lemma 3.11, (3.63) and Lemma 3.20. □

To get results analogous to Lemma 3.13 we assume for the nonlinearity d the following stronger Lipschitz condition in addition.

Assumption 3.24. For all $M > 0$ there exists a positive constant $L_{d,M}$ such that

$$|d(x_1,y_1) - d(x_2,y_2)| \le L_{d,M}\left(|x_1 - x_2|^\sigma + |y_1 - y_2|\right)$$

for all $x_i \in \Omega$ and $y_i \in \mathbb{R}$ with $|y_i| \le M$, $i = 1,2$, and some $\sigma \in (0,1)$ specified below.

Remark 3.25. The function $\tilde d$ fulfills Assumption 3.24 as well. In fact, one obtains by straightforward calculations

$$\begin{aligned}|\tilde d(x_1,y_1) - \tilde d(x_2,y_2)| &\le |d(x_1,y_1) - d(x_2,y_2)| + |d(x_1,0) - d(x_2,0)| + |\alpha(x_1)y_1 - \alpha(x_2)y_2| \\ &\le |d(x_1,y_1) - d(x_2,y_2)| + |d(x_1,0) - d(x_2,0)| + |\alpha(x_1)|\,|y_1 - y_2| \\ &\quad + |\alpha(x_1) - \alpha(x_2)|\,|y_2| \\ &\le c\left(|x_1 - x_2|^\sigma + |y_1 - y_2|\right)\end{aligned}$$

with $c = c(E_\Omega, c_\Omega, L_{d,M}, M)$, where we used the boundedness and Hölder continuity of α in the last step.

Corollary 3.26. *Let Assumptions 3.18 and 3.24 be fulfilled. Furthermore, let $\vec\gamma \in \mathbb{R}^m$, $\vec\delta \in \mathbb{R}^m$ and $\sigma \in (0,1)$ satisfy the conditions stated in Lemma 3.13. Then for $f \in N^{0,\sigma}_{\vec\delta}(\Omega)$ and $g \in N^{1,\sigma}_{\vec\delta}(\Gamma)$ the unique weak solution of problem (3.56) satisfies*

$$\|y\|_{W^{2,\infty}_{\vec\gamma}(\Omega)} + \|y\|_{W^{2,\infty}_{\vec\gamma}(\Gamma)} \le c\left(\|y\|_{C^2(\bar\Omega^0)} + \sum_{j=1}^{m}\sum_{|\alpha|\le 2}\|r_j^{\gamma_j}D^\alpha y\|_{C^0(\bar\Omega_{R_j})}\right) \le c \tag{3.64}$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{N^{0,\sigma}_{\vec\delta}(\Omega)}, \|g\|_{N^{1,\sigma}_{\vec\delta}(\Gamma)}) > 0$.

Proof. We can argue as for (3.41) to deduce the existence of parameters β_j with $\max(0,1-\lambda_j) < \beta_j < 1$ for $j = 1,\dots,m$ such that

$$N^{0,\sigma}_{\vec\delta}(\Omega) \hookrightarrow W^{0,2}_{\vec\beta}(\Omega) \quad \text{and} \quad N^{1,\sigma}_{\vec\delta}(\Gamma) \hookrightarrow W^{1/2,2}_{\vec\beta}(\Gamma). \tag{3.65}$$

Then we know from Corollary 3.23 that the solution y belongs to $W^{2,2}_{\vec{\beta}}(\Omega)$. Now, we show that the right hand side of (3.61) fulfills the requirements of Lemma 3.13. According to Lemma 2.32 we know that y can be split into

$$y = u + \sum_{i=1}^{m} \eta_j y(x^{(j)}), \tag{3.66}$$

where the function u belongs to $V^{2,2}_{\vec{\beta}}(\Omega)$ and the functions η_j are smooth cut-off functions in $\bar{\Omega}$ equal to one in $\Omega_{R_j/64}$ and $\operatorname{supp}\eta_j \subset \Omega_{R_j}$. Due to Lemma 2.36, $\tilde{d}(x,0) = 0$, the Lipschitz properties of $\tilde{d}$ and the splitting (3.66), we can conclude

$$\begin{aligned}
&\|\tilde{d}(\cdot,y)\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega_{R_j})} \\
&\leq c\left(\|r_j^{\delta_j-\sigma}(\tilde{d}(\cdot,y)-\tilde{d}(\cdot,0))\|_{C^0(\bar{\Omega}_{R_j})} + \sup_{\substack{x_1,x_2\in\Omega_{R_j}\\ |x_1-x_2|\leq\frac{1}{2}r_j(x_1)}} r_j(x_1)^{\delta_j}\frac{\left|\tilde{d}(x_1,y(x_1))-\tilde{d}(x_2,y(x_2))\right|}{|x_1-x_2|^\sigma}\right) \\
&\leq c\left(\|r_j^{\delta_j-\sigma} y\|_{C^0(\bar{\Omega}_{R_j})} + \sup_{\substack{x_1,x_2\in\Omega_{R_j}\\ |x_1-x_2|\leq\frac{1}{2}r_j(x_1)}} r_j(x_1)^{\delta_j}\frac{|y(x_1)-y(x_2)|}{|x_1-x_2|^\sigma} + \sup_{\substack{x_1,x_2\in\Omega_{R_j}\\ |x_1-x_2|\leq\frac{1}{2}r_j(x_1)}} r_j(x_1)^{\delta_j}\frac{|x_1-x_2|^\sigma}{|x_1-x_2|^\sigma}\right) \\
&\leq c(\|y\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega_{R_j})}+1) \leq c\left(\|u\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega_{R_j})} + \left|y(x^{(j)})\right| \|\eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega_{R_j})} + 1\right),
\end{aligned} \tag{3.67}$$

where the positive constant c depends on E_Ω, c_Ω, $\|f - d(\cdot,0)\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}$ and $\|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}$ according to Lemma 3.20 and (3.65). Analogously to (3.44) and (3.47) we get

$$\|u\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \sum_{i=1}^{m}\left|y(x^{(j)})\right| \|\eta_j\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} \leq c\left(\|u\|_{V^{2,2}_{\vec{\beta}}(\Omega)} + \sum_{i=1}^{m}\left|y(x^{(j)})\right|\right) \leq c\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}, \tag{3.68}$$

where we used Lemma 2.32 in the last step. Next, we observe that $C^{0,1}(\bar{\Omega}) \hookrightarrow N^{0,\sigma}_{\vec{\delta}}(\Omega)$ according to Lemma 2.36. Thus, we have $d(\cdot,0) \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$. In summary, we have proven $f - d(\cdot,0) - \tilde{d}(\cdot,y) \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$. The estimate (3.64) is now a consequence of Lemma 3.13, (3.67), (3.68), Corollary 3.23 and (3.65). □

Corollary 3.27. *Let Assumptions 3.18 and 3.24 be fulfilled. Moreover, let* $\vec{\delta} \in \mathbb{R}^m$, $\vec{\tau} \in \mathbb{R}^m$ *and* $\sigma \in (0,1)$ *satisfy the conditions stated in Corollary 3.14. Then for every* $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ *the unique weak solution* y *of problem* (3.56) *belongs to* $W^{1,\infty}_{\vec{\tau}}(\Gamma)$ *and fulfills the estimate*

$$\|y\|_{W^{1,\infty}_{\vec{\tau}}(\Gamma)} \leq c$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}, \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}) > 0$.

Corollary 3.28. *Suppose that Assumptions 3.18 and 3.24 are satisfied. Furthermore, let* $\vec{\delta} \in \mathbb{R}^m$, $\vec{\kappa} \in \mathbb{R}^m$ *and* $\sigma \in (0,1)$ *fulfill the conditions stated in Corollary 3.15. Then for every*

$f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ and $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ the unique weak solution y of problem (3.56) *is an element of $W^{2,2}_{\vec{\kappa}}(\Gamma)$ and satisfies the estimate*

$$\|y\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \le c$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}, \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}) > 0$.

Proof of Corollary 3.27 and Corollary 3.28. The proof follows the same steps as the proofs of Corollary 3.14 and Corollary 3.15 using the results of Corollary 3.26 instead of Lemma 3.13. □

As in the linear elliptic case let us end this section with some Lipschitz estimates which will frequently be used in Section 4.4.

Lemma 3.29. *Let Assumption 3.18 be satisfied, some arbitrary $r > 1$ and $s > 1$ be given and let $d(\cdot,0) \in L^r(\Omega)$. Furthermore, let $y_1 \in H^1(\Omega) \cap C^0(\bar{\Omega})$ and $y_2 \in H^1(\Omega) \cap C^0(\bar{\Omega})$ be the weak solutions of* (3.56) *with right hand sides $f_1 \in L^r(\Omega)$ and $f_2 \in L^r(\Omega)$ and Neumann boundary data $g_1 \in L^s(\Gamma)$ and $g_2 \in L^s(\Gamma)$, respectively. Then there is the estimate*

$$\|y_1 - y_2\|_{L^2(\Omega)} \le c\left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right)$$

with a constant $c = c(E_\Omega, c_\Omega, \|f_1 - d(\cdot,0)\|_{L^r(\Omega)}, \|f_2 - d(\cdot,0)\|_{L^r(\Omega)}, \|g_1\|_{L^s(\Gamma)}, \|g_2\|_{L^s(\Gamma)}) > 0$. Moreover, the estimate

$$\|y_1 - y_2\|_{H^1(\Omega)} \le c\left(\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)}\right)$$

holds true with a positive constant $c = c(E_\Omega, c_\Omega)$ independent of d, f_1, f_2, g_1 and g_2.

Proof. To prove this lemma, we proceed as for the proof of Lemma 3.16. For the first assertion we have to choose an appropriate dual problem. Let w be the weak solution of

$$\begin{aligned} -\Delta w + (\alpha + \psi) w &= y_1 - y_2 && \text{in } \Omega, \\ \partial_n w &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.69}$$

with

$$\psi(x) = \begin{cases} \frac{\tilde{d}(x,y_1(x)) - \tilde{d}(x,y_2(x))}{y_1(x) - y_2(x)} & \text{if } y_1(x) \neq y_2(x), \\ 0 & \text{otherwise.} \end{cases}$$

The function $\alpha+\psi$ fulfills Assumption 3.1 (A1) with $\alpha+\psi \ge c_\Omega$ on E_Ω and $\|\alpha+\psi\|_{L^\infty(\Omega)} \le L_{d,M}$ due to Assumption 3.18, where the Lipschitz constant $L_{d,M}$ depends on $\|f_1 - d(\cdot,0)\|_{L^r(\Omega)}$, $\|f_2 - d(\cdot,0)\|_{L^r(\Omega)}$, $\|g_1\|_{L^s(\Gamma)}$ and $\|g_2\|_{L^s(\Gamma)}$ according to Lemma 3.20. Therefore, we can conclude according to Lemma 3.4 and Remark 3.7 that

$$\begin{aligned} \|y_1 - y_2\|^2_{L^2(\Omega)} &= \tilde{a}(y_1 - y_2, w) + \int_\Omega \psi(y_1 - y_2) w = \tilde{a}(y_1 - y_2, w) + \int_\Omega (\tilde{d}(\cdot, y_1) - \tilde{d}(\cdot, y_2)) w \\ &= \int_\Omega (f_1 - f_2) w + \int_\Gamma (g_1 - g_2) w \le \left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right) \|w\|_{C^0(\bar{\Omega})} \\ &\le c\left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right) \|y_1 - y_2\|_{L^2(\Omega)} \end{aligned}$$

with a positive constant $c = c(E_\Omega, c_\Omega, \|f_1 - d(\cdot,0)\|_{L^r(\Omega)}, \|f_2 - d(\cdot,0)\|_{L^r(\Omega)}, \|g_1\|_{L^s(\Gamma)}, \|g_2\|_{L^s(\Gamma)})$. Dividing by $\|y_1 - y_2\|_{L^2(\Omega)}$ yields the first assertion. For the second one we use the coercivity of the bilinear form $\tilde{a}$ and the monotonicity of $\tilde{d}$ to get

$$\begin{aligned}\|y_1 - y_2\|^2_{H^1(\Omega)} &\le c\tilde{a}(y_1 - y_2, y_1 - y_2)\\ &\le c\left(\tilde{a}(y_1 - y_2, y_1 - y_2) + \int_\Omega \left(\tilde{d}(\cdot, y_1) - \tilde{d}(\cdot, y_2)\right)(y_1 - y_2)\right)\\ &= c\left(\int_\Omega (f_1 - f_2)(y_1 - y_2) + \int_\Gamma (g_1 - g_2)(y_1 - y_2)\right)\end{aligned}$$

with a positive constant $c = c(E_\Omega, c_\Omega)$ independent of d, f_1, f_2, g_1 and g_2. The second assertion can now be deduced by the Hölder inequality, Theorem 2.7 and Theorem 2.8, cf. the proof of Lemma 3.16. □

3.2 Discretization and error estimates

In this section we analyze the discretization of the linear elliptic boundary value problem (3.1) as well as the semilinear elliptic boundary value problem (3.56) by piecewise linear finite elements. We focus on the derivation of finite element error estimates in the domain and on the boundary on quasi-uniform and graded triangulations, which are introduced in Section 3.2.1 below. In preparation for the error analysis we prove error estimates for several interpolation operators on such triangulations in Section 3.2.2. Afterwards, the finite element error estimates in the domain and on the boundary for the linear problem are established in Section 3.2.3 and Section 3.2.4, respectively. For the proof of the estimates in the domain we rely on standard techniques such as Cea's Lemma and the Aubin-Nitsche method. On the contrary, the estimates on the boundary need a more sophisticated analysis which is new in this context. It is based on a dyadic partitioning of the domain and local finite element error estimates in different norms. For more details we refer to the road map in Section 3.2.4. The transfer of all these results to semilinear boundary value problems is contained in Section 3.2.6. Furthermore, at the end of Section 3.2.3 and Section 3.2.6, we derive for each problem Lipschitz estimates as in the continuous case, which are needed in Section 4.4 for the numerical analysis of the semilinear Neumann boundary control problems. In Sections 3.2.5 and 3.2.7 one can find numerical experiments for linear and semilinear problems, respectively, which confirm our theoretical findings.

Finally, let us remark that in the meantime the usage of gradually refined meshes is a well established method in order to compensate the negative effects of the corner singularities on the quality of finite element solutions. It already started with the very early contributions [94, 15, 93, 16, 97, 98].

3.2.1 Gradually refined triangulations

Before we begin with the error analysis, we first introduce a family of graded triangulations $\{\mathcal{T}_h\}$ of the domain Ω which is admissible in the sense of Ciarlet [34]. We denote by h the global mesh parameter. Furthermore, we assume $h \le h_0 < 1$. We denote by $\mu_j \in (0,1]$, $j = 1,\dots,m$, the mesh grading parameters which are collected in the vector $\vec{\mu}$. The distance of a triangle $T \in \mathcal{T}_h$ to the corner $x^{(j)}$ is defined by $r_{T,j} := \inf_{x\in T} |x - x^{(j)}|$. We assume that the element size $h_T := \operatorname{diam} T$ of each $T \in \mathcal{T}_h$ satisfies

$$\begin{aligned} c_1 h^{1/\mu_j} \le h_T \le c_2 h^{1/\mu_j} \quad & \text{for } r_{T,j} = 0, \\ c_1 h r_{T,j}^{1-\mu_j} \le h_T \le c_2 h r_{T,j}^{1-\mu_j} \quad & \text{for } 0 < r_{T,j} \le R_j, \\ c_1 h \le h_T \le c_2 h \quad & \text{for } r_{T,j} > R_j \end{aligned} \tag{3.70}$$

for $j = 1,\dots,m$ with the radii R_j which we have defined in Section 2.2. Furthermore, for $j = 1,\dots,m$ let $\mathcal{T}_{h,j}$ be a sub-triangulation of $\mathcal{T}_h$ such that $\bigcup_{T\in\mathcal{T}_{h,j}} \bar{T} \subset \Omega_{R_j}$ and $T \cap \Omega_{R_j} \ne T$ for all $T \notin \mathcal{T}_{h,j}$. We set $\mathcal{T}_{h,0} := \mathcal{T}_h \setminus \bigcup_{j=1}^m \mathcal{T}_{h,j}$. Next, we introduce the space V_h as the space of all piecewise linear and globally continuous functions in $\bar{\Omega}$,

$$V_h := \{y_h \in C^0(\bar{\Omega}) : y_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h\},$$

where $\mathcal{P}_k(T)$ denotes the space of polynomials of degree less than or equal to k on T. In the sequel we will denote by $X^{(i)}$ the nodes of the triangulation $\mathcal{T}_h$, by I_X the index set of the nodes and by ϕ_i the nodal basis functions of V_h.

Due to the definition of the triangulation $\mathcal{T}_h$, there is a segmentation $\mathcal{E}_h$ of the boundary naturally induced by the triangulation $\mathcal{T}_h$. We define the distance of an edge $E \in \mathcal{E}_h$ to the corner $x^{(j)}$ by $r_{E,j} := \inf_{x\in E} |x - x^{(j)}|$ and the element size h_E by $h_E := \operatorname{diam} E$. According to (3.70) for each $E \in \mathcal{E}_h$ there holds

$$\begin{aligned} c_1 h^{1/\mu_j} \le h_E \le c_2 h^{1/\mu_j} \quad & \text{for } r_{E,j} = 0, \\ c_1 h r_{E,j}^{1-\mu_j} \le h_E \le c_2 h r_{E,j}^{1-\mu_j} \quad & \text{for } 0 < r_{E,j} \le R_j, \\ c_1 h \le h_E \le c_2 h \quad & \text{for } r_{E,j} > R_j \end{aligned} \tag{3.71}$$

for $j = 1,\dots,m$. Furthermore, for $j = 1,\dots,m$ let $\mathcal{E}_{h,j}$ be the sub-triangulation of $\mathcal{E}_h$ such that $\bigcup_{E\in\mathcal{E}_{h,j}} \bar{E} \subset \Gamma_j^\pm$ and $E \cap \Gamma_j^\pm \ne E$ for all $E \notin \mathcal{E}_{h,j}$. We set $\mathcal{E}_{h,0} := \mathcal{E}_h \setminus \bigcup_{j=1}^m \mathcal{E}_{h,j}$. We define the space V_h^∂ as the restriction of V_h to the boundary Γ, and we denote by I_X^∂ the index set of the nodes on the boundary and by ψ_i the nodal basis functions of V_h^∂. Furthermore, we introduce the space U_h as the space of all piecewise constant functions on the boundary, i.e.,

$$U_h := \{u_h \in L^\infty(\Gamma) : u_h|_E \in \mathcal{P}_0(E)\ \forall E \in \mathcal{E}_h\}.$$

Next, let I_E be the index set of elements on the boundary. Then for $i \in I_E$ we denote by e_i a basis function of U_h which is equal to one on the element $E_i \in \mathcal{E}_h$ and equal to zero on $E_j \in \mathcal{E}_h$ with $j \in I_E$ and $j \ne i$.

Remark 3.30. For $\vec{\mu} = \vec{1}$ the mesh is called quasi-uniform. In contrast, if $\mu_j < 1$ for some $j \in \{1,\dots,m\}$ then the mesh is a graded one. We notice that the number of elements of $\mathcal{T}_h$ and $\mathcal{E}_h$ is of order h^{-2} and h^{-1}, respectively, independent of the choice of $\vec{\mu}$, see e.g. [10].

Finally, we emphasize once again that the generic constant $c > 0$ is always independent of the discretization parameter h in all what follows.

3.2.2 Interpolation error estimates on quasi-uniform and graded meshes

This section is devoted to the derivation of interpolation error estimates on quasi-uniform and graded triangulations. We start with local interpolation error estimates for elements $E \in \mathcal{E}_h$ and $T \in \mathcal{T}_h$. We do this for both kinds of elements at once. For that reason, let K be either an element of $\mathcal{E}_h$ or $\mathcal{T}_h$ and let n be the dimension of K. Note, for all $K \in \mathcal{E}_h \cup \mathcal{T}_h$ there is a reference element $\hat{K} \subset \mathbb{R}^2$ of dimension n with $|\hat{K}| \sim 1$ and an affine mapping F_K such that

$$K \ni x = F_K(\hat{x}) = A_K\hat{x} + a_K \quad \forall \hat{x} \in \hat{K} \tag{3.72}$$

with $A_K \in \mathbb{R}^{2\times 2}$ and $a_K \in \mathbb{R}^2$. Furthermore, we set $\hat{v} := v \circ F_K$ for every function v defined on K. Next, we introduce local interpolation operators. For every $K \in \mathcal{E}_h \cup \mathcal{T}_h$ let $l \in \mathbb{N}$ and let I_K be an operator with domain $D_I(K)$ and range $R_I(K)$ such that

$$\mathcal{P}_{l-1}(K) \subset D_I(K), \tag{3.73}$$
$$\mathcal{P}_{l-1}(K) \subset R_I(K), \tag{3.74}$$
$$I_K p_{l-1} = p_{l-1} \qquad \forall p_{l-1} \in \mathcal{P}_{l-1}(K), \tag{3.75}$$
$$I_K(v_1 + v_2) = I_K v_1 + I_K v_2 \quad \forall v_1, v_2 \in D_I(K), \tag{3.76}$$
$$(I_K v) \circ F_K = I_{\hat{K}}\hat{v} \qquad \forall v \in D_I(K). \tag{3.77}$$

Furthermore, we assume that there exist parameters $k \in \{0,\dots,l-1\}$, $p, q \in [1,\infty]$, $p' \in [1,p]$ if $p < \infty$, $p' \in [1,\infty)$ if $p = \infty$, and $\beta_j' \in (-n/p', n+1-n/p')$ for $j = 1,\dots,m$ such that

$$|\hat{v}|_{W^{k,q}(\hat{K})} + |I_{\hat{K}}\hat{v}|_{W^{k,q}(\hat{K})} \le c\|\hat{v}\|_{W^{l,p}(\hat{K})} \quad \forall v \in W^{l,p}(K) \tag{3.78}$$

if $r_{K,j} > 0$ for $j \in \{1,\dots,m\}$, and

$$|\hat{v}|_{W^{k,q}(\hat{K})} + |I_{\hat{K}}\hat{v}|_{W^{k,q}(\hat{K})} \le c \sum_{|\alpha|\le l} \|\hat{r}_j^{\beta_j'} D^\alpha \hat{v}\|_{L^{p'}(\hat{K})} \quad \forall v \in W^{l,p'}_{\beta_j'}(K) \tag{3.79}$$

if $r_{K,j} = 0$ for some $j \in \{1,\dots,m\}$. Note that $D^\alpha\hat{v}$ denotes the usual weak derivative of $\hat{v}$ on $\hat{K}$ for $K \in \mathcal{T}_h$ and the weak tangential derivative of $\hat{v}$ on $\hat{K}$ in case of $K \in \mathcal{E}_h$.

Lemma 3.31. *Let $I_K : D_I(K) \to R_I(K)$ be an operator which fulfills the requirements* (3.73)–(3.79)*. Then for every $K \in \mathcal{E}_{h,0} \cup \mathcal{T}_{h,0}$ there is the estimate*

$$|v - I_K v|_{W^{k,q}(K)} \le c h^{n/q-n/p+l-k} |v|_{W^{l,p}(K)} \tag{3.80}$$

for all $v \in W^{l,p}(K)$. For every $K \in \mathcal{E}_{h,j} \cup \mathcal{T}_{h,j}$ with $j \in \{1,\dots,m\}$ and $r_{K,j} > 0$ there holds

$$|v - I_K v|_{W^{k,q}(K)} \le c h^{n/q-n/p+l-k} r_{K,j}^{(n/q-n/p+l-k)(1-\mu_j)-\beta_j} |v|_{W^{l,p}_{\beta_j}(K)} \tag{3.81}$$

for all $v \in W^{l,p}_{\beta_j}(K)$ with some arbitrary $\beta_j \in \mathbb{R}$. Furthermore, for every $K \in \mathcal{E}_{h,j} \cup \mathcal{T}_{h,j}$ with $j \in \{1,\dots,m\}$ and $r_{K,j} = 0$ the estimate

$$|v - I_K v|_{W^{k,q}(K)} \le c h^{(n/q-n/p+l-k-\beta_j)/\mu_j} |v|_{W^{l,p}_{\beta_j}(K)} \tag{3.82}$$

is valid for all $v \in W^{l,p}_{\beta_j}(K)$ with some arbitrary $\beta_j \in [-\infty, n/p' - n/p + \beta_j')$ if $p' < p$ and $\beta_j \in [-\infty, \beta_j']$ if $p' = p < \infty$.

Proof. We start with a general estimate which holds for all elements $K \in \mathcal{E}_h \cup \mathcal{T}_h$. Introducing an arbitrary polynomial $p_{l-1} \in \mathcal{P}_{l-1}(K)$ yields

$$
\begin{aligned}
|v - I_K v|_{W^{k,q}(K)} &= |v - p_{l-1} - I_K (v - p_{l-1})|_{W^{k,q}(K)} \\
&\leq |v - p_{l-1}|_{W^{k,q}(K)} + |I_K (v - p_{l-1})|_{W^{k,q}(K)},
\end{aligned}
$$

where we used the properties (3.73), (3.75) and (3.76) of the local interpolation operator. Next we can conclude by means of Theorem 15.1 of [34], together with (3.77),

$$
|v - I_K v|_{W^{k,q}(K)} \leq c|K|^{1/q} h_K^{-k} \left(|\hat{v} - \hat{p}_{l-1}|_{W^{k,q}(\hat{K})} + |I_{\hat{K}} (\hat{v} - \hat{p}_{l-1})|_{W^{k,q}(\hat{K})} \right), \tag{3.83}
$$

where $\hat{p}_{l-1}$ is an arbitrary polynomial of order $l-1$ on $\hat{K}$ since F_K is affine according to (3.72). Now, we distinguish between elements with positive distance to the corners and elements with direct contact to a corner. First, let $K \in \mathcal{E}_h \cup \mathcal{T}_h$ and $r_{K,j} > 0$ for $j \in \{1, \dots, m\}$. According to (3.78) we can conclude

$$
|v - I_K v|_{W^{k,q}(K)} \leq c|K|^{1/q} h_K^{-k} \|\hat{v} - \hat{p}_{l-1}\|_{W^{l,p}(\hat{K})}.
$$

The Deny-Lions Lemma [40] (or Bramble-Hilbert Lemma in e.g. [20, Lemma 4.3.8]) yields together with the transformation to the world element

$$
|v - I_K v|_{W^{k,q}(K)} \leq c|K|^{1/q} h_K^{-k} |\hat{v}|_{W^{l,p}(\hat{K})} \leq c|K|^{1/q-1/p} h_K^{l-k} |v|_{W^{l,p}(K)},
$$

cf. again Theorem 15.1 of [34]. The mesh conditions (3.70) and (3.71) lead with $|K| \sim h_K^n$ to

$$
|v - I_K v|_{W^{k,q}(K)} \leq c h^{n/q-n/p+l-k} |v|_{W^{l,p}(K)}
$$

for $K \in \mathcal{E}_{h,0} \cup \mathcal{T}_{h,0}$, and in case of $K \in \mathcal{E}_{h,j} \cup \mathcal{T}_{h,j}$ to

$$
|v - I_K v|_{W^{k,q}(K)} \leq c h_K^{n/q-n/p+l-k} |v|_{W^{l,p}(K)} \leq c h^{n/q-n/p+l-k} r_{K,j}^{(n/q-n/p+l-k)(1-\mu_j)-\beta_j} |v|_{W^{l,p}_{\beta_j}(K)}
$$

with some arbitrary $\beta_j \in \mathbb{R}$ since $\min_{x \in K} r_j(x) \sim \max_{x \in K} r_j(x)$ for all elements K with $r_{K,j} > 0$. Thus, (3.80) and (3.81) are proven. Next we consider elements K with $r_{K,j} = 0$ and $j \in \{1, \dots, m\}$. Using (3.79) we obtain from (3.83)

$$
|v - I_K v|_{W^{k,q}(K)} \leq c|K|^{1/q} h_K^{-k} \sum_{|\alpha| \leq l} \|\hat{r}_j^{\beta'_j} D^\alpha (\hat{v} - \hat{p}_{l-1})\|_{L^{p'}(\hat{K})}.
$$

Next, we observe, that $\hat{K}$ is either a polygonal domain or a side of a polygonal domain, where the weight is related to its corner $F_K^{-1}(x^{(j)})$. Thus, an application of Lemma 2.31 yields

$$
\begin{aligned}
|v - I_K v|_{W^{k,q}(K)} &\leq c|K|^{1/q} h_K^{-k} \left(\sum_{|\alpha|=l} \|\hat{r}_j^{\beta'_j} D^\alpha (\hat{v} - \hat{p}_{l-1})\|_{L^{p'}(\hat{K})} + \sum_{|\alpha|<l} \left| \int_{\hat{K}} D^\alpha (\hat{v} - \hat{p}_{l-1}) \right| \right) \\
&= c|K|^{1/q} h_K^{-k} \left(\sum_{|\alpha|=l} \|\hat{r}_j^{\beta'_j} D^\alpha \hat{v}\|_{L^{p'}(\hat{K})} + \sum_{|\alpha|<l} \left| \int_{\hat{K}} D^\alpha (\hat{v} - \hat{p}_{l-1}) \right| \right).
\end{aligned} \tag{3.84}
$$

The last step holds since $\hat{p}_{l-1}$ is a polynomial of order $l-1$. Next, we choose $\hat{p}_{l-1}$ such that the last sum in (3.84) vanishes, which is possible without any restriction. By this we get

$$|v - I_K v|_{W^{k,q}(K)} \leq c|K|^{1/q} h_K^{-k} \sum_{|\alpha|=l} \|\hat{r}_j^{\beta'_j} D^\alpha \hat{v}\|_{L^{p'}(\hat{K})} \leq c|K|^{1/q} h_K^{-k} \sum_{|\alpha|=l} \|\hat{r}_j^{\beta_j} D^\alpha \hat{v}\|_{L^p(\hat{K})},$$

where we applied Lemma 2.29 in the last step. The transformation to the world element yields

$$|v - I_K v|_{W^{k,q}(K)} \leq c|K|^{1/q-1/p} h_K^{l-k-\beta_j} |v|_{W^{l,p}_{\beta_j}(K)},$$

since $\hat{r}_j \sim h_K^{-1} r_j$. Finally, we can conclude using the mesh conditions (3.70) and (3.71)

$$|v - I_K v|_{W^{k,q}(K)} \leq c h^{(n/q-n/p+l-k-\beta_j)/\mu_j} |v|_{W^{l,p}_{\beta_j}(K)},$$

which completes the proof. □

Remark 3.32. For functions $v \in V^{l,p}_{\beta_j}(K)$ with $K \in \mathcal{E}_{h,j} \cup \mathcal{T}_{h,j}$ and $j \in \{1, \ldots, m\}$ one could proceed in the same way to derive local interpolation error estimates in the neighborhood of the corners since the seminorms of the spaces $V^{l,p}_{\beta_j}(K)$ and $W^{l,p}_{\beta_j}(K)$ coincide. Alternatively, one could also assume for the local interpolation operator I_K

$$|\hat{v}|_{W^{k,q}(\hat{K})} + |I_{\hat{K}} \hat{v}|_{W^{k,q}(\hat{K})} \leq c \sum_{|\alpha|\leq l} \|\hat{r}_j^{\beta'_j - l + |\alpha|} D^\alpha \hat{v}\|_{L^{p'}(\hat{K})}$$

instead of (3.79). Then there is no need to use Lemma 2.31 for elements K with $r_{K,j} = 0$ since the transformation to the world element yields with $\hat{r}_j \sim h_K^{-1} r_j$

$$\sum_{|\alpha|\leq l} \|\hat{r}_j^{\beta'_j - l + |\alpha|} D^\alpha \hat{v}\|_{L^{p'}(\hat{K})} \leq c|K|^{-1/p'} h_K^{l-\beta'_j} \|v\|_{V^{l,p'}_{\beta'_j}(K)},$$

cf. e.g. [10], [4] and the references therein. In case of $W^{l,p}_{\beta_j}(K)$-regularity one could also circumvent the usage of Lemma 2.31 in some specific situations. In [7] local interpolation error estimates are derived for functions $v \in W^{2,p}_{\beta_j}(K)$ in two dimensional domains K with $r_{K,j} = 0$ under the assumption

$$|\hat{v}|_{W^{k,q}(\hat{K})} + |I_{\hat{K}} \hat{v}|_{W^{k,q}(\hat{K})} \leq c\|\hat{v}\|_{W^{2,p'}(\hat{K})}$$

with some arbitrarily small $p' > 1$. Then one can directly apply the Deny-Lions Lemma and part two of Lemma 2.29 afterwards to get a result comparable to that stated in Lemma 3.31.

In the remaining part of this section we check for some globally defined interpolation operators that they fulfill locally the requirements of Lemma 3.31 and state corresponding error estimates which are needed for the numerical analysis of the boundary value problems and the optimal control problems in the sequel. We start with the nodal Lagrange interpolant of order one in the domain and on the boundary which is defined by

$$(I_h v)(x) := \sum_{i\in I_X} v(X^{(i)})\phi_i(x) \quad \text{and} \quad (I_h^\partial v)(x) := \sum_{i\in I_X^\partial} v(X^{(i)})\psi_i(x),$$

respectively.

Corollary 3.33. *Let $j \in \{0, 1, \dots . m\}$ and $p \in (1, \infty]$. Then for every element $T \in \mathcal{T}_{h,j}$ there are the estimates*

$$\|v - I_h v\|_{L^2(T)} + h(|v - I_h v|_{H^1(T)} + \|v - I_h v\|_{L^\infty(T)}) \leq ch^{3-2/p}|v|_{W^{2,p}(T)}$$

if $j = 0$ and $v \in W^{2,p}(T)$,

$$\|v - I_h v\|_{L^2(T)} + hr_{T,j}^{1-\mu_j}(|v - I_h v|_{H^1(T)} + \|v - I_h v\|_{L^\infty(T)}) \leq ch^{3-2/p} r_{T,j}^{(3-2/p)(1-\mu_j)-\beta_j}|v|_{W^{2,p}_{\beta_j}(T)}$$

for $j \in \{1, \dots, m\}$, $r_{T,j} > 0$ and $v \in W^{2,p}_{\beta_j}(T)$ with some arbitrary $\beta_j \in \mathbb{R}$, and

$$\|v - I_h v\|_{L^2(T)} + h^{1/\mu_j}(|v - I_h v|_{H^1(T)} + \|v - I_h v\|_{L^\infty(T)}) \leq ch^{(3-2/p-\beta_j)/\mu_j}|v|_{W^{2,p}_{\beta_j}(T)}$$

if $j \in \{1, \dots, m\}$, $r_{T,j} = 0$ and $v \in W^{2,p}_{\beta_j}(T)$ with some arbitrary $\beta_j \in (-2 + 2/p, 2 - 2/p)$.

Proof. First, we observe that there is a local interpolant $I_T^h : C^0(\bar{T}) \to \mathcal{P}_1(T)$ defined by

$$(I_T^h v)(x) := \sum_{i \in I_{X,T}} v(X^{(i)})\phi_i(x) \quad \text{for } x \in T,$$

where $I_{X,T}$ denotes the index set of the nodes of the element T, such that

$$I_h v|_T = I_T^h v \quad \text{for } T \in \mathcal{T}_h.$$

This interpolant fulfills (3.73)–(3.77) by construction. Next, we show the properties (3.78) and (3.79). Let us set $\beta_j' = 0$, $p \in (1, \infty]$, $\beta_j \in (-2 + 2/p, 2 - 2/p)$ and $p' = 4p/(2p + 2 + \beta_j p)$. Thus, there holds $1 < p' < p$ and $\beta_j' \in (-2/p', 3-2/p')$, i.e., $0 \in (-1-1/p-\beta_j/2, 2-1/p-\beta_j/2)$. Furthermore, we can conclude according to Theorem 2.7 for every function $\hat{v} \in W^{2,p'}(\hat{T})$

$$\|\hat{v}\|_{H^1(\hat{T})} + \|\hat{v}\|_{L^\infty(\hat{T})} \leq c\|\hat{v}\|_{W^{2,p'}(\hat{T})}. \tag{3.85}$$

Moreover, using the norm equivalence for functions in finite dimensional spaces, the boundedness of the interpolation operator $I_{\hat{T}}^h$ from $L^\infty(\hat{T})$ to $L^\infty(\hat{T})$ and (3.85), we can show

$$\|I_{\hat{T}}^h \hat{v}\|_{H^1(\hat{T})} + \|I_{\hat{T}}^h \hat{v}\|_{L^\infty(\hat{T})} \leq c\|I_{\hat{T}}^h \hat{v}\|_{L^\infty(\hat{T})} \leq c\|\hat{v}\|_{L^\infty(\hat{T})} \leq c\|\hat{v}\|_{W^{2,p'}(\hat{T})}. \tag{3.86}$$

Thus, all requirements of Lemma 3.31 are fulfilled and the assertion follows, since $\beta_j < 2/p' - 2/p + \beta_j'$ is equivalent to $\beta_j < 2 - 2/p$. □

Corollary 3.34. *Let $j \in \{0, 1, \dots . m\}$ and let $E \in \mathcal{E}_{h,j}$. Then for $j = 0$ there are the estimates*

$$\|v - I_h^\partial v\|_{L^\infty(E)} \leq \begin{cases} ch|v|_{W^{1,\infty}(E)} & \text{if } v \in W^{1,\infty}(E), \\ ch^{3/2}|v|_{W^{2,2}(E)} & \text{if } v \in W^{2,2}(E). \end{cases}$$

For $j \in \{1, \dots, m\}$ and $r_{E,j} > 0$ one has

$$\|v - I_h^\partial v\|_{L^\infty(E)} \leq \begin{cases} chr_{E,j}^{1-\mu_j-\beta_j}|v|_{W^{1,\infty}_{\beta_j}(E)} & \text{if } v \in W^{1,\infty}_{\beta_j}(E), \beta_j \in \mathbb{R}, \\ ch^{3/2} r_{E,j}^{3(1-\mu_j)/2-\beta_j}|v|_{W^{2,2}_{\beta_j}(E)} & \text{if } v \in W^{2,2}_{\beta_j}(E), \beta_j \in \mathbb{R}. \end{cases}$$

For $j \in \{1,\dots,m\}$ and $r_{E,j}=0$ there holds

$$\|v - I_h^\partial v\|_{L^\infty(E)} \le \begin{cases} ch^{(1-\beta_j)/\mu_j}|v|_{W^{1,\infty}_{\beta_j}(E)} & \text{if } v \in W^{1,\infty}_{\beta_j}(E), \beta_j \in (-1,1), \\ ch^{(3/2-\beta_j)/\mu_j}|v|_{W^{2,2}_{\beta_j}(E)} & \text{if } v \in W^{2,2}_{\beta_j}(E), \beta_j \in (-1/2,3/2). \end{cases}$$

Proof. We proceed similar to the proof of Corollary 3.33. Let $I_E^\partial : C^0(\bar{E}) \to \mathcal{P}_1(E)$ be the local interpolant defined by

$$(I_E^\partial v)(x) := \sum_{i\in I_{X,E}} v(X^{(i)})\psi_i(x) \quad \text{for } x \in E,$$

where $I_{X,E}$ denotes the index set of the nodes of the element E. This interpolant fulfills

$$I_h^\partial v|_E = I_E^\partial v \quad \text{for } E \in \mathcal{E}_h$$

and the properties (3.73)–(3.77). It remains to prove (3.78) and (3.79) for the different regularity assumptions. We begin with functions v belonging $W^{1,\infty}(E)$ or $W^{1,\infty}_{\beta_j}(E)$. Let us set $\beta_j' = 0$, $\beta_j \in (-1,1)$ and $p' = 2/(1+\beta_j)$. With this choice there holds $1 < p' < \infty$ and $\beta_j' = 0 \in (-1/p', 2-1/p')$. Furthermore, using the boundedness of $I_{\hat{E}}^\partial$ from $L^\infty(\hat{E})$ to $L^\infty(\hat{E})$ we can conclude according to Theorem 2.7 for every function $\hat{v} \in W^{1,p'}(\hat{E})$

$$\|I_{\hat{E}}^\partial \hat{v}\|_{L^\infty(\hat{E})} \le \|\hat{v}\|_{L^\infty(\hat{E})} \le c\|\hat{v}\|_{W^{1,p'}(\hat{E})}. \tag{3.87}$$

Thus, Lemma 3.31 can be applied. The condition $\beta_j < 1/p' - 1/p + \beta_j'$ is equivalent to $\beta_j < 1$ in this case. Next we assume that v belongs to $W^{2,2}(E)$ or $W^{2,2}_{\beta_j}(E)$. Now, we set $\beta_j' = 1$, $\beta_j \in (-1/2, 3/2)$ and $p' = 8/(2\beta_j+5)$. It is easy to check that $1 < p' < 2$ and $\beta_j' = 1 \in (-1/p', 2-1/p')$. Moreover, using (3.87) and Theorem 2.7 we obtain for elements E with $r_{E,j} > 0$

$$\|I_{\hat{E}}^\partial \hat{v}\|_{L^\infty(\hat{E})} \le \|\hat{v}\|_{L^\infty(\hat{E})} \le c\|\hat{v}\|_{W^{1,p'}(\hat{E})} \le \|\hat{v}\|_{W^{2,p'}(\hat{E})},$$

which shows (3.78). In the same manner we can show for elements E with $r_{E,j} = 0$ using Theorem 2.7 and Lemma 2.29, that

$$\|I_{\hat{E}}^\partial \hat{v}\|_{L^\infty(\hat{E})} \le \|\hat{v}\|_{L^\infty(\hat{E})} \le c\|\hat{v}\|_{W^{1,p'}(\hat{E})} \le c \sum_{|\alpha|\le 2} \|\hat{r}_j D^\alpha \hat{v}\|_{L^{p'}(\hat{E})},$$

which proves (3.79). Again, we can conclude the assertion by means of Lemma 3.31. Note that the condition $\beta_j < 1/p' - 1/p + \beta_j'$ is equivalent $\beta_j < 3/2$. □

For the discretization error analysis of the optimal control problem we also need some results for the 0-interpolator on U_h. Let S_E be the midpoint of the edge $E \in \mathcal{E}_h$. The projection operator R_h is defined by

$$(R_h v)(x) := v(S_E) \quad \text{if } x \in E.$$

Corollary 3.35. *(i) Let $E \in \mathcal{E}_h$ and $v \in H^1(E)$. Then the estimate*

$$\|v - R_h v\|_{L^2(E)} + h_E^{1/2} \|v - R_h v\|_{L^\infty(E)} \le c h_E |v|_{H^1(E)}$$

holds.

(ii) Let $j \in \{0, 1, \dots, m\}$ and let $E \in \mathcal{E}_{h,j}$. Then for $j = 0$ the following estimates hold true

$$\left| \int_E (v - R_h v) \right| \le \begin{cases} ch|E||v|_{W^{1,\infty}(E)} & \text{if } v \in W^{1,\infty}(E), \\ ch^2 |E|^{1/2} |v|_{W^{2,2}(E)} & \text{if } v \in W^{2,2}(E). \end{cases}$$

For $j \in \{1, \dots, m\}$ and $r_{E,j} > 0$ the following estimates are valid

$$\left| \int_E (v - R_h v) \right| \le \begin{cases} ch|E| r_{E,j}^{1-\mu_j-\beta_j} |v|_{W^{1,\infty}_{\beta_j}(E)} & \text{if } v \in W^{1,\infty}_{\beta_j}(E), \beta_j \in \mathbb{R}, \\ ch^2 |E|^{1/2} r_{E,j}^{2(1-\mu_j)-\beta_j} |v|_{W^{2,2}_{\beta_j}(E)} & \text{if } v \in W^{2,2}_{\beta_j}(E), \beta_j \in \mathbb{R}. \end{cases}$$

For $j \in \{1, \dots, m\}$ and $r_{E,j} = 0$ the following estimates

$$\left| \int_E (v - R_h v) \right| \le \begin{cases} ch^{(1-\beta_j)/\mu_j} |E||v|_{W^{1,\infty}_{\beta_j}(E)} & \text{if } v \in W^{1,\infty}_{\beta_j}(E), \beta_j \in (-1, 1), \\ ch^{(2-\beta_j)/\mu_j} |E|^{1/2} |v|_{W^{2,2}_{\beta_j}(E)} & \text{if } v \in W^{2,2}_{\beta_j}(E), \beta_j \in (-1/2, 3/2) \end{cases}$$

hold.

Proof. (i) We set $p = p' = 2$ and $\beta'_j = \beta_j = 0$. There holds analogously to (3.85) and (3.86)

$$\|\hat{v}\|_{L^2(\hat{E})} + \|\hat{v}(S_{\hat{E}})\|_{L^2(\hat{E})} + \|\hat{v}\|_{L^\infty(\hat{E})} + \|\hat{v}(S_{\hat{E}})\|_{L^\infty(\hat{E})} \le c\|\hat{v}\|_{L^\infty(\hat{E})} \le c\|\hat{v}\|_{H^1(\hat{E})}.$$

Now, the result can easily be deduced from Lemma 3.31.

(ii) First, we observe that the integral vanishes for any polynomial p of order one, hence

$$\begin{aligned} \left| \int_E (v - R_h v) \right| &= \left| \int_E (v - p - R_h(v - p)) \right| \le |E| \left(\|v - p\|_{L^\infty(E)} + \|R_h(v - p)\|_{L^\infty(E)} \right) \\ &\le c|E| \|v - p\|_{L^\infty(E)}. \end{aligned} \tag{3.88}$$

In the last step we used that R_h is a bounded operator from $L^\infty(E)$ to $L^\infty(E)$. We choose $p = I_h^\partial v|_E$. Now, the assertion follows from Corollary 3.34 together with $|E| \sim h_E$ and the mesh conditions (3.70) and (3.71). □

The good approximation properties of the operator R_h in Corollary 3.35 (ii) depend on the special choice of the evaluation point S_E. For a different point we can not expect such an approximation order, which we are going to show next. Let X_E an arbitrary point of the edge $E \in \mathcal{E}_h$. Furthermore, let $K_1 \subset \Gamma$ and $K_2 \subset \Gamma$ be two disjoint sets of unions of edges $E \in \mathcal{E}_h$ such that $\bar{K}_1 \cup \bar{K}_2 = \Gamma$. The operator $\tilde{R}_h$ is defined by

$$(\tilde{R}_h v)(x) := \begin{cases} v(X_E) & \text{if } x \in E, E \subset K_1, \\ (R_h v)(x) & \text{if } x \in E, E \subset K_2. \end{cases}$$

Corollary 3.36. *(i) Let $E \in \mathcal{E}_h$ and $v \in H^1(E)$. Then the estimate*

$$\|v - \tilde{R}_h v\|_{L^2(E)} + h_E^{1/2} \|v - \tilde{R}_h v\|_{L^\infty(E)} \le c h_E |v|_{H^1(E)}$$

holds.

(ii) Let $\mathcal{S} = \{1, \ldots, m\}$ and $j \in \{0\} \cup \mathcal{S}$. For $E \in \mathcal{E}_{h,j} \cap K_1$ the following estimates hold true

$$\left| \int_E (v - \tilde{R}_h v) \right| \le \begin{cases} ch|E||v|_{W^{1,\infty}(E)} & \text{if } j = 0,\ v \in W^{1,\infty}(E), \\ ch|E| r_{E,j}^{1-\mu_j-\beta_j} |v|_{W^{1,\infty}_{\beta_j}(E)} & \text{if } j \in \mathcal{S}, r_{E,j} > 0, v \in W^{1,\infty}_{\beta_j}(E), \beta_j \in \mathbb{R}, \\ ch^{(1-\beta_j)/\mu_j} |E||v|_{W^{1,\infty}_{\beta_j}(E)} & \text{if } j \in \mathcal{S}, r_{E,j} = 0, v \in W^{1,\infty}_{\beta_j}(E), \beta_j \in (-1, 1). \end{cases}$$

For $E \in \mathcal{E}_{h,j} \cap K_2$ the following estimates are valid

$$\left| \int_E (v - \tilde{R}_h v) \right| \le \begin{cases} ch^2|E|^{1/2}|v|_{W^{2,2}(E)} & \text{if } j = 0,\ v \in W^{2,2}(E), \\ ch^2|E|^{1/2} r_{E,j}^{2(1-\mu_j)-\beta_j} |v|_{W^{2,2}_{\beta_j}(E)} & \text{if } j \in \mathcal{S}, v \in W^{2,2}_{\beta_j}(E), \beta_j \in \mathbb{R}, \\ ch^{(2-\beta_j)/\mu_j} |E|^{1/2}|v|_{W^{2,2}_{\beta_j}(E)} & \text{if } j \in \mathcal{S}, v \in W^{2,2}_{\beta_j}(E), \beta_j \in (-1/2, 3/2). \end{cases}$$

Proof. (i) The proof of Corollary 3.35 (i) can easily be adopted to get the desired result for the modified operator $\tilde{R}_h$.

(ii) For elements $E \in K_2$ the estimates are a consequence of Corollary 3.35 (ii). For elements $E \in K_1$ we do not have that the integral vanishes for any polynomial p of order one. However, there holds

$$\left| \int_E (v - \tilde{R}_h v) \right| \le c|E| \|v - \tilde{R}_h v\|_{L^\infty(E)}.$$

Now, following the steps of the first part of the proof of Corollary 3.34 allows us to deduce the assertion for elements $E \subset K_1$. □

Finally, we define the L^2-projection of a function $v \in L^2(\Gamma)$ as the piecewise constant function $Q_h v$ in U_h that fulfills

$$Q_h v|_E = \frac{1}{|E|} \int_E v$$

on any element $E \in \mathcal{E}_h$.

Corollary 3.37. *For any element $E \in \mathcal{E}_h$ and any function $v \in H^1(E)$ the estimate*

$$\|v - Q_h v\|_{L^2(E)} \le c h_E |v|_{H^1(E)}$$

is valid.

Proof. First we observe that $Q_h p = p$ for any $p \in \mathcal{P}_0(E)$. Thus we can write

$$\begin{aligned} \|v - Q_h v\|_{L^2(E)} &= \|v - R_h v - Q_h(v - R_h v)\|_{L^2(E)} \le \|v - R_h v\|_{L^2(E)} + \|Q_h(v - R_h v)\|_{L^2(E)} \\ &\le c\|v - R_h v\|_{L^2(E)}, \end{aligned}$$

where we simply used the definition of the L^2-projection to get the boundedness. The estimate follows now from Corollary 3.35. Alternatively, one can also check, that Q_h fulfills the requirements of Lemma 3.31. □

Corollary 3.38. *For any element $E \in \mathcal{E}_h$ and any functions $v \in H^1(E)$ and $w \in H^1(E)$, the estimate*

$$(v - Q_h v, w)_{L^2(E)} \le c h_E^2 |v|_{H^1(E)} |w|_{H^1(E)}$$

is valid.

Proof. Due to the definition of Q_h we have the orthogonality $(v - Q_h v, p)_{L^2(E)} = 0$ for all $p \in \mathcal{P}_0(E)$. Thus we get

$$\begin{aligned}(v - Q_h v, w)_{L^2(E)} &= (v - Q_h v, w - Q_h w)_{L^2(E)} \le \|v - Q_h v\|_{L^2(E)} \|w - Q_h w\|_{L^2(E)} \\ &\le c h_E^2 |v|_{H^1(E)} |w|_{H^1(E)},\end{aligned}$$

where we used the Cauchy-Schwarz inequality and Corollary 3.37. □

3.2.3 Finite element error estimates for linear elliptic problems

In this section we derive finite element error estimates in the domain and state error estimates on the boundary. The proof of the estimates on the boundary is postponed to Section 3.2.4. We start with the definition of discrete solutions of (3.1).

Definition 3.39. Let $f \in H^1(\Omega)^*$ and $g \in H^{1/2}(\Gamma)^*$ and let Assumption 3.1 (A1) be fulfilled. A discrete solution of (3.1) is an element $y_h \in V_h \subset H^1(\Omega)$ that satisfies

$$a(y_h, v_h) = \int_\Omega f v_h + \int_\Gamma g v_h \quad \forall v_h \in V_h \tag{3.89}$$

with the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ from (3.3).

Lemma 3.40. *Suppose that the Assumption 3.1 (A1) is fulfilled. Then problem* (3.89) *has a unique solution $y_h \in V_h$ for*

(i) $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$. Furthermore, there is the estimate

$$\|y_h\|_{H^1(\Omega)} \le c \left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)} \right)$$

with a positive constant $c = c(E_\Omega, m)$ independent of f, g and α.

(ii) $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ with $0 \le \beta_j < 1$ for $j = 1, \dots, m$. Moreover, there holds

$$\|y_h\|_{H^1(\Omega)} \le c \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right)$$

with a constant $c = c(E_\Omega, m) > 0$ independent of f, g and α.

Proof. This follows from the Lax-Milgram Theorem as in the continuous case, cf. Lemma 3.4. □

Now we concentrate on the derivation of finite element error estimates on quasi-uniform and graded triangulations.

Lemma 3.41. *Let Assumption 3.1 (A1) be fulfilled. Moreover, let y and y_h be the weak solution of* (3.1) *and the solution of* (3.89)*, respectively. The discretization error can be estimated by*

$$\|y - y_h\|_{L^2(\Omega)} \le ch\|y - y_h\|_{W^{1,2}(\Omega)} \le ch^2\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le ch^2 \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right) \tag{3.90}$$

and

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch|\ln h|^{1/2}\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le ch|\ln h|^{1/2} \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right)$$

with a positive constant $c = c(E_\Omega, m, M)$ independent of f, g and α provided that $\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1} - \vec{\mu}$, $\vec{\beta} \ge \vec{0}$, $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$.

Proof. To prove the assertion we use standard techniques for estimates on finite element errors combined with the interpolation error estimates on graded triangulations given in Corollary 3.33. We can conclude by means of Theorem 2.8.1 of [20] (Céa's Lemma)

$$\|y - y_h\|_{W^{1,2}(\Omega)} \le \frac{\mu_1}{\mu_2} \inf_{v_h \in V_h} \|y - v_h\|_{W^{1,2}(\Omega)}, \tag{3.91}$$

where μ_1 denotes the continuity constant of (3.6) and μ_2 the coercivity constant of (3.7). Thus, there is a positive constant $c = c(E_\Omega, m, M)$ independent of f, g and α such that

$$\|y - y_h\|_{W^{1,2}(\Omega)} \le c \inf_{v_h \in V_h} \|y - v_h\|_{W^{1,2}(\Omega)} \le c\|e_h\|_{W^{1,2}(\Omega)}, \tag{3.92}$$

where $e_h := y - I_h y$ and $I_h y \in V_h$ denotes the linear Lagrangian interpolant of y. To derive an estimate for the interpolation error, we use the estimates of Corollary 3.33 with $p = 2$. Hence, we get for $\vec{\beta} \in (-1, 1)^m$

$$\begin{aligned}
\|e_h\|_{W^{1,2}(\Omega)} &\le c \left(\sum_{T \in \mathcal{T}_h} \|e_h\|^2_{W^{1,2}(T)} \right)^{1/2} \\
&\le c \left(\left(\sum_{T \in \mathcal{T}_{h,0}} \|e_h\|^2_{W^{1,2}(T)} \right)^{1/2} + \sum_{j=1}^m \left(\sum_{\substack{T \in \mathcal{T}_{h,j} \\ r_{T,j}=0}} \|e_h\|^2_{W^{1,2}(T)} + \sum_{\substack{T \in \mathcal{T}_{h,j} \\ r_{T,j}>0}} \|e_h\|^2_{W^{1,2}(T)} \right)^{1/2} \right) \\
&\le c \left(\left(\sum_{T \in \mathcal{T}_{h,0}} h^2 |y|^2_{W^{2,2}(T)} \right)^{1/2} \right. \\
&\quad \left. + \sum_{j=1}^m \left(\sum_{\substack{T \in \mathcal{T}_{h,j} \\ r_{T,j}=0}} h^{2(1-\beta_j)/\mu_j} |y|^2_{W^{2,2}_{\beta_j}(T)} + \sum_{\substack{T \in \mathcal{T}_{h,j} \\ r_{T,j}>0}} h^2 r_{T,j}^{2(1-\mu_j-\beta_j)} |y|^2_{W^{2,2}_{\beta_j}(T)} \right)^{1/2} \right)
\end{aligned}$$

$$\leq c \left(h|y|_{W^{2,2}(\Omega^0)} + \sum_{j=1}^{m} \max \left(h^{(1-\beta_j)/\mu_j}, \max_{\substack{T\in\mathcal{T}_{h,j}\\ r_{T,j}>0}} hr_{T,j}^{1-\mu_j-\beta_j} \right) |y|_{W^{2,2}_{\beta_j}(\Omega_{R_j})} \right). \tag{3.93}$$

If we set $\vec{1} - \vec{\lambda} < \vec{\beta} \leq \vec{1} - \vec{\mu}$ and $\vec{\beta} \geq \vec{0}$ we can conclude from (3.92) and (3.93) together with Lemma 3.11

$$\|y - y_h\|_{W^{1,2}(\Omega)} \leq ch\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq ch \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right) \tag{3.94}$$

with a positive constant $c = c(E_\Omega, m, M)$ independent of f, g and α. To get an error estimate in $L^2(\Omega)$ we can use the Aubin-Nitsche method, see e.g. [16]. Let $w \in H^1(\Omega)$ be the weak solution of

$$\begin{aligned} -\Delta w + \alpha w &= y - y_h \quad \text{in } \Omega, \\ \partial_n w &= 0 \qquad\quad \text{on } \Gamma_j, \quad j = 1, \ldots, m. \end{aligned}$$

Due to the Galerkin orthogonality $a(y - y_h, v_h) = 0 \ \forall v_h \in V_h$ and the continuity in $H^1(\Omega)$ of the bilinear form a, cf. (3.6), we can conclude that there is a positive constant $c = c(M)$ such that

$$\|y - y_h\|^2_{L^2(\Omega)} = a(y - y_h, w - I_h w) \leq c\|y - y_h\|_{W^{1,2}(\Omega)}\|w - I_h w\|_{W^{1,2}(\Omega)}. \tag{3.95}$$

Arguing as in (3.93) we get using Lemma 3.11 and Lemma 2.29

$$\|w - I_h w\|_{W^{1,2}(\Omega)} \leq ch\|w\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq ch\|y - y_h\|_{W^{0,2}_{\vec{\beta}}(\Omega)} \leq ch\|y - y_h\|_{L^2(\Omega)} \tag{3.96}$$

with a positive constant $c = c(E_\Omega, m, M)$ independent of f, g and α, provided that $\vec{1} - \vec{\lambda} < \vec{\beta} \leq \vec{1} - \vec{\mu}$ and $\vec{\beta} \geq 0$. The first inequality of (3.90) is now a consequence of (3.95) and (3.96). The second one holds if we additionally apply (3.94). Next we derive the estimate in $L^\infty(\Omega)$. Introducing the intermediate function $I_h y$ and applying the discrete Sobolev inequality of e.g. [20, Lemma 4.9.2] yields

$$\begin{aligned} \|y - y_h\|_{L^\infty(\Omega)} &\leq \|y - I_h y\|_{L^\infty(\Omega)} + \|I_h y - y_h\|_{L^\infty(\Omega)} && (3.97) \\ &\leq \|y - I_h y\|_{L^\infty(\Omega)} + c(1 + |\ln h|)^{1/2}\|I_h y - y_h\|_{H^1(\Omega)} && (3.98) \\ &\leq \|y - I_h y\|_{L^\infty(\Omega)} + c(1 + |\ln h|)^{1/2} \left(\|y - I_h y\|_{H^1(\Omega)} + \|y - y_h\|_{H^1(\Omega)} \right), && (3.99) \end{aligned}$$

where we inserted y in the last step. Now, we assume that $y - I_h y$ admits its maximum at some point $x_0 \in \bar{T}_* \in \mathcal{T}_h$. Next, if we argue as in (3.93), we can conclude for the first term of (3.99) using Corollary 3.33

$$\|y - I_h y\|_{L^\infty(\Omega)} = \|y - I_h y\|_{L^\infty(T_*)} \leq ch\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq ch \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right), \tag{3.100}$$

provided that $\vec{1} - \vec{\lambda} < \vec{\beta} \leq \vec{1} - \vec{\mu}$ and $\vec{\beta} \geq \vec{0}$. The positive constant $c = c(E_\Omega, m, M)$ is again independent of f, g and α. Finally, the estimates (3.99), (3.100), (3.93) and (3.94) yield the last inequality of the assertion. □

Corollary 3.42. *Let Assumption 3.1 (A1) be fulfilled and let $\mu_j = 1$ for $j = 1, \dots, m$ (quasi-uniform mesh). Furthermore, let $f \in W^{0,2}_{\vec\beta}(\Omega)$ and $g \in W^{1/2,2}_{\vec\beta}(\Gamma)$ with $\vec\beta = \vec1 - \vec\lambda + \vec\epsilon$, $\vec\beta \geq \vec0$ and $\vec\epsilon \in \mathbb{R}^m$ with $\vec0 < \vec\epsilon < \vec\lambda$. Then the discretization error can be estimated by*

$$\|y - y_h\|_{L^2(\Omega)} \leq ch^\lambda \|y - y_h\|_{W^{1,2}(\Omega)} \leq ch^{2\lambda}\|y\|_{W^{2,2}_{\vec\beta}(\Omega)} \leq ch^{2\lambda}\left(\|f\|_{W^{0,2}_{\vec\beta}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\beta}(\Gamma)}\right)$$

and

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^\lambda |\ln h|^{1/2}\|y\|_{W^{2,2}_{\vec\beta}(\Omega)} \leq ch^\lambda|\ln h|^{1/2}\left(\|f\|_{W^{0,2}_{\vec\beta}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec\beta}(\Gamma)}\right),$$

where $\lambda = \min(1, \min(\vec\lambda - \vec\epsilon))$ and $c = c(E_\Omega, m, M)$ is a positive constant independent of f, g and α.

Proof. Analogously to (3.93) and (3.100) one can show on quasi-uniform meshes

$$\|y - y_h\|_{W^{1,2}(\Omega)} + \|y - I_h y\|_{L^\infty(\Omega)} \leq ch^\lambda\|y\|_{W^{2,2}_{\vec\beta}(\Omega)}.$$

Instead of (3.96) we deduce

$$\|w - I_h w\|_{W^{1,2}(\Omega)} \leq ch^\lambda\|w\|_{W^{2,2}_{\vec\beta}(\Omega)} \leq ch^\lambda\|y - y_h\|_{W^{0,2}_{\vec\beta}(\Omega)} \leq ch^\lambda\|y - y_h\|_{L^2(\Omega)}.$$

Using these estimates in the proof of Lemma 3.41 yields the assertion. □

Remark 3.43. In Corollary 3.42 we proved a convergence rate of $\min(1, \min(\vec\lambda - \vec\epsilon))$ in $H^1(\Omega)$ on quasi-uniform meshes using regularity in weighted Sobolev spaces. However, if one would employ regularity results in Besov spaces, one could also show the order $\min(1, \min(\vec\lambda))$, cf. [17].

Remark 3.44. The finite element error estimates in the $L^\infty(\Omega)$-norm of Lemma 3.41 and Corollary 3.42 are only suboptimal in case of regularity higher than $W^{2,2}_{\vec\beta}(\Omega)$. In case of $W^{2,\infty}_{\vec\gamma}(\Omega)$-regularity optimal ones of order close to two can be deduced by the techniques used in Section 3.2.4, cf. [105, 8] for graded meshes and [106, 104] for quasi-uniform meshes.

Next, we are going to show that one can also expect convergence for $g \notin W^{1/2,2}_{\vec\beta}(\Omega)$. It relies on the fact, that the solution also belongs to $H^t(\Omega)$ with some $t > 1$ for $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$, cf. Corollary 3.6. We will need this results for the convergence analysis of semilinear Neumann boundary control problems in Section 4.4.

Corollary 3.45. *Suppose that Assumption 3.1 (A1) is fulfilled and let $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r \in (1, 4/3)$ and $s \in (1, 2)$. Furthermore, let $t = \min(3 - 2/r, 2 - 1/s)$, $\vec\epsilon \in \mathbb{R}^m$ with $\vec0 < \vec\epsilon < \vec\lambda$ and $\lambda = \min(1, \min(\vec\lambda - \vec\epsilon))$. Then the discretization error can be estimated by*

$$\begin{aligned}\|y - y_h\|_{L^2(\Omega)} + h^\lambda\|y - y_h\|_{W^{1,2}(\Omega)} + h^\lambda|\ln h|^{-1/2}\|y - y_h\|_{L^\infty(\Omega)} \\ \leq ch^{t-1+\lambda}\|y\|_{H^t(\Omega)} \leq ch^{t-1+\lambda}\left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right),\end{aligned}$$

where the constant $c = c(E_\Omega, m, M)$ is independent of f, g and α.

Proof. For every function $y \in H^t(T)$, $T \in \mathcal{T}_h$, we obtain from Theorem 6.1 of [46]

$$\|y - I_h y\|_{L^\infty(T)} + \|y - I_h y\|_{W^{1,2}(T)} \le ch^{t-1}|y|_{H^t(T)},$$

see also Example 3 in Section 8 of [46]. Using these estimates together with the regularity results of Corollary 3.6, one can mimic the proofs of Lemma 3.41 and Corollary 3.42 to get the validity of the assertion. □

Remark 3.46. For $f \in L^{4/3}(\Omega)$ and $g \in L^2(\Gamma)$ there holds $t = 3/2 - \epsilon'$ with some arbitrary $1/2 > \epsilon' > 0$. Furthermore, assume without loss of generality that $\pi/\omega_j - \epsilon_j = \min(\vec{\lambda} - \vec{\epsilon}) \le 1$ with some $j = 1, \dots, m$ and some arbitrary $\epsilon_j \in (0, \lambda_j)$. Next, let us choose ϵ' and ϵ_j such that there additionally holds $0 < \epsilon' + \epsilon_j < \pi/\omega_j - 1/2$, which is definitely possible since $\omega_j \in (0, 2\pi)$. Then we have

$$t - 1 + \lambda = \frac{3}{2} - \epsilon' - 1 + \frac{\pi}{\omega_j} - \epsilon_j > 1.$$

Thus, if we set $\epsilon = \pi/\omega_j - 1/2 - \epsilon' - \epsilon_j$, we have shown the existence of an $\epsilon > 0$ such that $t - 1 + \lambda \ge 1 + \epsilon$.

Based on the results of Corollary 3.45 we can show that the discrete solution is uniformly bounded in $L^\infty(\Omega)$ independent of the mesh parameter h.

Corollary 3.47. *Let Assumption 3.1 (A1) be satisfied. Then the solution $y_h \in V_h$ of problem (3.89) fulfills for $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$ the estimate*

$$\|y_h\|_{L^\infty(\Omega)} \le c\left(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right)$$

with a constant $c = c(E_\Omega, m, M) > 0$ independent of f, g and α.

Proof. One can estimate

$$\|y_h\|_{L^\infty(\Omega)} \le \|y_h - y\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Omega)}.$$

The assertion is now a consequence of Corollary 3.45, Corollary 3.6 and Remark 3.7. □

Now, we present the main results of this section, the finite element error estimates on the boundary.

Theorem 3.48. *Suppose that Assumption 3.1 (A2) is fulfilled. Let $\varrho \in [0, 1/2]$, $\vec{\mu} \in (\varrho/2, 1]^m$, $\vec{2} - \vec{\lambda} < \vec{\gamma} \le \vec{2} + \vec{\varrho} - 2\vec{\mu}$, $\vec{\gamma} \ge \vec{0}$ and let $\vec{\delta}$ and σ fulfill the conditions stated in Lemma 3.13. Furthermore, let $\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1} - \vec{\mu}$ and $\vec{\beta} \ge \vec{0}$. Then the finite element error on the boundary admits the estimate*

$$\begin{aligned}\|y - y_h\|_{L^2(\Gamma)} &\le ch^2|\ln h|^{1+\varrho}\left(\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}\right)\\ &\le ch^2|\ln h|^{1+\varrho}\left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}\right),\end{aligned}$$

provided that $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega) \cap W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma) \cap W^{1/2,2}_{\vec{\beta}}(\Gamma)$, where $c = c(E_\Omega, m, M)$ is a positive constant independent of f, g and α.

Corollary 3.49. *Let Assumption 3.1 (A2) be satisfied and let* $\vec{\mu} = \vec{1}$ *(quasi-uniform mesh). Furthermore, let* $\varrho \in [0, 1/2]$, $\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\epsilon}$, $\vec{\gamma} \geq \vec{0}$ *with* $\vec{0} < \vec{\epsilon} < \vec{\lambda}$ *and let* $\vec{\delta}$ *and* σ *fulfill the conditions stated in Lemma 3.13. Then for* $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ *there is the estimate*

$$\|y - y_h\|_{L^2(\Gamma)} \leq ch^\rho |\ln h|^{1+\varrho} \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \leq ch^\rho |\ln h|^{1+\varrho} \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \right),$$

where $\rho = \min(2, \min(\vec{\varrho} + \vec{\lambda} - \vec{\epsilon}))$ *and* $c = c(E_\Omega, m, M)$ *is a positive constant independent of* f*,* g *and* α*.*

Let us add some discussion about the previous assertions.

Remark 3.50. Of course, one can show by means of Lemma 2.39 that

$$N^{0,\sigma}_{\vec{\delta}}(\Omega) \hookrightarrow W^{0,2}_{\vec{\beta}}(\Omega) \quad \text{and} \quad N^{1,\sigma}_{\vec{\delta}}(\Gamma) \hookrightarrow W^{1/2,2}_{\vec{\beta}}(\Gamma)$$

if $\vec{\gamma} < \vec{1} + \vec{\beta}$ with $\vec{\gamma} = \vec{\delta} - \vec{\sigma}$. However, for $\vec{\gamma} = 2 + \vec{\varrho} - 2\vec{\mu}$ and $\vec{\beta} = \vec{1} - \vec{\mu}$ this implies $\vec{\mu} > \vec{\varrho}$, which is a more restrictive constraint compared to the condition $\vec{\mu} > \vec{\varrho}/2$ in Theorem 3.48.

Remark 3.51. To get optimal approximation rates in the domain, one only needs a graded mesh with grading parameters $\vec{\mu} < \vec{\lambda}$ if the largest interior angle in the domain is greater than π. However, the stronger condition $\vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$ is required to guarantee a finite element error estimate on the boundary of order $O(h^2 |\ln h|^{3/2})$. This implies that one needs gradually refined meshes for interior angles greater than or equal to $2\pi/3$. Numerical experiments also indicate that this condition is sharp, see Section 3.2.5.

Remark 3.52. Optimal finite element error estimates in the L^2-norm on a strip at the boundary with width h are closely related to the error estimate of Theorem 3.48 and Corollary 3.49. In [83] the authors prove an optimal estimate on a strip for the Dirichlet problem in convex polygonal and polyhedral domains using quasi-uniform meshes. Whereas the general approach in [83] as well as in the present work relies on local finite element error estimates as described in [109, 39], the regularity theory used for the numerical analysis differs fundamentally. In [83] weighted and anisotropic spaces are used, which employ the distance to the boundary. In contrast, our analysis is based on weighted spaces with respect to the corners, which allow the usage of graded meshes with local grading parameters μ_j depending on the interior angles ω_j of each particular corner.

Remark 3.53. In non-convex domains a result, comparable to that of Corollary 3.49, can also be obtained by the Aubin-Nitsche method employing regularity results in classical Sobolev-Slobodetskij spaces, cf. [77].

Before we pay our attention to the proof of Theorem 3.48 and Corollary 3.49, we present a couple of Lipschitz estimates, which are required in Section 4.4.

Lemma 3.54. *Let Assumption 3.1 (A1) be fulfilled and let some arbitrary* $r > 1$ *and* $s > 1$ *be given. Furthermore, let* $y_{1,h} \in V_h$ *and* $y_{2,h} \in V_h$ *be the solutions of* (3.89) *with right hand* *sides $f_1 \in L^r(\Omega)$ and $f_2 \in L^r(\Omega)$ and Neumann boundary data $g_1 \in L^s(\Gamma)$ and $g_2 \in L^s(\Gamma)$, respectively. Then the estimate*

$$\|y_{1,h} - y_{2,h}\|_{L^2(\Omega)} \le c\left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right)$$

holds with a positive constant $c = c(E_\Omega, m, M)$ independent of f_1, f_2, g_1, g_2 and α. Furthermore, we have

$$\|y_{1,h} - y_{2,h}\|_{H^1(\Omega)} \le c\left(\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)}\right)$$

with a positive constant $c = c(E_\Omega, m)$ independent of f_1, f_2, g_1, g_2 and α.

Proof. The proof is similar to that of Lemma 3.16. Let w be the weak solution of

$$\begin{aligned} -\Delta w + \alpha w &= y_{1,h} - y_{2,h} && \text{in } \Omega, \\ \partial_n w &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m. \end{aligned}$$

Moreover, let w_h denote its discrete solution. Now, we can conclude

$$\begin{aligned} \|y_{1,h} - y_{2,h}\|^2_{L^2(\Omega)} &= a(y_{1,h} - y_{2,h}, w_h) = \int_\Omega (f_1 - f_2) w_h + \int_\Gamma (g_1 - g_2) w_h \\ &\le \left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right) \|w_h\|_{L^\infty(\Omega)} \\ &\le c\left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)}\right) \|y_{1,h} - y_{2,h}\|_{L^2(\Omega)}, \end{aligned}$$

where we used the results Corollary 3.47 in the last step. Note that the positive constant $c = c(E_\Omega, m, M)$ is independent of f_1, f_2, g_1, g_2 and α. Dividing by $\|y_{1,h} - y_{2,h}\|_{L^2(\Omega)}$ yields the first assertion. The proof of the second estimate of the assertion is a word by word repetition of the second part of the proof of Lemma 3.16 using the discrete variational formulation (3.89) instead of the continuous one (3.2). ☐

Lemma 3.55. *Let E_Ω be a subset of Ω with $|E_\Omega| > 0$ and let m, M_1, M_2 be constants greater than zero. Furthermore, let the functions $\alpha_i \in L^\infty(\Omega)$, $i \in \{1, 2\}$, fulfill $\alpha_i(x) \ge 0$ for a.a. $x \in \Omega$, $\alpha_i \ge m$ for a.a. $x \in E_\Omega$ and $\|\alpha_i\|_{L^\infty(\Omega)} \le M_i$. Then the solutions of*

$$\int_\Omega (\nabla y_{1,h} \nabla v_h + \alpha_1 y_{1,h} v_h) = \int_\Omega f_1 v_h + \int_\Gamma g_1 v_h \quad \forall v_h \in V_h \tag{3.101}$$

and

$$\int_\Omega (\nabla y_{2,h} \nabla v_h + \alpha_2 y_{2,h} v_h) = \int_\Omega f_2 v_h + \int_\Gamma g_2 v_h \quad \forall v_h \in V_h \tag{3.102}$$

with $f_1, f_2 \in L^r(\Omega)$ and $g_1, g_2 \in L^s(\Gamma)$, $r, s > 1$, satisfy the estimate

$$\begin{aligned} \|y_{1,h} - y_{2,h}\|_{H^1(\Omega)} \le c\Big(&\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \\ &+ \|\alpha_1 - \alpha_2\|_{L^t(\Omega)}\left(\|f_1\|_{L^r(\Omega)} + \|g_1\|_{L^s(\Gamma)}\right)\Big) \end{aligned}$$

with some arbitrary $t > 1$ and the positive constant $c = c(E_\Omega, m)$ is independent of α_1, α_2, f_1, f_2, g_1 and g_2.

Proof. The proof can be done as the proof of Lemma 3.17 using (3.101), (3.102) and Lemma 3.40 instead of (3.53), (3.54) and Lemma 3.4, respectively. ☐

Corollary 3.56. *Let E_Ω be a subset of Ω with $|E_\Omega| > 0$ and let m, M_1, M_2 be constants greater than zero. Furthermore, let the functions $\alpha_i \in L^\infty(\Omega)$, $i \in \{1, 2\}$ satisfy $\alpha_i(x) \geq 0$ for a.a. $x \in \Omega$, $\alpha_i \geq m$ for a.a. $x \in E_\Omega$ and $\|\alpha_i\|_{L^\infty(\Omega)} \leq M_i$. Then the weak solution y_1 of*

$$\begin{aligned} -\Delta y_1 + \alpha_1 y_1 &= f_1 \quad \text{in } \Omega, \\ \partial_n y_1 &= g_1 \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.103}$$

and the discrete solution $y_{2,h}$ of

$$\int_\Omega (\nabla y_{2,h} \nabla v_h + \alpha_2 y_{2,h} v_h) = \int_\Omega f_2 v_h + \int_\Gamma g_2 v_h \quad \forall v_h \in V_h \tag{3.104}$$

with $f_1, f_2 \in L^r(\Omega)$ and $g_1, g_2 \in L^s(\Gamma)$, $r, s > 1$, satisfy the estimate

$$\begin{aligned} \|y_1 - y_{2,h}\|_{L^2(\Omega)} \leq c \Big(& \|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \\ & + \left(h^{1+\epsilon} + \|\alpha_1 - \alpha_2\|_{L^t(\Omega)}\right) \left(\|f_1\|_{L^r(\Omega)} + \|g_1\|_{L^s(\Gamma)}\right)\Big) \end{aligned}$$

with some $\epsilon > 0$, $t > 1$ and a positive constant $c = c(E_\Omega, m, M_1)$ independent of α_1, α_2, f_1, f_2, g_1 and g_2.

Proof. Let us denote by $y_{1,h}$ the discrete solution of (3.103). Then one can apply Theorem 2.7, Corollary 3.45 together with Remark 3.46 and Lemma 3.55 to deduce the desired result, i.e.,

$$\begin{aligned} \|y_1 - y_{2,h}\|_{L^2(\Omega)} &\leq \|y_1 - y_{1,h}\|_{L^2(\Omega)} + \|y_{1,h} - y_{2,h}\|_{L^2(\Omega)} \\ &\leq \|y_1 - y_{1,h}\|_{L^2(\Omega)} + c\|y_{1,h} - y_{2,h}\|_{H^1(\Omega)} \\ &\leq c \Big(h^{1+\epsilon} \left(\|f_1\|_{L^r(\Omega)} + \|g_1\|_{L^s(\Gamma)}\right) + \|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \\ &\quad + \|\alpha_1 - \alpha_2\|_{L^t(\Omega)} \left(\|f_1\|_{L^r(\Omega)} + \|g_1\|_{L^s(\Gamma)}\right)\Big) \end{aligned}$$

with some $\epsilon > 0$, $t > 1$ and a positive constant $c = c(E_\Omega, m, M_1)$ independent of α_1, α_2, f_1, f_2, g_1 and g_2. ☐

3.2.4 Proof of Theorem 3.48 and Corollary 3.49

This section is devoted to the proofs of Theorem 3.48 and Corollary 3.49. The general approach is inspired by [105, 8], where $L^\infty(\Omega)$-finite element error estimates on graded triangulations are proven. In fact, $L^\infty(\Omega)$- and $L^2(\Gamma)$-finite element error estimates are closely related as we will see in the sequel.

Almost in the complete section we assume $\alpha \equiv 1$ with α being the coefficient function in (3.1) and (3.89), respectively. This enables us to ensure that the constants in the estimates do not depend on α. Only in the proofs of Theorem 3.48 and Corollary 3.49 on page 76 ff. we show how the results for $\alpha \equiv 1$ can be extended to a general function α satisfying Assumption 3.1 (A2).

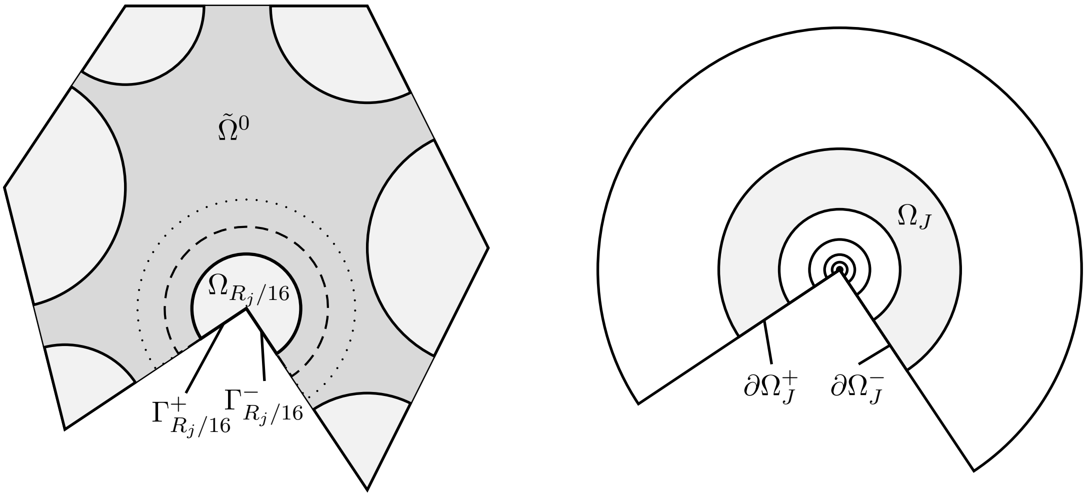

Figure 3.1: Partition of Ω with subdomains $\Omega_{R_j/16}$ (left) and partition of Ω_R with subdomains Ω_J (right)

In Definition 2.19 we have already introduced a very specific partitioning of the domain into circular sectors, which is illustrated exemplarily in Figure 3.1. The dashed and the dotted lines indicate (not to scale) the domains $\Omega_{R_j/8}$ and Ω_{R_j}, respectively. Now we will proceed for every corner in the same way. Let the corner $x^{(j_0)}$ be the corner under consideration. We assume for the sake of simplicity but without loss of generality that the corner $x^{(j_0)}$ is located at the origin and $R_{j_0} = 1$. For the general case we refer to the proof of Lemma 3.9. Furthermore, we suppress the subscript j_0 in the following, i.e., $\Omega_R = \Omega_{R_{j_0}}$, $\Omega_{R/2} = \Omega_{R_{j_0}/2}$, etc. Next, we divide the domain Ω_R into subsets Ω_J,

$$\Omega_R = \bigcup_{J=0}^{I} \Omega_J,$$

where $\Omega_J := \{x : d_{J+1} \le |x| \le d_J\}$ for $J = 0, \dots, I-1$ and $\Omega_I := \{x : |x| \le d_I\}$. The radii d_J are set to 2^{-J} and the index I is chosen such that

$$2^{-(I+k+1)} \le c_2 h^{1/\mu} \le 2^{-(I+k)}$$

for some fixed $k \in \mathbb{N}_0$ and c_2 from (3.70). Thus, $I \sim |\ln h|$ for some $h \le h_0 < 1$. Obviously, there exists some constant $c_I \in \mathbb{R}$ with

$$c_2 2^k \le c_I \le c_2 2^{k+1} \tag{3.105}$$

such that

$$d_I = 2^{-I} = c_I h^{1/\mu}. \tag{3.106}$$

For the moment we only assume that the parameter k is chosen such that $c_I \ge 1$. It will be exactly specified in the proof of Lemma 3.61. The boundary parts of Ω_J which coincide with the boundary of Ω are denoted by $\partial\Omega_J^+$ for $\varphi = \omega$ and by $\partial\Omega_J^-$ for $\varphi = 0$. We set $\partial\Omega_J^\pm = \partial\Omega_J^+ \cup \partial\Omega_J^-$. Figure 3.1 shows such a division. Note that

$$\Omega_{R/2} = \bigcup_{J=1}^{I} \Omega_J,\ \Omega_{R/4} = \bigcup_{J=2}^{I} \Omega_J,\ \Omega_{R/8} = \bigcup_{J=3}^{I} \Omega_J,\ \text{etc.}$$

Moreover, we introduce the extended subsets Ω'_J for $J \geq 1$ and Ω''_J for $J \geq 2$ by

$$\Omega'_J := \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1}$$

and

$$\Omega''_J := \Omega'_{J-1} \cup \Omega'_J \cup \Omega'_{J+1},$$

respectively, with the obvious modifications for $J = I-1, I$. The boundary parts $\partial\Omega_J^{\pm\prime}$ are analogously defined with respect to Ω'_J.

Before going into detail let us elucidate the structure of our proof. As we will see on page 76, $L^2(\Gamma^{\pm}_{R/16})$-discretization error estimates are crucial ingredients of the proof of Theorem 3.48 (and Corollary 3.49). These are established in Lemma 3.64 (and Corollary 3.65). The proof requires $L^\infty(\Omega'_J)$-interpolation error estimates, see Lemma 3.58 and Remark 3.59, the weighted finite element error estimate of Lemma 3.61 (or Corollary 3.62), and some kind of an inverse inequality provided in Lemma 3.63. The proof of Lemma 3.61 (and Corollary 3.62) relies on a kick back argument, which is established by the special partition of the domain Ω_R, the $H^1(\Omega_J)$-interpolation error estimates of Lemma 3.58, and local $H^1(\Omega_J)$-finite element error estimates provided by Lemma 3.60. Lemma 3.58 and Remark 3.59 are also used in the proof of Lemma 3.60. All these arguments rely on the property that the mesh is quasi-uniform in the strips Ω'_J which we are going to prove first.

Lemma 3.57. *The element size h_T of the elements $T \subset \Omega'_J$ satisfies*

$$2^{-2(1-\mu)} c_1 h d_J^{1-\mu} \leq h_T \leq 2^{1-\mu} c_2 h d_J^{1-\mu} \qquad \text{if } 1 \leq J \leq I-2, \tag{3.107}$$

$$c_1 h^{1/\mu} \leq h_T \leq 2^{2(1-\mu)} c_2 h d_I^{1-\mu} = 2^{2(1-\mu)} c_2 c_I^{1-\mu} h^{1/\mu} \qquad \text{if } J = I, I-1 \tag{3.108}$$

with constants c_1 and c_2 from (3.70) *and c_I from* (3.106).

Proof. For any element $T \subset \Omega'_J$ and $J \leq I-2$ one has $d_{J+2} < r_T < d_{J-1}$. Thus, assertion (3.107) follows immediately with $d_{J+2} = 2^{-2} d_J$, $d_{J-1} = 2d_J$ and the mesh condition (3.70). Assertion (3.108) holds analogously since for any element $T \subset \Omega'_J$, $J = I, I-1$, one has $0 \leq r_T \leq d_{I-2} = 2^2 d_I = 2^2 c_I h^{1/\mu}$. □

As indicated above we will use a kick back argument in the proof of Lemma 3.61. This depends on the size of the constant c_I. For that purpose we distinguish between the generic constant c and the constant c_I in the following two lemmas.

Lemma 3.58. *Let $p \in [2,\infty]$ and $l = 0,1$.*

(i) For $1 \leq J \leq I-2$ the estimates

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \leq c h^{2-l} d_J^{(2-l)(1-\mu)+1-2/p-\beta} |v|_{W^{2,p}_\beta(\Omega'_J)}, \tag{3.109}$$

$$\|v - I_h v\|_{L^\infty(\Omega_J)} \leq c h^{2-2/p} d_J^{(2-2/p)(1-\mu)-\beta} |v|_{W^{2,p}_\beta(\Omega'_J)} \tag{3.110}$$

are valid if $v \in W^{2,p}_\beta(\Omega'_J)$ with $\beta \in \mathbb{R}$.

(ii) Let $\theta_l := \max\{0, (3-l-2/p)(1-\mu)-\beta)\}$ *and* $\theta_\infty := \max\{0, (2-2/p)(1-\mu)-\beta\}$. *Then for* $J = I, I-1$ *the inequalities*

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \le c c_I^{\theta_l + 1 - 2/p} h^{(3-l-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)}, \tag{3.111}$$

$$\|v - I_h v\|_{L^\infty(\Omega_J)} \le c c_I^{\theta_\infty} h^{(2-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)} \tag{3.112}$$

hold if $v \in W^{2,p}_\beta(\Omega'_J)$ *with* $2/p - 2 < \beta < 2 - 2/p$.

Proof. We begin with estimating $\|v - I_h v\|_{W^{l,2}(\Omega_J)}$ for $J = 0, \ldots, I$ and $l = 0, 1$. We distinguish between $1 \le J \le I-2$ and $J = I-1, I$. In the former case we get from Corollary 3.33

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \le \left(\sum_{T \subset \Omega'_J} \|v - I_h v\|^2_{W^{l,2}(T)} \right)^{1/2} \le c \left(\sum_{T \subset \Omega'_J} \left(h^{2-l} r_T^{(2-l)(1-\mu)-\beta} |v|_{W^{2,2}_\beta(T)} \right)^2 \right)^{1/2}$$
$$\le c h^{2-l} d_J^{(2-l)(1-\mu)-\beta} |v|_{W^{2,2}_\beta(\Omega'_J)} \le c h^{2-l} d_J^{(2-l)(1-\mu)+1-2/p-\beta} |v|_{W^{2,p}_\beta(\Omega'_J)},$$

where we used $r_T \sim d_J$ (cf. Lemma 3.57), the Hölder inequality with some $p \in [2, \infty]$ and $|\Omega'_J| \sim d_J^2$ in the last steps. In case that $J = I-1, I$ we can conclude using Corollary 3.33 with some $p \in [2, \infty]$ and $2/p - 2 < \beta < 2 - 2/p$

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \le \left(\sum_{\substack{T \subset \Omega'_J \\ r_T = 0}} \|v - I_h v\|^2_{W^{l,2}(T)} + \sum_{\substack{T \subset \Omega'_J \\ r_T > 0}} \|v - I_h v\|^2_{W^{l,2}(T)} \right)^{1/2}$$
$$\le c \left(\sum_{\substack{T \subset \Omega'_J \\ r_T = 0}} \left(h^{(3-l-\beta-2/p)/\mu} |v|_{W^{2,p}_\beta(T)} \right)^2 + \sum_{\substack{T \subset \Omega'_J \\ r_T > 0}} \left(h^{3-l-2/p} r_T^{(3-l-2/p)(1-\mu)-\beta} |v|_{W^{2,p}_\beta(T)} \right)^2 \right)^{1/2}$$
$$\le c \left(\sum_{\substack{T \subset \Omega'_J \\ r_T = 0}} \left(h^{(3-l-\beta-2/p)/\mu} |v|_{W^{2,p}_\beta(T)} \right)^2 + \sum_{\substack{T \subset \Omega'_J \\ r_T > 0}} \left(c_I^{\theta_l} h^{(3-l-\beta-2/p)/\mu} |v|_{W^{2,p}_\beta(T)} \right)^2 \right)^{1/2}$$
$$\le c c_I^{\theta_l} h^{(3-l-\beta-2/p)/\mu} \left(\sum_{T \subset \Omega'_J} |v|^2_{W^{2,p}_\beta(T)} \right)^{1/2} \le c c_I^{\theta_l} h^{(3-l-\beta-2/p)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)} \left(\sum_{T \subset \Omega'_J} 1 \right)^{1/2-1/p}, \tag{3.113}$$

where we used $h^{1/\mu} \le r_T \le c d_I = c c_I h^{1/\mu}$, if $r_T > 0$, and the discrete Hölder inequality. Since $|\Omega'_J| \sim d_I^2$, $d_I = c_I h^{1/\mu}$ and $\min_{T \subset \Omega'_J} h_T \sim h^{1/\mu}$ for $J = I, I-1$ we get that

$$\left(\sum_{T \subset \Omega'_J} 1 \right)^{1/2-1/p} \le \left(\frac{|\Omega'_J|}{\min_{T \subset \Omega'_J} h_T^2} \right)^{1/2-1/p} \le c \left(c_I^2 \right)^{1/2-1/p} = c c_I^{1-2/p}.$$

Thus, we obtain for $p \in [2, \infty]$ and $2/p - 2 < \beta < 2 - 2/p$

$$\|v - I_h v\|_{W^{l,2}(\Omega_J)} \leq c c_I^{\theta_l + 1 - 2/p} h^{(3-l-\beta-2/p)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)}.$$

It remains to prove the L^∞-error estimates. Now we suppose that $v - I_h v$ admits its maximum in Ω_J at some point $x_0 \in \bar{T}_* \subset \Omega'_J$. If $1 \leq J \leq I-2$ we obtain for $p \in (1, \infty]$ using Corollary 3.33 and $r_T \sim d_J$

$$\begin{aligned}\|v - I_h v\|_{L^\infty(\Omega_J)} = \|v - I_h v\|_{L^\infty(T_*)} &\leq c h^{2-2/p} d_J^{(2-2/p)(1-\mu)-\beta} |v|_{W^{2,p}_\beta(T_*)} \\ &\leq c h^{2-2/p} d_J^{(2-2/p)(1-\mu)-\beta} |v|_{W^{2,p}_\beta(\Omega'_J)}.\end{aligned}$$

In case that $J = I-1, I$ we get for $r_{T_*} = 0$ according to Corollary 3.33

$$\|v - I_h v\|_{L^\infty(\Omega_J)} = \|v - I_h v\|_{L^\infty(T_*)} \leq c h^{(2-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(T_*)} \leq c h^{(2-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)},$$

which holds for $p \in (1, \infty]$ and $2/p - 2 < \beta < 2 - 2/p$. In case that $r_{T_*} > 0$ we can conclude analogously to (3.113) using Corollary 3.33

$$\begin{aligned}\|v - I_h v\|_{L^\infty(\Omega_J)} = \|v - I_h v\|_{L^\infty(T_*)} &\leq c h^{2-2/p} r_{T_*}^{(2-2/p)(1-\mu)-\beta} |v|_{W^{2,p}_\beta(T_*)} \\ &\leq c c_I^{\theta_\infty} h^{2-2/p} h^{((2-2/p)(1-\mu)-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)} = c c_I^{\theta_\infty} h^{(2-2/p-\beta)/\mu} |v|_{W^{2,p}_\beta(\Omega'_J)}.\end{aligned}$$

□

Remark 3.59. The inequalities (3.109)–(3.112) hold as well if we replace Ω_J with Ω'_J and Ω'_J with Ω''_J, respectively. In that case we have to distinguish $2 \leq J \leq I-3$ and $J = I-2, I-1, I$.

Lemma 3.60. *Let $\alpha \equiv 1$. The following assertions hold:*

(i) For $2 \leq J \leq I-3$ the estimate

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left(h d_J^{2-\mu-\beta} |y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right)$$

is valid for $y \in W^{2,\infty}_\beta(\Omega_R)$ with $\beta \in \mathbb{R}$.

(ii) For $J \geq I-2$ the inequality

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left(c_I^5 h^{(2-\beta)/\mu} |y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right)$$

holds true for $y \in W^{2,\infty}_\beta(\Omega_R)$ with $-2 < \beta < 2$.

Proof. The proof relies on local finite element error estimates stated in [39] and on the interpolation error estimates given in Lemma 3.58. For $J = 0, \ldots, I$ we get from Theorem 3.4 of [39]

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c \left(\|y - I_h y\|_{H^1(\Omega'_J)} + d_J^{-1} \|y - I_h y\|_{L^2(\Omega'_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right),$$

where the constant c does not depend on c_I. In case that $2 \le J \le I-3$ one gets with Lemma 3.58 and Remark 3.59

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(hd_J^{2-\mu-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + h^2 d_J^{2-2\mu-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right).$$

Since $hd_J^{-\mu} \le hd_I^{-\mu} = c_I^{-\mu} \le 1$ we arrive at

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(hd_J^{2-\mu-\beta}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right).$$

This is the first inequality of the assertion. For $J \ge I-2$ we proceed in an analogous way. But now we use the interpolation error estimates from Lemma 3.58, having regard to Remark 3.59, which are stated there for domains close to or at the corner. Let $\theta_l := \max\{0,(3-l)(1-\mu)-\beta\}$ for $l=0,1$. By this we obtain

$$\begin{aligned}\|y-y_h\|_{H^1(\Omega_J)} &\le c\Big(c_I^{\theta_1+1}h^{(2-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + c_I^{\theta_0+1}d_J^{-1}h^{(3-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)}\\ &\quad + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\Big)\\ &\le c\left((c_I^{\theta_1+1}+c_I^{\theta_0})h^{(2-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)\\ &\le c\left(c_I^5 h^{(2-\beta)/\mu}|y|_{W^{2,\infty}_\beta(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right),\end{aligned}$$

where we used $d_J^{-1}h^{1/\mu} \le d_I^{-1}h^{1/\mu} = c_I^{-1}$, $\theta_1 \le 4$ and $\theta_0 \le 5$ in the last steps. □

Lemma 3.61. *Let* $\alpha \equiv 1$ *and let* $\max(0,1-\lambda) < \tau < 1$, $\gamma \le 3-\tau-2\mu$ *and* $-2<\gamma<2$. *Then for* $y \in W^{2,\infty}_\gamma(\Omega_R)$ *the inequality*

$$\|(r+d_I)^{-\tau}(y-y_h)\|_{L^2(\Omega_{R/8})} \le c\left(h^2|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)}\right)$$

holds.

Proof. We define the weight function $\sigma = r+d_I$ where r denotes the distance to the center of Ω_R. Furthermore, let χ be the characteristic function, which is equal to one in $\Omega_{R/8}$ and equal to zero in $\Omega\setminus \mathrm{cl}(\Omega_{R/8})$. Next, we introduce the boundary value problem

$$\begin{aligned}-\Delta w + w &= \sigma^{-2\tau}(y-y_h)\chi \quad &&\text{in } \Omega,\\ \partial_n w &= 0 &&\text{on } \Gamma_j, \quad j=1,\dots,m,\end{aligned}$$

with its weak formulation

$$a(\varphi,w) = (\sigma^{-2\tau}(y-y_h)\chi,\varphi)_{L^2(\Omega)} \quad \forall\varphi\in H^1(\Omega). \tag{3.114}$$

Since $r^{2\tau}(r+d_I)^{-4\tau} \le r^{-2\tau}$ and $(y-y_h)\in H^1(\Omega)$ we can conclude using Lemma 2.28

$$\|\sigma^{-2\tau}(y-y_h)\chi\|_{W^{0,2}_\tau(\Omega_R)} \le \|\sigma^{-2\tau}(y-y_h)\|_{W^{0,2}_\tau(\Omega_R)} \le \|y-y_h\|_{W^{0,2}_{-\tau}(\Omega_R)} \le \|y-y_h\|_{H^1(\Omega_R)}$$

or more precisely $\sigma^{-2\tau}(y-y_h)\chi \in W^{0,2}_\tau(\Omega_R)$. Thus, we get according to Lemma 3.11 that the solution w belongs to $W^{2,2}_\tau(\Omega_R)$ for any τ satisfying $\max(0,1-\lambda) < \tau < 1$. Moreover, if we use the inequality $r < r + d_I$ we obtain the validity of the a priori estimate

$$\|w\|_{W^{2,2}_\tau(\Omega_R)} \le c\|\sigma^{-2\tau}\,(y-y_h)\,\|_{W^{0,2}_\tau(\Omega_{R/8})} \le c\|\sigma^{-\tau}\,(y-y_h)\,\|_{L^2(\Omega_{R/8})}. \tag{3.115}$$

Using Lemma 2.28 we can also show that

$$\|w\|_{H^1(\Omega_R)} = \|w\|_{W^{1,2}_0(\Omega_R)} \le c\|w\|_{W^{2,2}_\tau(\Omega_R)} \le c\|\sigma^{-\tau}\,(y-y_h)\,\|_{L^2(\Omega_{R/8})}. \tag{3.116}$$

Now, let η be an infinitely differentiable function in $\bar{\Omega}$, which is equal to one in $\Omega_{R/8}$, $\operatorname{supp}\eta \subset \Omega_{R/4}$ and $\partial_n\eta = 0$ on $\partial\Omega_R$ with $\|\eta\|_{W^{k,\infty}(\Omega)} \le c$ for $k \in \mathbb{N}_0$. By setting $\varphi = \eta v$ in (3.114) with some $v \in H^1(\Omega)$ one can show that $\tilde{w} = \eta w$ fulfills the equation

$$a_{\Omega_R}(v,\tilde{w}) = (\eta\sigma^{-2\tau}\,(y-y_h)\,\chi - \Delta\eta w - 2\nabla\eta\cdot\nabla w, v)_{L^2(\Omega_R)} \quad \forall v \in H^1(\Omega),$$

where the bilinear form $a_{\Omega_R} : H^1(\Omega_R)\times H^1(\Omega_R) \to \mathbb{R}$ is defined by

$$a_{\Omega_R}(\varphi,w) := \int_{\Omega_R} (\nabla\varphi\cdot\nabla w + \varphi w)\,.$$

By this we get

$$\begin{aligned}
\|\sigma^{-\tau}\,(y-y_h)\|^2_{L^2(\Omega_{R/8})} &= (\eta\sigma^{-2\tau}(y-y_h)\chi, y-y_h)_{L^2(\Omega_R)} \\
&= a_{\Omega_R}(y-y_h,\tilde{w}) + (\Delta\eta w, y-y_h)_{L^2(\Omega_R)} + 2(\nabla\eta\cdot\nabla w, y-y_h)_{L^2(\Omega_R)} \\
&\le a_{\Omega_R}(y-y_h,\tilde{w}) + \Big(\|\Delta\eta w\|_{L^2(\Omega_R)} + 2\|\nabla\eta\cdot\nabla w\|_{L^2(\Omega_R)}\Big)\,\|y-y_h\|_{L^2(\Omega_R)} \\
&\le a_{\Omega_R}(y-y_h,\tilde{w}) + c\|w\|_{H^1(\Omega_R)}\|y-y_h\|_{L^2(\Omega_R)} \\
&\le a_{\Omega_R}(y-y_h,\tilde{w}) + c\|\sigma^{-\tau}\,(y-y_h)\,\|_{L^2(\Omega_{R/8})}\|y-y_h\|_{L^2(\Omega_R)},
\end{aligned} \tag{3.117}$$

where we used the Cauchy-Schwarz inequality and (3.116) in the last steps. It remains to estimate the first term in (3.117). Since $\tilde{w}$ is equal to zero in $\Omega_R\setminus \operatorname{cl}(\Omega_{R/4})$ we can use the Galerkin orthogonality of $y-y_h$, i.e., $a_{\Omega_R}(y-y_h, I_h\tilde{w}) = a(y-y_h, I_h\tilde{w}) = 0$. This yields together with an application of the Cauchy-Schwarz inequality

$$a_{\Omega_R}(y-y_h,\tilde{w}) = a_{\Omega_R}(y-y_h,\tilde{w}-I_h\tilde{w}) \le c\sum_{J=2}^{I} \|y-y_h\|_{H^1(\Omega_J)}\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)}. \tag{3.118}$$

Remember that $\tilde{w}-I_h\tilde{w} \equiv 0$ in Ω_J for $J=0,1$. Now each term on the right hand side of (3.118) is estimated separately. We distinguish between $2 \le J \le I-3$ and $J = I, I-1, I-2$ as it has already been done in the previous lemmas. We get for $2 \le J \le I-3$ with Lemma 3.60

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(hd_J^{2-\mu-\gamma}|y|_{W^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)$$

and with Lemma 3.58

$$\|\tilde{w}-I_h\tilde{w}\|_{H^1(\Omega_J)} \le chd_J^{1-\tau-\mu}|\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)}.$$

By means of these two estimates one can conclude for $2 \leq J \leq I-3$

$$\begin{aligned}
\|y - y_h\|_{H^1(\Omega_J)} \|\tilde{w} - I_h\tilde{w}\|_{H^1(\Omega_J)} \\
&\leq c\left(h^2 d_J^{3-\tau-2\mu-\gamma} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + h d_J^{-\mu} \|d_J^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)} \\
&\leq c\left(h^2 d_J^{3-\tau-2\mu-\gamma} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{-\mu} \|d_J^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)}, \qquad (3.119)
\end{aligned}$$

where we used $h d_J^{-\mu} \leq h d_I^{-\mu} = c_I^{-\mu}$. For $J = I, I-1, I-2$ we get from Lemma 3.60 for $-2 < \gamma < 2$

$$\|y - y_h\|_{H^1(\Omega_J)} \leq c\left(c_I^5 h^{(2-\gamma)/\mu} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y - y_h\|_{L^2(\Omega'_J)}\right)$$

and from Lemma 3.58

$$\|\tilde{w} - I_h\tilde{w}\|_{H^1(\Omega_J)} \leq c c_I^{\max\{0,1-\tau-\mu\}} h^{(1-\tau)/\mu} |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)}.$$

We can combine the last two estimates to arrive at

$$\begin{aligned}
&\|y - y_h\|_{H^1(\Omega_J)} \|\tilde{w} - I_h\tilde{w}\|_{H^1(\Omega_J)} \\
&\leq c\left(c_I^6 h^{(3-\tau-\gamma)/\mu} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{\max\{0,1-\tau-\mu\}} (h^{1/\mu} d_J^{-1})^{1-\tau} \|d_J^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)} \\
&\leq c\left(c_I^6 h^{(3-\tau-\gamma)/\mu} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{\max\{-1+\tau,-\mu\}} \|d_J^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)}, \qquad (3.120)
\end{aligned}$$

where we used $\max\{0, 1-\tau-\mu\} < 1$ and $h^{1/\mu} d_J^{-1} \leq h^{1/\mu} d_I^{-1} = c_I^{-1}$. Let $\theta := \max\{-1+\tau, -\mu\}$. Inserting the inequalities (3.119) and (3.120) into (3.118) yields

$$\begin{aligned}
&a_{\Omega_R}(y - y_h, \tilde{w}) \\
&\leq c \sum_{J=2}^{I-3} \left(h^2 d_J^{3-\tau-2\mu-\gamma} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^{-\mu} \|d_J^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)} \\
&+ c \sum_{J=I-2}^{I} \left(c_I^6 h^{(3-\tau-\gamma)/\mu} |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^\theta \|d_J^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)}. \qquad (3.121)
\end{aligned}$$

If we additionally set $\gamma \leq 3 - \tau - 2\mu$ we can conclude using $c_I^{-\mu} < c_I^\theta$ $(c_I \geq 1)$ and $d_J^{-1} \leq c\sigma^{-1}$

$$a_{\Omega_R}(y - y_h, \tilde{w}) \leq c \sum_{J=2}^{I} \left(c_I^6 h^2 |y|_{W^{2,\infty}_\gamma(\Omega''_J)} + c_I^\theta \|\sigma^{-\tau}(y - y_h)\|_{L^2(\Omega'_J)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega'_J)}. \qquad (3.122)$$

Now we get with $\sum_{J=2}^{I} 1 \sim |\ln h|$ and the discrete Cauchy-Schwarz inequality

$$a_{\Omega_R}(y - y_h, \tilde{w}) \leq c\left(c_I^6 h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^\theta \|\sigma^{-\tau}(y - y_h)\|_{L^2(\Omega_R)}\right) |\tilde{w}|_{W^{2,2}_\tau(\Omega_R)}.$$

Since $|\tilde{w}|_{W^{2,2}_\tau(\Omega_R)} \leq c\|w\|_{W^{2,2}_\tau(\Omega_R)}$ we can apply the a priori estimate (3.115), which yields

$$\begin{aligned}
&a_{\Omega_R}(y - y_h, \tilde{w}) \\
&\leq c\left(c_I^6 h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^\theta \|\sigma^{-\tau}(y - y_h)\|_{L^2(\Omega_R)}\right) \|\sigma^{-\tau}(y - y_h)\|_{L^2(\Omega_{R/8})}. \qquad (3.123)
\end{aligned}$$

By inserting (3.123) into (3.117) and dividing by $\|\sigma^{-\tau}(y-y_h)\|_{L^2(\Omega_{R/8})}$ we obtain

$$\begin{aligned}
&\|\sigma^{-\tau}(y-y_h)\|_{L^2(\Omega_{R/8})} \\
&\le c\left(c_I^6 h^2|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^\theta\|\sigma^{-\tau}(y-y_h)\|_{L^2(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)}\right) \\
&\le c\left(c_I^6 h^2|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^\theta\|\sigma^{-\tau}(y-y_h)\|_{L^2(\Omega_{R/8})} + c_I^\theta\|y-y_h\|_{L^2(\Omega_R)}\right),
\end{aligned}$$

where we used $\sigma^{-\tau}=(r+d_I)^{-\tau}\le r^{-\tau}\le (R/8)^{-\tau}\le c$ if $r\ge R/8$. Finally, we get

$$\left(1-cc_I^\theta\right)\|\sigma^{-\tau}(y-y_h)\|_{L^2(\Omega_{R/8})} \le c\left(c_I^6h^2|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + c_I^\theta\|y-y_h\|_{L^2(\Omega_R)}\right).$$

If one has chosen the parameter k in (3.105) large enough such that

$$cc_I^\theta = cc_I^{\max\{-1+\tau,-\mu\}} \le c\left(c_2 2^k\right)^{\max\{-1+\tau,-\mu\}} < 1,$$

then the desired result follows. □

Corollary 3.62. *Let $\alpha\equiv 1$ and let $\max(0,1-\lambda)<\tau<1$, $-2<\gamma<2$ and $\mu=1$ (quasi-uniform mesh). Then for $y\in W^{2,\infty}_\gamma(\Omega_R)$ the inequality*

$$\|(r+d_I)^{-\tau}(y-y_h)\|_{L^2(\Omega_{R/8})} \le c\left(h^{\min(2,3-\tau-\gamma)}|\ln h|^{1/2}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)}\right)$$

is valid.

Proof. The assertion can be proven analogously to Lemma 3.61, but now we drop the assumption $\gamma\le 3-\tau-2\mu$ and set $\mu=1$. From (3.121) and (3.106) we deduce

$$a_{\Omega_R}(y-y_h,\tilde w) \le c\sum_{J=2}^{I}\left(c_I^6 h^{\min(2,3-\tau-\gamma)}|y|_{W^{2,\infty}_\gamma(\Omega_J'')} + c_I^\theta\|\sigma^{-\tau}(y-y_h)\|_{L^2(\Omega_J')}\right)|\tilde w|_{W^{2,2}_\tau(\Omega_J')}$$

instead of (3.122). All other steps remain unchanged. □

In the remainder of this section the constant c_I is hidden in the generic constant c.

Lemma 3.63. *For $v_h\in V_h$ and $1\le p\le\infty$ there exists a constant $c>0$ such that*

$$\begin{aligned}
\|v_h\|_{L^p(\partial\Omega_J^\pm)} &\le ch^{-1/p}d_J^{-(1-\mu)/p}\|v_h\|_{L^p(\Omega_J')} && \text{for } 1\le J\le I-2,\\
\|v_h\|_{L^p(\partial\Omega_J^\pm)} &\le ch^{-1/(p\mu)}\|v_h\|_{L^p(\Omega_J')} && \text{for } J=I-1,I.
\end{aligned}$$

Proof. Let $E\in\mathcal{E}_h$ with $E\subset\partial\Omega_J^{\pm\prime}$ and let $T\subset\Omega_J'$ be the corresponding triangle. By an affine change of variables to the reference edge $\hat E$ and reference triangle $\hat T$, respectively, using the continuity of $\hat v_h$ on $\mathrm{cl}(\hat T)$ and the norm equivalence in finite dimensional spaces we obtain

$$\begin{aligned}
\|v_h\|_{L^p(E)} &\le ch_T^{1/p}\|\hat v_h\|_{L^p(\hat E)} \le ch_T^{1/p}\|\hat v_h\|_{L^\infty(\hat E)} \le ch_T^{1/p}\|\hat v_h\|_{L^\infty(\hat T)}\\
&\le ch_T^{1/p}\|\hat v_h\|_{L^p(\hat T)} \le ch_T^{-1/p}\|v_h\|_{L^p(T)}.
\end{aligned}$$

Now we can sum up to get

$$\|v_h\|^p_{L^p(\partial\Omega^\pm_J)} \le \sum_{E\subset\partial\Omega^{\pm\prime}_J} \|v_h\|^p_{L^p(E)} \le c \sum_{T\subset\Omega'_J} \left(h_T^{-1}\|v_h\|^p_{L^p(T)}\right) \le c \min_{T\subset\Omega'_J} h_T^{-1} \sum_{T\subset\Omega'_J} \|v_h\|^p_{L^p(T)}.$$

One can conclude the desired result with Lemma 3.57. □

Lemma 3.64. *Let $\alpha \equiv 1$ and let $0 \le \varrho \le 1/2$, $\gamma \le 2 + \varrho - 2\mu$ and $-2 < \gamma < 2$. Then for $y \in W^{2,\infty}_\gamma(\Omega_R)$ the estimate*

$$\|y - y_h\|_{L^2(\Gamma^\pm_{R/16})} \le c\left(h^2|\ln h|^{1+\varrho}|y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y-y_h\|_{L^2(\Omega_R)}\right)$$

is valid.

Proof. Note that $\Gamma^\pm_{R/16} = \bigcup_{J=4}^I \partial\Omega^\pm_J$. It holds for $J = I-1, I$

$$\begin{aligned}\|y-y_h\|_{L^2(\partial\Omega^\pm_J)} &\le \|y - I_hy\|_{L^2(\partial\Omega^\pm_J)} + \|I_hy - y_h\|_{L^2(\partial\Omega^\pm_J)}\\ &\le cd_J^{1/2}\|y-I_hy\|_{L^\infty(\partial\Omega^\pm_J)} + \|I_hy-y_h\|_{L^2(\partial\Omega^\pm_J)},\end{aligned}$$

where we have used $|\partial\Omega^\pm_J| \sim d_J$. The continuity of $y - I_hy$ on $\mathrm{cl}(\Omega_J)$ and Lemma 3.63 with $p = 2$ yields

$$\|y-y_h\|_{L^2(\partial\Omega^\pm_J)} \le cd_J^{1/2}\|y-I_hy\|_{L^\infty(\Omega_J)} + ch^{-1/(2\mu)}\|I_hy-y_h\|_{L^2(\Omega'_J)}.$$

Since $d_J \sim h^{1/\mu}$ for $J = I-1, I$ and $|\Omega'_J| \sim d_J^2$ we can proceed with

$$\begin{aligned}\|y-y_h\|_{L^2(\partial\Omega^\pm_J)} &\le cd_J^{1/2}\|y-I_hy\|_{L^\infty(\Omega_J)} + cd_J^{-1/2}\|y-I_hy\|_{L^2(\Omega'_J)} + cd_J^{-1/2}\|y-y_h\|_{L^2(\Omega'_J)}\\ &\le cd_J^{1/2}\|y-I_hy\|_{L^\infty(\Omega'_J)} + cd_J^{-1/2}\|y-y_h\|_{L^2(\Omega'_J)}.\end{aligned} \tag{3.124}$$

Next we consider the case $4 \le J \le I-2$. Again we use $|\partial\Omega^\pm_J| \sim d_J$ and the continuity of $y - I_hy$ on $\mathrm{cl}(\Omega_J)$. Thus we can write

$$\|y-y_h\|_{L^2(\partial\Omega^\pm_J)} \le cd_J^{1/2}\|y-y_h\|_{L^\infty(\partial\Omega^\pm_J)} \le cd_J^{1/2}\|y-y_h\|_{L^\infty(\Omega_J)}.$$

Since each subdomain Ω'_J has a positive distance to the corner for $4 \le J \le I-2$, we can use Theorem 10.1 in [109] with $s = 0$ (or Corollary 5.1 of [103]) to get

$$\|y-y_h\|_{L^2(\partial\Omega^\pm_J)} \le cd_J^{1/2}|\ln h|\|y-I_hy\|_{L^\infty(\Omega'_J)} + cd_J^{-1/2}\|y-y_h\|_{L^2(\Omega'_J)}. \tag{3.125}$$

Actually, the estimate of Theorem 10.1 of [109] was proven for interior domains, but in Example 10.1 of [109] the author showed that this result is also applicable for the domains Ω'_J, i.e., for domains which abut on the boundary but contain no corner point. Using (3.124) and (3.125) we arrive at

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R/16})} = \left(\sum_{J=4}^{I} \|y - y_h\|^2_{L^2(\partial\Omega^{\pm}_J)} \right)^{1/2}$$
$$\le c \left(\sum_{J=4}^{I} \left(d_J^{1/2} |\ln h| \|y - I_h y\|_{L^\infty(\Omega'_J)} + d_J^{-1/2} \|y - y_h\|_{L^2(\Omega'_J)} \right)^2 \right)^{1/2}$$
$$\le c|\ln h| \max_{4\le J\le I} \left(d_J^{\varrho} \|y - I_h y\|_{L^\infty(\Omega'_J)} \right) \left(\sum_{J=4}^{I} d_J^{1-2\varrho} \right)^{1/2} + c \left(\sum_{J=4}^{I} \|d_J^{-1/2} (y - y_h)\|^2_{L^2(\Omega'_J)} \right)^{1/2}.$$

An application of the discrete Hölder inequality yields

$$\left(\sum_{J=4}^{I} d_J^{1-2\varrho} \right)^{1/2} \le \left(\sum_{J=4}^{I} d_J \right)^{(1-2\varrho)/2} \left(\sum_{J=4}^{I} 1 \right)^{\varrho} \le c|\ln h|^{\varrho},$$

where we have used $\sum_{J=4}^{I} d_J \sim |\Gamma^{\pm}_{R/16}|$ and $\sum_{J=4}^{I} 1 \sim |\ln h|$ in the last step. Thus, we obtain

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R/16})} \le c|\ln h|^{1+\varrho} \max_{4\le J\le I} \left(d_J^{\varrho} \|y - I_h y\|_{L^\infty(\Omega'_J)} \right) + c\|(r + d_I)^{-1/2}(y - y_h)\|_{L^2(\Omega_{R/8})}. \tag{3.126}$$

Finally, we get with Lemma 3.58, Lemma 3.57 and Lemma 3.61

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R/16})} \le ch^2 |\ln h|^{1+\varrho} \max_{4\le J\le I} |y|_{W^{2,\infty}_\gamma(\Omega''_J)}$$
$$+ c \left(h^2 |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y - y_h\|_{L^2(\Omega_R)} \right), \tag{3.127}$$

since $\gamma \le 2 + \varrho - 2\mu \le 5/2 - 2\mu$ for $\varrho \in [0, 1/2]$ and $1 - \lambda < 1/2$ for $\omega \in (0, 2\pi)$. □

Corollary 3.65. *Let $\alpha \equiv 1$ and let $0 \le \varrho \le 1/2$, $-2 < \gamma < 2$ and $\mu = 1$ (quasi-uniform mesh). Then for $y \in W^{2,\infty}_\gamma(\Omega_R)$ the estimate*

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R/16})} \le c \left(h^{\min(2,2+\varrho-\gamma)} |\ln h|^{1+\varrho} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y - y_h\|_{L^2(\Omega_R)} \right)$$

is valid.

Proof. The proof follows the same steps as the proof of Lemma 3.64. Only, the step after (3.126) has to be adjusted. Instead of (3.127), we can deduce from Lemma 3.58 and Corollary 3.62

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R/16})} \le ch^{\min(2,2+\varrho-\gamma)} |\ln h|^{1+\varrho} \max_{4\le J\le I} |y|_{W^{2,\infty}_\gamma(\Omega''_J)}$$
$$+ c \left(h^{\min(2,5/2-\gamma)} |\ln h|^{1/2} |y|_{W^{2,\infty}_\gamma(\Omega_R)} + \|y - y_h\|_{L^2(\Omega_R)} \right),$$

which is the desired result. □

Now we are able to prove Theorem 3.48.

Proof of Theorem 3.48. As already indicated we will first show the result of Theorem 3.48 for $\alpha \equiv 1$ and afterwards we will extend it to a general function α satisfying Assumption 3.1 (A2).

Now, let us start with the case $\alpha \equiv 1$. We split the error on the boundary into the already introduced boundary parts,

$$\|y - y_h\|_{L^2(\Gamma)} \le c \left(\sum_{j=1}^{m} \|y - y_h\|_{L^2(\Gamma^{\pm}_{R_j/16})} + \|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \right). \tag{3.128}$$

For each boundary part $\Gamma^{\pm}_{R_j/16}$, $j = 1, \ldots, m$, we get from Lemma 3.64

$$\|y - y_h\|_{L^2(\Gamma^{\pm}_{R_j/16})} \le c \left(h^2 |\ln h|^{1+\varrho} \|y\|_{W^{2,\infty}_{\gamma_j}(\Omega_{R_j})} + \|y - y_h\|_{L^2(\Omega_{R_j})} \right), \tag{3.129}$$

provided that $0 \le \varrho \le 1/2$, $\gamma_j \le 2 + \varrho - 2\mu_j$ and $-2 < \gamma_j < 2$. If we set $\mu_j > \varrho/2$ we get that (3.129) is valid for $-2 < \gamma_j \le 2 + \varrho - 2\mu_j$ with some arbitrary $\varrho \in [0, 1/2]$. Next, we estimate the last term on the right hand side of (3.128). We can conclude from the embedding $L^\infty(\tilde{\Gamma}^0) \hookrightarrow L^2(\tilde{\Gamma}^0)$ and the fact that $y - y_h$ is a continuous function on $\mathrm{cl}(\tilde{\Omega}^0)$

$$\|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \le c \|y - y_h\|_{L^\infty(\tilde{\Gamma}^0)} \le c \|y - y_h\|_{L^\infty(\tilde{\Omega}^0)}.$$

Next we use Theorem 10.1 in [109] with $s = 0$ to get

$$\|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \le c \left(|\ln h| \|y - I_h y\|_{L^\infty(\check{\Omega}^0)} + \|y - y_h\|_{L^2(\check{\Omega}^0)} \right).$$

Compare the proof of Lemma 3.64 for the applicability of this theorem in that case. Since the domain $\check{\Omega}^0 \subset \Omega^0$ has a constant, positive distance to the corner, we can conclude using Corollary 3.33

$$\|y - y_h\|_{L^2(\tilde{\Gamma}^0)} \le c \left(h^2 |\ln h| \|y\|_{W^{2,\infty}(\Omega^0)} + \|y - y_h\|_{L^2(\check{\Omega}^0)} \right). \tag{3.130}$$

Combining the inequalities (3.128), (3.129) and (3.130) we obtain for $-2 < \gamma_j \le 2 + \varrho - 2\mu_j$ with $\varrho \in [0, 1/2]$ and $\mu_j \in (\varrho/2, 1]$ that

$$\|y - y_h\|_{L^2(\Gamma)} \le c \left(h^2 |\ln h|^{1+\varrho} \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y - y_h\|_{L^2(\Omega)} \right). \tag{3.131}$$

Using Lemma 3.13 we conclude for $2 - \lambda_j < \gamma_j < 2$ and $\gamma_j \ge 0$ that

$$\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \le c \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \right) \tag{3.132}$$

with $\delta_j = \gamma_j + \sigma$ and $\sigma \in (0, 1)$. For $1 - \lambda_j < \beta_j \le 1 - \mu_j$ and $\beta_j \ge 0$ Lemma 3.41 implies

$$\|y - y_h\|_{L^2(\Omega)} \le c h^2 \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le c h^2 \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right). \tag{3.133}$$

Finally, inequalities (3.131), (3.132) and (3.133) yield the desired result for $\alpha \equiv 1$.

Next, using the results proven so far for $\alpha \equiv 1$, we are going to prove the assertion for a general function α which fulfills Assumption 3.1 (A2). First, we introduce the bilinear form $\tilde{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$\tilde{a}(y, v) := \int_\Omega (\nabla y \cdot \nabla v + yv)$$

and the linear form $F : H^1(\Omega) \to \mathbb{R}$ by

$$F(v) := \int_\Omega fv + \int_\Gamma gv.$$

Note that the solution $y \in H^1(\Omega)$ of (3.1) and $y_h \in V_h$ of (3.89) also satisfy

$$\tilde{a}(y, v) = F(v) + \int_\Omega (1 - \alpha)yv \quad \forall v \in H^1(\Omega)$$

and

$$\tilde{a}(y_h, v_h) = F(v_h) + \int_\Omega (1 - \alpha)y_h v_h \quad \forall v_h \in V_h,$$

respectively. Next, we introduce some kind of semi-discretization of y, which is the unique element $\tilde{y}_h \in V_h$ that satisfies

$$\tilde{a}(\tilde{y}_h, v_h) = F(v_h) + \int_\Omega (1 - \alpha)yv_h \quad \forall v_h \in V_h. \tag{3.134}$$

Inserting $\tilde{y}_h$ as an intermediate function into the finite element error on the boundary yields

$$\|y - y_h\|_{L^2(\Gamma)} \le \|y - \tilde{y}_h\|_{L^2(\Gamma)} + \|\tilde{y}_h - y_h\|_{L^2(\Gamma)}. \tag{3.135}$$

For the first term in (3.135) we can employ the estimate proven above for $\alpha \equiv 1$. Let the parameters $\vec{\gamma}$ and $\vec{\beta}$ be chosen as before. Then we obtain

$$\|y - \tilde{y}_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{1+\varrho} \left(\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \right) \tag{3.136}$$

with a constant $c > 0$ independent of f, g, α, m and M. By means of Lemma 3.11 and Lemma 3.13 we can conclude

$$\|y - \tilde{y}_h\|_{L^2(\Gamma)} \le ch^2 |\ln h|^{1+\varrho} \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} + \|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right), \tag{3.137}$$

where the positive constant $c = c(E_\Omega, m, M)$ is independent of f, g and α. For the last term of (3.135) we first apply Theorem 2.8 to get

$$\|\tilde{y}_h - y_h\|_{L^2(\Gamma)} \le c\|\tilde{y}_h - y_h\|_{H^1(\Omega)}. \tag{3.138}$$

Next, we use Lemma 3.54. This yields

$$\|\tilde{y}_h - y_h\|_{H^1(\Omega)} \le c\|(1 - \alpha)(y - y_h)\|_{L^2(\Omega)} \le c\|y - y_h\|_{L^2(\Omega)} \tag{3.139}$$

with a constant $c = c(M)$. We continue with Lemma 3.41

$$\|y - y_h\|_{L^2(\Omega)} \le ch^2\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le ch^2 \left(\|f\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right), \tag{3.140}$$

where the positive constant $c = c(E_\Omega, m, M)$ does not depend on f, g and α. The estimates (3.135), (3.137), (3.138), (3.139) and (3.140) yield the assertion for a general function α satisfying Assumption 3.1 (A2). □

Proof of Corollary 3.49. The proof is a word by word repetition of the proof of Theorem 3.48 using Corollary 3.65 and Corollary 3.42 instead of Lemma 3.64 and Lemma 3.41, respectively. Let us point out the differences in detail. In the first part of the proof, where we assumed $\alpha \equiv 1$, we get from Corollary 3.65

$$\|y - y_h\|_{L^2(\Gamma^\pm_{R_j/16})} \le c \left(h^{\min(2,2+\varrho-\gamma_j)} |\ln h|^{1+\varrho} |y|_{W^{2,\infty}_{\gamma_j}(\Omega_{R_j})} + \|y - y_h\|_{L^2(\Omega_{R_j})} \right)$$

with $0 \le \varrho \le 1/2$ and $-2 < \gamma_j < 2$ instead of (3.129) and consequently

$$\|y - y_h\|_{L^2(\Gamma)} \le c \left(h^{\min(2,\min(\vec{2}+\vec{\varrho}-\vec{\gamma}))} |\ln h|^{1+\varrho} \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y - y_h\|_{L^2(\Omega)} \right)$$

instead of (3.131). Moreover, using Lemma 3.13, we replace (3.132) by

$$\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \le c \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \right) \tag{3.141}$$

with $\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\epsilon}$, $\vec{\gamma} \ge \vec{0}$, some $\vec{0} < \vec{\epsilon} < \vec{\lambda}$, $\delta_j = \gamma_j + \sigma$ and some $\sigma \in (0,1)$. Next, let $\vec{\beta} = \vec{1} - \vec{\lambda} + \vec{\epsilon}_\lambda$ with some $\vec{\epsilon} < \vec{\epsilon}_\lambda < \vec{\lambda}$, $\vec{\beta} \ge \vec{0}$ and $\lambda = \min(1, \min(\vec{\lambda} - \vec{\epsilon}_\lambda))$. Then one obtains from Corollary 3.42

$$\|y - y_h\|_{L^2(\Omega)} \le ch^{2\lambda} \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le ch^{2\lambda} \|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \le ch^{2\lambda} \left(\|f\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)} + \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)} \right)$$

instead of (3.133), where we applied Lemma 2.29 and (3.141) in the last steps. The proof for a general function α, which satisfies Assumption 3.1 (A2), proceeds as before using the previous results. □

3.2.5 Numerical example

This section is devoted to the numerical verification of the theoretical convergence results of Section 3.2.3. To this aim we present two numerical examples. In both examples the computational domain Ω_ω depending on an interior angle $\omega \in (0, 2\pi)$ is defined by

$$\Omega_\omega := (-1,1)^2 \cap \{x \in \mathbb{R}^2 : (r(x), \varphi(x)) \in (0, \sqrt{2}] \times [0, \omega]\}, \tag{3.142}$$

where r and φ stand for the polar coordinates located at the origin. The boundary of Ω_ω is denoted by Γ_ω which is decomposed into straight line segments Γ_j, $j = 1, \ldots, m(\omega)$, counting counterclockwise beginning at the origin. We solve the problem

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega_\omega, \\ \partial_n y &= g \quad \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned} \tag{3.143}$$

numerically by using a finite element method with piecewise linear finite elements. Thus, we compute the solution $\vec{y} = (y_1, \ldots, y_N)^T \in \mathbb{R}^N$, $N = \#I_X$, of the linear systems of equations

$$\sum_{k \in I_X} y_k \int_{\Omega_\omega} (\nabla \phi_k \cdot \nabla \phi_i + \phi_k \phi_i) = \int_{\Omega_\omega} f \phi_i + \int_{\Gamma_\omega} g \phi_i, \quad \forall i \in I_X, \tag{3.144}$$

where the implementation is realized in a Matlab-code similar to that in [2]. In the first example the data f and g are set such that the exact solution is known whereas in the second one the data is arbitrary and we take for the purpose of comparison a reference solution which is computed on a very fine mesh. But note that in both examples the data are chosen such that the regularity of the solution is dominated by the influence of the corner singularities. Then we know according to the results of Section 3.2.3 that there are border line angles for the error in $L^2(\Omega)$, $H^1(\Omega)$ and $L^2(\Gamma)$, such that the convergence rates in the corresponding norms decreases if the interior angle increases. We will illustrate this effect by varying the interior angle ω, i.e., we will choose $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$. Furthermore, we will use meshes with differently strong grading to illustrate how one can use such meshes to compensate the lower convergence rates in the different norms. Here the question arises in which way one can generate meshes satisfying the mesh condition (3.70). In the first example we generate quasi-uniform meshes by a uniform refinement of a coarse quasi-uniform start mesh as described in [33]. Afterwards, depending on the grading parameter μ we transform the mesh by moving all nodes $X^{(i)}$ within a circular sector S_R with radius R around the origin according to

$$X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1} \qquad \forall X^{(i)} \in \Omega_\omega \cap S_R.$$

By this we obtain a graded mesh which satisfies the mesh grading condition (3.70), cf. [4, Section 4.2.2]. Note that this procedure trivially preserves the complexity of the initial quasi-uniform mesh. In the second example a newest vertex bisection algorithm as described in [33] is applied to a coarse start mesh in order to construct a mesh which fulfills the mesh grading condition (3.70). Within this algorithm we mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

$$h_T > h \quad \text{or} \quad h_T > h \left(\frac{r_{T,C}}{R} \right)^{1-\mu}$$

until the desired mesh size h is reached, where $r_{T,C}$ denotes the distance between the origin and the centroid of the triangle T. Of course, this procedure adds elements in order to generate graded meshes. This is different to the first one. But, one can show that the complexities of the resulting meshes are still of order $O(h^{-2})$, cf. [10, Remark 3.1]. Furthermore, the algorithm is implemented such that it produces nested meshes if we decrease the mesh size h or the grading parameter μ. This simplifies the calculation of the error in case of using a reference solution on a finer mesh. Exemplarily for $\omega = 3\pi/2$, $\mu = 0.5$ and $R = 0.4$ one can find such meshes in Figure 3.2.

Now, let us present the numerical examples.

Example 3.66. The data f and g are chosen in the following way

$$\begin{aligned} f &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g &= \partial_n \left(r^\lambda \cos(\lambda\varphi) \right) && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

with $\lambda = \pi/\omega$. Then the unique solution of (3.143) is given by

$$y = r^\lambda \cos(\lambda\varphi),$$

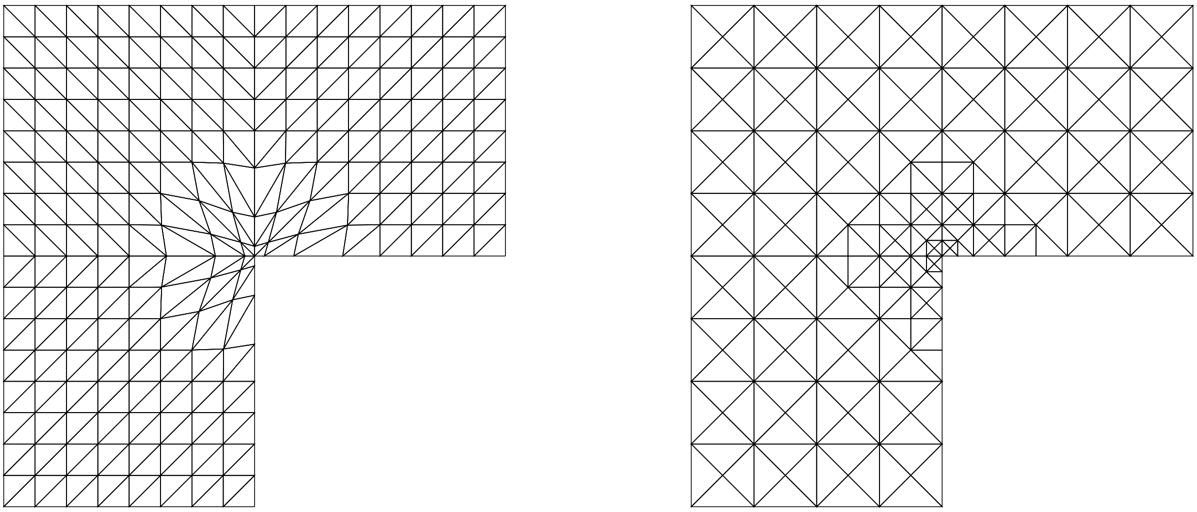

Figure 3.2: $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$) generated by transformation (left) and by bisection (right)

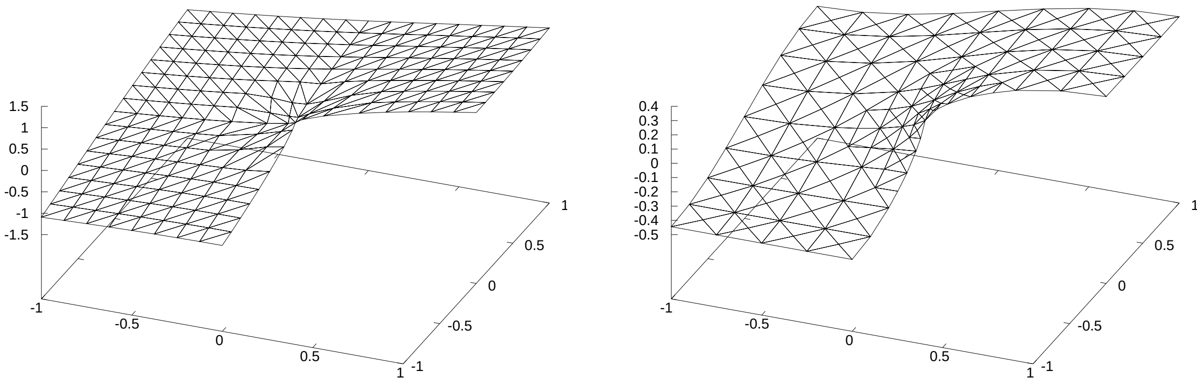

Figure 3.3: Solution y_h of Example 3.66 (left) and solution y_h of Example 3.67 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

which has exactly the singular behavior discussed in Section 3.1.1. In Figure 3.3 one can find the discrete solution y_h for $\omega = 3\pi/2$, which has been calculated on the transformed mesh illustrated in Figure 3.2. Moreover, we calculated the errors in the $L^2(\Omega_\omega)$-, $H^1(\Omega_\omega)$- and $L^2(\Gamma_\omega)$-norm for the angles $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes h and different mesh grading parameters μ as indicated in Tables 3.1–3.6. The experimental order of convergence $\mathrm{eoc}(X)$ is calculated for $X = L^2(\Omega_\omega)$, $X = H^1(\Omega_\omega)$ and $X = L^2(\Gamma_\omega)$ by

$$\mathrm{eoc}(X) := \frac{\ln(\|y - y_{h_{i-1}}\|_X / \|y - y_{h_i}\|_X)}{\ln(h_{i-1}/h_i)},$$

where h_{i-1} and h_i denote two consecutive mesh sizes defined by $h_i = \max_{T\in\mathcal{T}_{h_i}} h_T$. For an interior angle of $2\pi/3$ we observe on quasi-uniform meshes a convergence order of 2 in the $L^2(\Omega_\omega)$- and $L^2(\Gamma_\omega)$-norm and of 1 in the $H^1(\Omega_\omega)$-norm which fits to the theoretical results of Lemma 3.41 and Theorem 3.48. In case of an interior angle of $3\pi/4$ we observe for $\mu = 1$ the full order of convergence in $L^2(\Omega_\omega)$ and $H^1(\Omega_\omega)$ as expected, but only a convergence rate of about 1.83 in $L^2(\Gamma_\omega)$, which confirms our theoretical findings of Corollary 3.49 as well. If we choose $\mu = 0.83 < 0.92 \approx 1/4 + \lambda/2$ we retain the full order of convergence in $L^2(\Gamma_\omega)$ as proven in Theorem 3.48. Note, if we would set $\mu = 1/4 + \lambda/2 - \epsilon$ with an arbitrarily small $\epsilon > 0$, the requirements of Theorem 3.48 are fulfilled but it could take a long time until one can observe the proven convergence rates, since the constants in the estimates could be large. For an interior angle of $3\pi/2$ the situation is even worse. On a quasi-uniform mesh we only have a convergence rate of about 1.34 in $L^2(\Omega_\omega)$, of about 0.66 in $H^1(\Omega_\omega)$ and of about 1.16 in $L^2(\Gamma_\omega)$ as shown in Corollary 3.42 and Corollary 3.49. If we choose $\mu = 0.6 < 0.67 \approx \lambda$ we observe approximately the full convergence rate of 2 in $L^2(\Omega_\omega)$ and of 1 in $H^1(\Omega_\omega)$. But to get a convergence rate close to 2 in the $L^2(\Gamma_\omega)$-norm, this grading does not suffice. According to Theorem 3.48 we set $\mu = 0.5 < 0.58 \approx 1/4 + \lambda/2$ which allows us to achieve the full order of convergence in the $L^2(\Gamma_\omega)$-norm.

Example 3.67. Let $\omega > \pi/4$. We define

$$b(x) := \left(\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2\right)^{1/2}, \quad x = (x_1, x_2) \in \Omega_\omega,$$

and set the data f and g as follows

$$\begin{aligned} f &= b^{1/10}\cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

with $\lambda = \pi/\omega$. In Figure 3.3 the discrete solution y_h is illustrated for $\omega = 3\pi/2$ on the mesh given in Figure 3.2, which was produced by bisection. The discretization errors in different norms for $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes and different mesh grading parameters μ are presented in Tables 3.7–3.12. Since the exact solution of this problem is unknown, we calculated a reference solution y_{ref} on a mesh with mesh size h_{ref} and mesh grading parameter μ_{ref} as specified in the different tables. By interpolation of the solutions y_h on the reference mesh we are able to calculate an approximate experimental order of convergence $\mathrm{eoc}(X)$ for $X = L^2(\Omega_\omega)$, $X = H^1(\Omega_\omega)$ and $X = L^2(\Gamma_\omega)$ by

$$\mathrm{eoc}(X) := \frac{\ln(\|y_{ref} - y_{h_{i-1}}\|_X / \|y_{ref} - y_{h_i}\|_X)}{\ln(h_{i-1}/h_i)},$$

where h_{i-1} and h_i denote again two consecutive mesh sizes. The observations are as in Example 3.66. Of course, the theoretical convergence rates are not reproduced as perfectly as in Example 3.66, since we only compare the discrete solutions with a reference solution and not with the exact solution.

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.577350	1.50e-02		1.94e-01		5.87e-02	
0.288675	3.98e-03	1.91	1.00e-01	0.95	1.66e-02	1.82
0.144338	1.02e-03	1.97	5.09e-02	0.98	4.50e-03	1.88
0.072169	2.57e-04	1.99	2.56e-02	0.99	1.20e-03	1.91
0.036084	6.44e-05	1.99	1.28e-02	1.00	3.16e-04	1.92
0.018042	1.61e-05	2.00	6.43e-03	1.00	8.29e-05	1.93
0.009021	4.03e-06	2.00	3.22e-03	1.00	2.16e-05	1.94
0.004511	1.01e-06	2.00	1.61e-03	1.00	5.62e-06	1.94
0.002255	2.52e-07	2.00	8.04e-04	1.00	1.46e-06	1.95

Table 3.1: Discretization errors $e_h = y - y_h$ for Example 3.66 with $\omega = 2\pi/3$ and $\mu = 1$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	2.37e-02		1.98e-01		8.00e-02	
0.353553	6.17e-03	1.94	1.04e-01	0.92	2.36e-02	1.76
0.176777	1.55e-03	1.99	5.37e-02	0.96	6.76e-03	1.80
0.088388	3.85e-04	2.01	2.73e-02	0.98	1.92e-03	1.82
0.044194	9.56e-05	2.01	1.38e-02	0.99	5.40e-04	1.83
0.022097	2.37e-05	2.01	6.94e-03	0.99	1.52e-04	1.83
0.011049	5.91e-06	2.01	3.48e-03	0.99	4.27e-05	1.83
0.005524	1.47e-06	2.01	1.74e-03	1.00	1.20e-05	1.83
0.002762	3.67e-07	2.00	8.74e-04	1.00	3.36e-06	1.83

Table 3.2: Discretization errors $e_h = y - y_h$ for Example 3.66 with $\omega = 3\pi/4$ and $\mu = 1$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	2.37e-02		1.98e-01		8.00e-02	
0.370133	6.36e-03	2.03	1.06e-01	0.97	2.34e-02	1.90
0.195646	1.65e-03	2.11	5.41e-02	1.05	6.39e-03	2.04
0.103664	4.19e-04	2.16	2.74e-02	1.07	1.68e-03	2.11
0.052560	1.05e-04	2.04	1.38e-02	1.01	4.33e-04	2.00
0.026439	2.63e-05	2.02	6.89e-03	1.01	1.10e-04	1.99
0.013258	6.58e-06	2.01	3.45e-03	1.00	2.80e-05	1.99
0.006639	1.65e-06	2.00	1.73e-03	1.00	7.06e-06	1.99
0.003324	4.12e-07	2.00	8.63e-04	1.00	1.78e-06	1.99

Table 3.3: Discretization errors $e_h = y - y_h$ for Example 3.66 with $\omega = 3\pi/4$ and $\mu = 0.83$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	7.31e-02		2.42e-01		1.41e-01	
0.353553	2.84e-02	1.36	1.57e-01	0.62	6.38e-02	1.15
0.176777	1.10e-02	1.37	1.01e-01	0.64	2.90e-02	1.14
0.088388	4.26e-03	1.37	6.44e-02	0.65	1.31e-02	1.14
0.044194	1.66e-03	1.36	4.10e-02	0.65	5.94e-03	1.15
0.022097	6.46e-04	1.36	2.60e-02	0.66	2.67e-03	1.15
0.011049	2.53e-04	1.35	1.64e-02	0.66	1.20e-03	1.16
0.005524	9.96e-05	1.35	1.04e-02	0.66	5.37e-04	1.16
0.002762	3.93e-05	1.34	6.54e-03	0.66	2.40e-04	1.16

Table 3.4: Discretization errors $e_h = y - y_h$ for Example 3.66 with $\omega = 3\pi/2$ and $\mu = 1$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	7.31e-02		2.42e-01		1.41e-01	
0.403914	2.96e-02	1.61	1.65e-01	0.68	6.14e-02	1.49
0.233893	1.03e-02	1.94	1.01e-01	0.90	2.26e-02	1.83
0.135498	3.14e-03	2.17	5.74e-02	1.03	7.37e-03	2.05
0.070628	8.97e-04	1.92	3.14e-02	0.93	2.26e-03	1.81
0.036008	2.47e-04	1.91	1.67e-02	0.93	6.68e-04	1.81
0.018176	6.67e-05	1.92	8.77e-03	0.94	1.93e-04	1.82
0.009131	1.77e-05	1.93	4.56e-03	0.95	5.46e-05	1.83
0.004587	4.64e-06	1.94	2.35e-03	0.96	1.53e-05	1.85

Table 3.5: Discretization errors $e_h = y - y_h$ for Example 3.66 with $\omega = 3\pi/2$ and $\mu = 0.6$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	7.31e-02		2.42e-01		1.41e-01	
0.425046	3.12e-02	1.67	1.70e-01	0.69	6.29e-02	1.59
0.258029	1.15e-02	2.00	1.06e-01	0.94	2.37e-02	1.95
0.156360	3.56e-03	2.34	6.08e-02	1.11	7.49e-03	2.30
0.083008	9.86e-04	2.03	3.26e-02	0.99	2.13e-03	1.98
0.042742	2.61e-04	2.00	1.69e-02	0.99	5.78e-04	1.97
0.021687	6.75e-05	1.99	8.64e-03	0.99	1.52e-04	1.97
0.010923	1.72e-05	1.99	4.38e-03	0.99	3.94e-05	1.97
0.005496	4.35e-06	2.00	2.21e-03	1.00	1.01e-05	1.98

Table 3.6: Discretization errors $e_h = y - y_h$ for Example 3.66 with $\omega = 3\pi/2$ and $\mu = 0.5$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	8.76e-03		4.70e-02		1.72e-02	
0.250000	2.34e-03	1.90	2.60e-02	0.85	4.67e-03	1.88
0.125000	6.01e-04	1.96	1.36e-02	0.94	1.25e-03	1.91
0.062500	1.52e-04	1.99	6.93e-03	0.97	3.29e-04	1.92
0.031250	3.80e-05	2.00	3.50e-03	0.99	8.62e-05	1.93
0.015625	9.37e-06	2.02	1.75e-03	1.00	2.23e-05	1.95
0.007812	2.22e-06	2.08	8.62e-04	1.02	5.60e-06	1.99

Table 3.7: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.67 with $\omega = 2\pi/3$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.26e-02		6.69e-02		3.02e-02	
0.250000	3.28e-03	1.95	3.48e-02	0.94	7.99e-03	1.92
0.125000	8.41e-04	1.96	1.79e-02	0.96	2.14e-03	1.90
0.062500	2.13e-04	1.98	9.15e-03	0.97	5.74e-04	1.90
0.031250	5.35e-05	1.99	4.63e-03	0.98	1.55e-04	1.89
0.015625	1.32e-05	2.02	2.32e-03	1.00	4.17e-05	1.89
0.007812	3.12e-06	2.09	1.15e-03	1.02	1.11e-05	1.91

Table 3.8: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.67 with $\omega = 3\pi/4$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.93e-02		6.86e-02		4.22e-02	
0.250000	4.65e-03	2.05	3.39e-02	1.02	1.05e-02	2.01
0.125000	1.46e-03	1.67	1.68e-02	1.01	3.20e-03	1.71
0.062500	3.50e-04	2.06	8.40e-03	1.00	7.75e-04	2.04
0.031250	8.75e-05	2.00	4.21e-03	0.99	1.95e-04	1.99
0.015625	2.42e-05	1.86	2.09e-03	1.01	5.29e-05	1.88
0.007812	5.68e-06	2.09	1.02e-03	1.03	1.25e-05	2.09

Table 3.9: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.67 with $\omega = 3\pi/4$, $\mu = 0.83$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	3.41e-02		1.87e-01		6.71e-02	
0.250000	1.19e-02	1.51	1.11e-01	0.76	2.80e-02	1.26
0.125000	4.37e-03	1.45	6.70e-02	0.72	1.21e-02	1.20
0.062500	1.65e-03	1.41	4.11e-02	0.70	5.37e-03	1.18
0.031250	6.34e-04	1.38	2.55e-02	0.69	2.39e-03	1.17
0.015625	2.47e-04	1.36	1.59e-02	0.68	1.07e-03	1.16
0.007812	9.65e-05	1.35	9.94e-03	0.68	4.78e-04	1.16

Table 3.10: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.67 with $\omega = 3\pi/2$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	2.72e-02		1.62e-01		5.21e-02	
0.250000	7.15e-03	1.93	7.95e-02	1.03	1.40e-02	1.90
0.125000	1.87e-03	1.94	4.11e-02	0.95	3.94e-03	1.83
0.062500	5.10e-04	1.87	2.18e-02	0.92	1.15e-03	1.77
0.031250	1.28e-04	1.99	1.09e-02	1.00	2.94e-04	1.97
0.015625	3.28e-05	1.97	5.56e-03	0.97	8.15e-05	1.85
0.007812	8.36e-06	1.97	2.83e-03	0.97	2.29e-05	1.83

Table 3.11: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.67 with $\omega = 3\pi/2$, $\mu = 0.6$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	2.53e-02		1.47e-01		4.74e-02	
0.250000	6.64e-03	1.93	7.41e-02	0.99	1.25e-02	1.92
0.125000	1.70e-03	1.97	3.75e-02	0.98	3.21e-03	1.97
0.062500	4.30e-04	1.98	1.90e-02	0.98	8.20e-04	1.97
0.031250	1.08e-04	1.99	9.54e-03	0.99	2.09e-04	1.97
0.015625	2.70e-05	2.01	4.77e-03	1.00	5.23e-05	2.00
0.007812	6.50e-06	2.05	2.34e-03	1.02	1.26e-05	2.05

Table 3.12: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.67 with $\omega = 3\pi/2$, $\mu = 0.5$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

3.2.6 Finite element error estimates for semilinear elliptic problems

The aim of this section is to derive finite element error estimates for semilinear elliptic equations. We begin with the definition of discrete solutions of (3.56).

Definition 3.68. Let Assumption 3.18 be fulfilled. Furthermore, let $d(\cdot,0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ and $g \in L^s(\Gamma)$ with $r, s > 1$. A discrete solution of (3.56) is an element $y_h \in V_h$ that fulfills

$$a(y_h, v_h) + \int_\Omega d(\cdot, y_h) v_h = \int_\Omega f v_h + \int_\Gamma g v_h \quad \forall v_h \in V_h \tag{3.145}$$

with the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ from (3.58).

As for the weak formulation (3.57) of problem (3.56) in Section 3.1.2, we introduce an equivalent formulation of (3.145) which will simplify the numerical analysis in the sequel. Let α and $\tilde{d}$ be the functions defined in the beginning of Section 3.1.2. Furthermore, let $\tilde{a}$ be the bilinear form (3.60). Then the variational equation (3.145) can be stated equivalently as

$$\tilde{a}(y_h, v_h) + \int_\Omega \tilde{d}(\cdot, y_h) v_h = \int_\Omega (f - d(\cdot, 0)) v_h + \int_\Gamma g v_h \quad \forall v_h \in V_h. \tag{3.146}$$

The next result is devoted to the existence and uniqueness of a solution of (3.145).

Lemma 3.69. *Let Assumption 3.18 be fulfilled. Then problem* (3.145) *has a unique solution* $y_h \in V_h$ *for*

(i) $d(\cdot,0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ *and* $g \in L^s(\Gamma)$ *with* $r, s > 1$. *Furthermore, there is the estimate*

$$\|y_h\|_{H^1(\Omega)} \le c \left(\|f - d(\cdot,0)\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)} \right)$$

with a constant $c = c(E_\Omega, c_\Omega) > 0$ *independent of* d, f *and* g.

(ii) $d(\cdot,0) \in W^{0,2}_{\vec{\beta}}(\Omega)$, $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ *and* $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ *with* $0 \le \beta_j < 1$ *for* $j = 1, \ldots, m$. *Moreover, there holds*

$$\|y_h\|_{H^1(\Omega)} \le c \left(\|f - d(\cdot,0)\|_{W^{0,2}_{\vec{\beta}}(\Omega)} + \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \right)$$

with a positive constant $c = c(E_\Omega, c_\Omega)$ *independent of* d, f *and* g.

Proof. The existence of a solution of (3.146) can easily be proven by Brouwer's fixed-point theorem employing the monotonicity of the nonlinearity $\tilde{d}$. The uniqueness can simply be deduced from the coercivity of the bilinear form $\tilde{a}$ and the monotonicity of the nonlinearity $\tilde{d}$, see also the steps below. The a priori estimate in the first case can be obtained by the coercivity of the bilinear form $\tilde{a}$, the monotonicity of the nonlinearity $\tilde{d}$, together with the property $\tilde{d}(x,0) = 0$, the Hölder inequality, Theorem 2.7 and Theorem 2.8. By this we can conclude with a positive constant $c = c(E_\Omega, c_\Omega)$ independent of d, f and g that

$$\begin{aligned}
\|y_h\|^2_{H^1(\Omega)} &\leq c\tilde{a}(y_h, y_h) \\
&\leq c\left(\tilde{a}(y_h, y_h) + \int_\Omega \left(\tilde{d}(\cdot, y_h) - \tilde{d}(\cdot, 0)\right)(y_h - 0)\right) \\
&= c\left(\tilde{a}(y_h, y_h) + \int_\Omega \tilde{d}(\cdot, y_h) y_h\right) \\
&= c\left(\int_\Omega (f - d(\cdot, 0)) y_h + \int_\Gamma g y_h\right) \\
&\leq c\left(\|f - d(\cdot, 0)\|_{L^r(\Omega)} \|y_h\|_{L^{r/(r-1)}(\Omega)} + \|g\|_{L^s(\Gamma)} \|y_h\|_{L^{s/s-1}(\Gamma)}\right) \\
&\leq c\left(\|f - d(\cdot, 0)\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}\right) \|y_h\|_{H^1(\Omega)}.
\end{aligned}$$

The second a priori estimate is a consequence of Lemma 2.39. □

Now, we are going to derive finite element error estimates for the semilinear problems (3.56) and (3.145). In [25] a generalization of Cea's Lemma and the Aubin-Nitsche method is proven, which yield error estimates in $H^1(\Omega)$ and $L^2(\Omega)$, respectively. However, we proceed in a different way inspired by [84], which allows us to simply deduce error estimates in the $L^\infty(\Omega)$- and $L^2(\Gamma)$-norm. For further literature concerning finite element error estimates for semilinear and quasilinear elliptic partial differential equations we refer to e.g. [45], [50], [44], [55, Section 3.5], [18], [20, Section 8.7], [30] and [23].

First, we observe that the solution $y \in H^1(\Omega)$ of (3.56) fulfills

$$\tilde{a}(y, v) = \int_\Omega (f - d(\cdot, 0) - \tilde{d}(\cdot, y)) v + \int_\Gamma g v \quad \forall v \in H^1(\Omega).$$

Next, we introduce the function $\tilde{y}_h \in V_h$ as solution of the linear discrete equation

$$\tilde{a}(\tilde{y}_h, v_h) = \int_\Omega (f - d(\cdot, 0) - \tilde{d}(\cdot, y)) v_h + \int_\Gamma g v_h \quad \forall v_h \in V_h. \tag{3.147}$$

It is easy to conclude $\tilde{d}(\cdot, y) \in L^r(\Omega)$. Indeed, by employing the Lipschitz continuity of d, together with Lemma 3.20 to deduce the boundedness of y in $C^0(\bar{\Omega})$, the definition of α, and Theorem 2.7, we obtain

$$\begin{aligned}
\|\tilde{d}(\cdot, y))\|_{L^r(\Omega)} &= \|d(\cdot, y) - d(\cdot, 0) - \alpha y\|_{L^r(\Omega)} \\
&\leq c\|d(\cdot, y) - d(\cdot, 0)\|_{L^r(\Omega)} + \|\alpha y\|_{L^r(\Omega)} \\
&\leq c\|y\|_{L^r(\Omega)} \leq c
\end{aligned}$$

with a positive constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot, 0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)})$. Thus, the solution of (3.147) exists and is unique according to Lemma 3.40.

The result of the following lemma enables us to reduce the finite element error estimates for semilinear problems to the corresponding estimates for linear problems.

Lemma 3.70. *Let Assumption 3.18 be satisfied. Then for the weak solution y of* (3.56)*, the solution y_h of* (3.145) *and the solution $\tilde{y}_h$ of* (3.147) *there is the estimate*

$$\|\tilde{y}_h - y_h\|_{H^1(\Omega)} \le c\|y - \tilde{y}_h\|_{L^2(\Omega)}$$

with a positive constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)})$ independent of d, f and g.

Proof. Due to the coercivity of the bilinear form $\tilde{a}$ with coercivity constant $c_* = c_*(E_\Omega, c_\Omega)$, the variational equations (3.146) and (3.147), and the monotonicity of $\tilde{d}$ we can conclude

$$\begin{aligned}
c_*\|\tilde{y}_h - y_h\|^2_{H^1(\Omega)} &\le \tilde{a}(\tilde{y}_h - y_h, \tilde{y}_h - y_h) = \int_\Omega (\tilde{d}(\cdot, y_h) - \tilde{d}(\cdot, y))(\tilde{y}_h - y_h) \\
&= \int_\Omega (\tilde{d}(\cdot, y_h) - \tilde{d}(\cdot, \tilde{y}_h))(\tilde{y}_h - y_h) + \int_\Omega (\tilde{d}(\cdot, \tilde{y}_h) - \tilde{d}(\cdot, y))(\tilde{y}_h - y_h) \\
&\le \int_\Omega (\tilde{d}(\cdot, \tilde{y}_h) - \tilde{d}(\cdot, y))(\tilde{y}_h - y_h) \le \|\tilde{d}(\cdot, \tilde{y}_h) - \tilde{d}(\cdot, y)\|_{L^2(\Omega)}\|\tilde{y}_h - y_h\|_{L^2(\Omega)} \\
&\le c\|\tilde{y}_h - y\|_{L^2(\Omega)}\|\tilde{y}_h - y_h\|_{H^1(\Omega)},
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality, the Lipschitz continuity of $\tilde{d}$ and Theorem 2.7 in the last steps. Note that the positive constant c only depends on the constant c_Ω and the Lipschitz constant $L_{d,M}$ stated in Assumption 3.18. The latter constant is bounded if $\|f - d(\cdot,0)\|_{L^r(\Omega)}$ and $\|g\|_{L^s(\Gamma)}$ are bounded according to Lemma 3.20 and Lemma 3.47. Dividing by $\|\tilde{y}_h - y_h\|_{H^1(\Omega)}$ yields the assertion. ☐

Remark 3.71. If one would additionally consider the semilinear boundary condition $\partial_n y + b(x,y) = g$ with assumptions for the nonlinearity b similar to those for the nonlinearity d, one would end up with

$$\|\tilde{y}_h - y_h\|_{H^1(\Omega)} \le c\left(\|y - \tilde{y}_h\|_{L^2(\Omega)} + \|y - \tilde{y}_h\|_{L^2(\Gamma)}\right)$$

in Lemma 3.70.

Now, let us prove the different finite element error estimates in the domain.

Corollary 3.72. *Let Assumption 3.18 be fulfilled. Furthermore, let $d(\cdot,0) \in W^{0,2}_{\vec{\beta}}(\Omega)$, $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$. Then the discretization error can be estimated by*

$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{W^{1,2}(\Omega)} + h|\ln h|^{-1/2}\|y - y_h\|_{L^\infty(\Omega)} \le ch^2\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le ch^2$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{W^{0,2}_{\vec{\beta}}(\Omega)}, \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)})$, provided that $\vec{1} - \vec{\lambda} < \vec{\beta} \le \vec{1} - \vec{\mu}$ and $\vec{\beta} \ge \vec{0}$.

Corollary 3.73. *Let Assumption 3.18 be satisfied and let $\mu_j = 1$ for $j = 1, \dots, m$ (quasi-uniform mesh). Furthermore, let $d(\cdot,0) \in W^{0,2}_{\vec{\beta}}(\Omega)$, $f \in W^{0,2}_{\vec{\beta}}(\Omega)$ and $g \in W^{1/2,2}_{\vec{\beta}}(\Gamma)$ with $\vec{\beta} = \vec{1} - \vec{\lambda} + \vec{\epsilon}$, $\vec{\beta} \ge \vec{0}$ and $\vec{0} < \vec{\epsilon} < \vec{\lambda}$. Then the discretization error can be estimated by*

$$\|y - y_h\|_{L^2(\Omega)} + h^\lambda\|y - y_h\|_{W^{1,2}(\Omega)} + h^\lambda|\ln h|^{-1/2}\|y - y_h\|_{L^\infty(\Omega)} \le ch^{2\lambda}\|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \le ch^{2\lambda},$$

where $\lambda = \min(1, \min(\vec{\lambda} - \vec{\epsilon}))$ *and* c *denotes a positive constant with* $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{W^{0,2}_{\vec{\beta}}(\Omega)}, \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)})$.

Corollary 3.74. *Suppose that Assumption 3.18 is fulfilled and let* $d(\cdot,0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ *and* $g \in L^s(\Gamma)$ *with* $r \in (1, 4/3)$ *and* $s \in (1,2)$. *Furthermore, let* $t = \min(3 - 2/r, 2 - 1/s)$, $\vec{\epsilon} \in \mathbb{R}^m$ *with* $\vec{0} < \vec{\epsilon} < \vec{\lambda}$ *and* $\lambda = \min(1, \min(\vec{\lambda} - \vec{\epsilon}))$. *Then there is the estimate*

$$\|y - y_h\|_{L^2(\Omega)} + h^\lambda \|y - y_h\|_{W^{1,2}(\Omega)} + h^\lambda |\ln h|^{-1/2} \|y - y_h\|_{L^\infty(\Omega)} \le c h^{t-1+\lambda} \|y\|_{H^t(\Omega)} \le c h^{t-1+\lambda}$$

with a positive constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)})$.

Proof of Corollaries 3.72, 3.73 and 3.74. Introducing the intermediate function $\tilde{y}_h$, cf. (3.147), and using Lemma 3.70 yields

$$\begin{aligned}
\|y - y_h\|_{L^2(\Omega)} &\le \|y - \tilde{y}_h\|_{L^2(\Omega)} + \|\tilde{y}_h - y_h\|_{L^2(\Omega)} \le c\|y - \tilde{y}_h\|_{L^2(\Omega)}, \\
\|y - y_h\|_{H^1(\Omega)} &\le \|y - \tilde{y}_h\|_{H^1(\Omega)} + \|\tilde{y}_h - y_h\|_{H^1(\Omega)} \le \|y - \tilde{y}_h\|_{H^1(\Omega)} + c\|y - \tilde{y}_h\|_{L^2(\Omega)}
\end{aligned}$$

with $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)}) > 0$. Analogously we obtain

$$\begin{aligned}
\|y - y_h\|_{L^\infty(\Omega)} &\le \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + \|\tilde{y}_h - y_h\|_{L^\infty(\Omega)} \\
&\le \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + c(1 + |\ln h|)^{1/2} \|\tilde{y}_h - y_h\|_{H^1(\Omega)} \\
&\le \|y - \tilde{y}_h\|_{L^\infty(\Omega)} + c(1 + |\ln h|)^{1/2} \|y - \tilde{y}_h\|_{L^2(\Omega)},
\end{aligned}$$

where we applied the discrete Sobolev inequality in between, cf. Lemma 4.9.2 of [20]. Finally, having regard to Corollary 3.22 and Corollary 3.23, we can conclude the assertion of Corollary 3.72, Corollary 3.73 and Corollary 3.74 by using Lemma 3.41, Corollary 3.42 and Corollary 3.45, respectively. □

As a consequence we can deduce that the y_h is uniformly bounded in $L^\infty(\Omega)$ independent of the mesh parameter h.

Corollary 3.75. *Let Assumption 3.18 be fulfilled. Then the solution* $y_h \in V_h$ *of problem* (3.145) *satisfies for* $d(\cdot,0) \in L^r(\Omega)$, $f \in L^r(\Omega)$ *and* $g \in L^s(\Gamma)$ *with* $r, s > 1$ *the estimate*

$$\|y_h\|_{L^\infty(\Omega)} \le c$$

with a constant $c = c(E_\Omega, c_\Omega, \|f - d(\cdot,0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)}) > 0$.

Proof. This is a consequence of Corollary 3.74 and Lemma 3.20 since one can estimate

$$\|y_h\|_{L^\infty(\Omega)} \le \|y_h - y\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Omega)}.$$

□

Next, we show that the finite element error estimates on the boundary extend to semilinear problems as well. The approach is as before.

Corollary 3.76. *Assume that Assumptions 3.18 and 3.24 are fulfilled. Let* $\varrho \in [0,1/2]$, $\vec{\mu} \in (\varrho/2,1]^m$, $\vec{2}-\vec{\lambda} < \vec{\gamma} \leq \vec{2}+\vec{\varrho}-2\vec{\mu}$, $\vec{\gamma} \geq \vec{0}$, $\vec{1}-\vec{\lambda} < \vec{\beta} \leq \vec{1}-\vec{\mu}$, $\vec{\beta} \geq \vec{0}$, *and let* $\vec{\delta}$ *and* σ *fulfill the conditions stated in Lemma 3.13. Moreover, let* $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega) \cap W^{0,2}_{\vec{\beta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma) \cap W^{1/2,2}_{\vec{\beta}}(\Gamma)$. *Then the discretization error on the boundary admits the estimate*

$$\|y-y_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{1+\varrho}\left(\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|y\|_{W^{2,2}_{\vec{\beta}}(\Omega)}\right) \leq ch^2|\ln h|^{1+\varrho}$$

with $c = c(E_\Omega, c_\Omega, \|f-d(\cdot,0)\|_{W^{0,2}_{\vec{\beta}}(\Omega)}, \|g\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)}, \|f-d(\cdot,0)\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}, \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}) > 0$.

Corollary 3.77. *Let Assumptions 3.18 and 3.24 be satisfied and let* $\vec{\mu} = \vec{1}$ *(quasi-uniform mesh). Furthermore, let* $\varrho \in [0,1/2]$, $\vec{\gamma} = \vec{2}-\vec{\lambda}+\vec{\epsilon}$, $\vec{\gamma} \geq \vec{0}$, $\vec{\epsilon} \in \mathbb{R}^m$ *with* $\vec{0} < \vec{\epsilon} < \vec{\lambda}$, *and let* $\vec{\delta}$ *and* σ *fulfill the conditions stated in Lemma 3.13. Then for* $f \in N^{0,\sigma}_{\vec{\delta}}(\Omega)$ *and* $g \in N^{1,\sigma}_{\vec{\delta}}(\Gamma)$ *there is the estimate*

$$\|y-y_h\|_{L^2(\Gamma)} \leq ch^\rho|\ln h|^{1+\varrho}\|y\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} \leq ch^\rho|\ln h|^{1+\varrho}$$

with $\rho = \min(2, \min(\vec{\varrho}+\vec{\lambda}-\vec{\epsilon}))$ *and a constant* $c = c(E_\Omega, c_\Omega\|f-d(\cdot,0)\|_{N^{0,\sigma}_{\vec{\delta}}(\Omega)}, \|g\|_{N^{1,\sigma}_{\vec{\delta}}(\Gamma)}) > 0$.

Proof of Corollaries 3.76 and 3.77. Again, we introduce the function $\tilde{y}_h$ from (3.147). Next, we apply Theorem 2.8 and Lemma 3.70. By this we get

$$\begin{aligned}\|y-y_h\|_{L^2(\Gamma)} &\leq \|y-\tilde{y}_h\|_{L^2(\Gamma)} + \|\tilde{y}_h-y_h\|_{L^2(\Gamma)} \leq \|y-\tilde{y}_h\|_{L^2(\Gamma)} + c\|\tilde{y}_h-y_h\|_{H^1(\Omega)}\\ &\leq \|y-\tilde{y}_h\|_{L^2(\Gamma)} + c\|y-\tilde{y}_h\|_{L^2(\Omega)}\end{aligned}$$

with a constant $c = c(E_\Omega, c_\Omega, \|f-d(\cdot,0)\|_{L^r(\Omega)}, \|g\|_{L^s(\Gamma)}) > 0$. We obtain the validity of the assertions from Theorem 3.48, Lemma 3.41, Corollary 3.23 and Corollary 3.26 in the first case, and from Corollary 3.49, Corollary 3.42, Lemma 2.29 and Corollary 3.26 in the second one. □

Remark 3.78. If there is a nonlinearity b located on the Neumann boundary, cf. Remark 3.71, one can proceed in the same way for the derivation of $H^1(\Omega)$-, $L^\infty(\Omega)$- and $L^2(\Gamma)$-error estimates possibly using Theorem 2.7 and Theorem 2.8. Only for the $L^2(\Omega)$-error estimates this approach would yield suboptimal results. In this case we can introduce the linear elliptic dual problem

$$\begin{aligned} -\Delta w + (\alpha + \psi_\Omega)w &= y - y_h && \text{in } \Omega,\\ \partial_n w + \psi_\Gamma w &= 0 && \text{on } \Gamma_j, \quad j = 1,\ldots,m,\end{aligned}$$

with

$$\psi_\Omega(x) := \begin{cases} \frac{\tilde{d}(x,y(x))-\tilde{d}(x,y_h(x))}{y(x)-y_h(x)} & \text{if } y(x)-y_h(x) \neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\psi_\Gamma(x) := \begin{cases} \frac{b(x,y(x))-b(x,y_h(x))}{y(x)-y_h(x)} & \text{if } y(x)-y_h(x) \neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Now, applying the Aubin-Nitsche method using this dual problem and the $H^1(\Omega)$-error estimates yields the desired estimate in $L^2(\Omega)$, cf. [25, Lemma 4].

Let us close this section with Lipschitz estimates for discrete solutions of (3.145) analogous to those of Lemma 3.29.

Lemma 3.79. *Let Assumption 3.18 be fulfilled, $r > 1$ and $s > 1$ be given and let $d(\cdot, 0) \in L^r(\Omega)$. Moreover, let $y_{1,h} \in V_h$ and $y_{2,h} \in V_h$ be discrete solutions of* (3.145) *with right hand sides $f_1 \in L^r(\Omega)$ and $f_2 \in L^r(\Omega)$ and Neumann boundary data $g_1 \in L^s(\Gamma)$ and $g_2 \in L^s(\Gamma)$, respectively. Then the estimate*

$$\|y_{1,h} - y_{2,h}\|_{L^2(\Omega)} \le c \left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)} \right)$$

holds with $c = c(E_\Omega, c_\Omega, \|f_1 - d(\cdot, 0)\|_{L^r(\Omega)}, \|f_2 - d(\cdot, 0)\|_{L^r(\Omega)}, \|g_1\|_{L^s(\Gamma)}, \|g_2\|_{L^s(\Gamma)})$. Furthermore, one has

$$\|y_{1,h} - y_{2,h}\|_{H^1(\Omega)} \le c \left(\|f_1 - f_2\|_{L^r(\Omega)} + \|g_1 - g_2\|_{L^s(\Gamma)} \right),$$

where the constant $c = c(E_\Omega, c_\Omega)$ is independent of d, f_1, f_2, g_1 and g_2.

Proof. We proceed as in the proofs of Lemma 3.54 and Lemma 3.29. Let w be the weak solution of

$$\begin{aligned} -\Delta w + (\alpha + \psi) w &= y_{1,h} - y_{2,h} && \text{in } \Omega, \\ \partial_n w &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.148}$$

with

$$\psi(x) = \begin{cases} \frac{\tilde{d}(x, y_{1,h}(x)) - \tilde{d}(x, y_{2,h}(x))}{y_{1,h}(x) - y_{2,h}(x)} & \text{if } y_{1,h}(x) \ne y_{2,h}(x), \\ 0 & \text{otherwise.} \end{cases}$$

For the function $\alpha+\psi$ there holds $\alpha+\psi \ge c_\Omega$ on E_Ω and $\|\alpha+\psi\|_{L^\infty(\Omega)} \le L_{d,M}$ due to Assumption 3.18, where the Lipschitz constant $L_{d,M}$ depends on $\|f_1 - d(\cdot,0)\|_{L^r(\Omega)}$, $\|f_2 - d(\cdot,0)\|_{L^r(\Omega)}$, $\|g_1\|_{L^s(\Gamma)}$ and $\|g_2\|_{L^s(\Gamma)}$ according to Corollary 3.75. Therefore, the elliptic problem (3.148) is well-posed according to Lemma 3.4. Furthermore, let us denote its discrete solution with w_h. Then we obtain

$$\begin{aligned} \|y_{1,h} - y_{2,h}\|^2_{L^2(\Omega)} &= \tilde{a}(y_{1,h} - y_{2,h}, w_h) + \int_\Omega \psi (y_{1,h} - y_{2,h}) w_h \\ &= \tilde{a}(y_{1,h} - y_{2,h}, w_h) + \int_\Omega (\tilde{d}(\cdot, y_{1,h}) - \tilde{d}(\cdot, y_{2,h})) w_h \\ &= \int_\Omega (f_1 - f_2) w_h + \int_\Gamma (g_1 - g_2) w_h \\ &\le \left(\|f_1 - f_2\|_{L^1(\Omega)} + \|g_1 - g_2\|_{L^1(\Gamma)} \right) \|w_h\|_{L^\infty(\Omega)}. \end{aligned}$$

The first estimate of the assertion follows from Corollary 3.47. For the second one we can argue as in the second part of the proof of Lemma 3.29 using (3.146) instead of (3.59). □

3.2.7 Numerical example

In this section we perform numerical tests in order to illustrate the theoretical results of the previous section. As in Section 3.2.5 we present two numerical examples. In the first one, the exact solution is known, whereas in the second one, we use a reference solution on a finer mesh to be able to state experimental orders of convergence. The computational domains are the domains Ω_ω defined in (3.142). In both examples, we numerically solve by a finite element method with linear finite elements the problem

$$\begin{aligned} -\Delta y + y + y^3 &= f \quad \text{in } \Omega_\omega, \\ \partial_n y &= g \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{3.149}$$

where the data f and g are again chosen such that the singular terms in the solution dominate the discretization errors. The resulting nonlinear system of equations is

$$\vec{F}(\vec{y}) = \vec{0} \tag{3.150}$$

with $\vec{y} = (y_1, \dots, y_N)^T$, $N = \#I_x$ and

$$F_i(\vec{y}) := \sum_{k \in I_X} y_k \int_{\Omega_\omega} (\nabla \phi_k \cdot \nabla \phi_i + \phi_k \phi_i) + \int_{\Omega_\omega} \left(\sum_{k \in I_X} y_k \phi_k \right)^3 \phi_i - \int_{\Omega_\omega} f \phi_i - \int_{\Gamma_\omega} g \phi_i$$

for $i \in I_X$. We approximately solve (3.150) by applying Newton's method. Thus, we need the Jacobian matrix $J_{\vec{F}}$ of $\vec{F}$ which is given by

$$(J_{\vec{F}}(\vec{y}))_{i,j} := \frac{\partial}{\partial y_j} F_i(\vec{y}) = \int_{\Omega_\omega} (\nabla \phi_j \cdot \nabla \phi_i + \phi_j \phi_i) + \int_{\Omega_\omega} 3 \left(\sum_{k \in I_X} y_k \phi_k \right)^2 \phi_j \phi_i$$

for $i, j \in I_X$. The new iterate $\vec{y}_{l+1}$ of Newton's method for our problem is the solution of

$$J_{\vec{F}}(\vec{y}_l)\vec{y}_{l+1} = J_{\vec{F}}(\vec{y}_l)\vec{y}_l - \vec{F}(\vec{y}_l), \tag{3.151}$$

where we set $\vec{y}_0 = \vec{0} \in \mathbb{R}^N$. As stopping criterion for Newton's method we choose

$$\frac{\| \sum_{k \in I_X} (y_{l+1,k} - y_{l,k}) \phi_k \|_{L^2(\Omega_\omega)}}{\| \sum_{k \in I_X} y_{l+1,k} \phi_k \|_{L^2(\Omega_\omega)}} < TOL$$

with $TOL = 10^{-8}$. The realization of the implementation of the finite element method is again similar to that in [2]. But note that we have to extend the algorithms to be able to calculate

$$\int_{\Omega_\omega} 3 \left(\sum_{k \in I_X} y_{l,k} \phi_k \right)^2 \phi_j \phi_i \quad \forall j, i \in I_X \tag{3.152}$$

and

$$\int_{\Omega_\omega} \left(\sum_{k \in I_X} y_{l,k} \phi_k \right)^3 \phi_i \quad \forall i \in I_X \tag{3.153}$$

in (3.151).

Now let us describe the specific numerical examples.

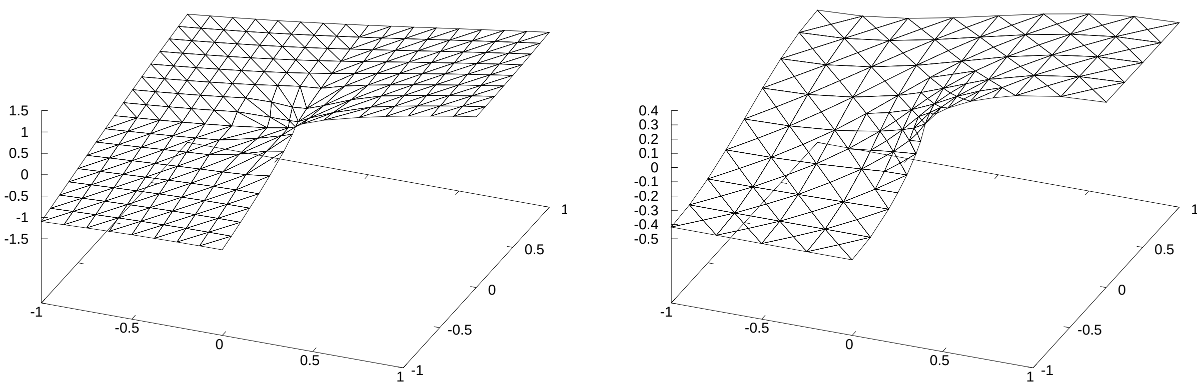

Figure 3.4: Solution y_h of Example 3.80 (left) and solution y_h of Example 3.81 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5$, $R = 0.4$)

Example 3.80. We set

$$f = r^\lambda \cos(\lambda\varphi) + \left(r^\lambda \cos(\lambda\varphi)\right)^3 \quad \text{in } \Omega_\omega,$$
$$g = \partial_n \left(r^\lambda \cos(\lambda\varphi)\right) \quad \text{on } \Gamma_j, \quad j = 1, \dots, m,$$

with $\lambda = \pi/\omega$. Then the unique solution of (3.149) is given by

$$y = r^\lambda \cos(\lambda\varphi),$$

which has exactly the regularity discussed in Section 3.1.2. In Figure 3.4 the discrete solution y_h for $\omega = 3\pi/2$ is illustrated on a graded mesh with $\mu = 0.5$ and $R = 0.4$ which has been generated by transformation of the nodes, see Section 3.2.5 for details. The discretization errors in the $L^2(\Omega_\omega)$-, $H^1(\Omega_\omega)$- and $L^2(\Gamma_\omega)$-norm for the angles $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes h and different mesh grading parameters μ are presented in Tables 3.13–3.18, where we generated graded meshes as in Example 3.66 by transformation of the nodes. The experimental orders of convergence are calculated as in Example 3.66 as well. The observations are equal to those for Example 3.66. Let us repeat them for the convenience of the reader. In case of $\omega = 2\pi/3$ we achieve on a quasi-uniform mesh a convergence rate of 2 and almost 2 in $L^2(\Omega_\omega)$ and $L^2(\Gamma_\omega)$, respectively, and of 1 in $H^1(\Omega_\omega)$, which underlines the theoretical findings of Corollary 3.72 and Corollary 3.76. For $\omega = 3\pi/4$ we observe on a quasi-uniform mesh the best possible approximation rate of 2 and 1 in $L^2(\Omega_\omega)$ and $H^1(\Omega_\omega)$, respectively, but only the reduced convergence rate of about 1.83 in $L^2(\Gamma_\omega)$, which confirms the estimates of Corollary 3.72 and Corollary 3.77. If we choose $\mu = 0.83 < 0.92 \approx 1/4 + \lambda/2$ we can compensate the negative influence of the corner singularities and retain a convergence order of almost 2 in $L^2(\Gamma_\omega)$. Next we consider the domains $\Omega_{3\pi/2}$. On a quasi-uniform mesh the convergence rate is lowered in all norms. In $L^2(\Omega_\omega)$, $H^1(\Omega_\omega)$ and $L^2(\Gamma_\omega)$ we observe the approximation rates 1.35, 0.66 and 1.15, respectively, which fits to the theoretical results of Corollary 3.73 and Corollary 3.77. Next, if we set $\mu = 0.6 < 0.67 \approx \lambda$ we observe approximately the full order of convergence in $L^2(\Omega_\omega)$ and $H^1(\Omega)$ according to Corollary 3.72. But to achieve a convergence order of almost 2 in $L^2(\Gamma_\omega)$ we choose the stronger mesh grading of $\mu = 0.5 < 0.58 \approx 1/4 + \lambda/2$, which confirms the estimate of Corollary 3.76.

Example 3.81. Let $\omega > \pi/4$. We define

$$b(x) := \left(\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2\right)^{1/2}, \quad x = (x_1, x_2) \in \Omega_\omega.$$

The data f and g are chosen as follows

$$\begin{aligned} f &= b^{1/10}\cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g &= 0 && \text{on } \Gamma_j, \quad j = 1, \ldots, m, \end{aligned}$$

with $\lambda = \pi/\omega$. In Figure 3.4 one can find the discrete solution y_h for $\omega = 3\pi/2$ on a graded mesh constructed by bisection as illustrated in Figure 3.2. Since we do not know the exact solution, we calculated a reference solution for each $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$ on a mesh with mesh size h_{ref} and with mesh grading parameter μ_{ref} as indicated in Tables 3.19–3.24 in order to be able to calculate approximate discretization errors and approximate experimental convergence orders as in Example 3.67. The results are presented in Tables 3.19–3.24. The observations do not differ fundamentally from those in the previous example which confirms the estimates of Section 3.2.6. Of course, the theoretical results are not such as perfectly reflected as in case of a known singular solution, but the proven influence of the corner singularities on the approximation properties of the discrete solution is apparent.

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.577350	1.31e-02		1.94e-01		5.79e-02	
0.288675	3.51e-03	1.90	1.00e-01	0.95	1.63e-02	1.83
0.144338	8.98e-04	1.97	5.09e-02	0.98	4.43e-03	1.88
0.072169	2.26e-04	1.99	2.56e-02	0.99	1.18e-03	1.91
0.036084	5.68e-05	1.99	1.28e-02	1.00	3.12e-04	1.92
0.018042	1.42e-05	2.00	6.43e-03	1.00	8.19e-05	1.93
0.009021	3.56e-06	2.00	3.22e-03	1.00	2.14e-05	1.94
0.004511	8.89e-07	2.00	1.61e-03	1.00	5.56e-06	1.94
0.002255	2.22e-07	2.00	8.04e-04	1.00	1.44e-06	1.95

Table 3.13: Discretization errors $e_h = y - y_h$ for Example 3.80 with $\omega = 2\pi/3$ and $\mu = 1$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	1.59e-02		1.97e-01		7.12e-02	
0.353553	4.42e-03	1.85	1.04e-01	0.92	2.17e-02	1.71
0.176777	1.12e-03	1.98	5.37e-02	0.96	6.36e-03	1.77
0.088388	2.78e-04	2.01	2.73e-02	0.98	1.83e-03	1.80
0.044194	6.87e-05	2.02	1.38e-02	0.99	5.21e-04	1.81
0.022097	1.70e-05	2.02	6.94e-03	0.99	1.48e-04	1.82
0.011049	4.20e-06	2.01	3.48e-03	0.99	4.17e-05	1.82
0.005524	1.04e-06	2.01	1.74e-03	1.00	1.18e-05	1.83
0.002762	2.59e-07	2.01	8.74e-04	1.00	3.32e-06	1.83

Table 3.14: Discretization errors $e_h = y - y_h$ for Example 3.80 with $\omega = 3\pi/4$ and $\mu = 1$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	1.59e-02		1.97e-01		7.12e-02	
0.370133	4.69e-03	1.89	1.05e-01	0.97	2.16e-02	1.84
0.195646	1.26e-03	2.06	5.41e-02	1.05	5.97e-03	2.01
0.103664	3.23e-04	2.14	2.74e-02	1.07	1.58e-03	2.09
0.052560	8.13e-05	2.03	1.38e-02	1.01	4.08e-04	1.99
0.026439	2.04e-05	2.01	6.89e-03	1.01	1.04e-04	1.99
0.013258	5.10e-06	2.01	3.45e-03	1.00	2.65e-05	1.99
0.006639	1.27e-06	2.00	1.73e-03	1.00	6.70e-06	1.99
0.003324	3.19e-07	2.00	8.63e-04	1.00	1.69e-06	1.99

Table 3.15: Discretization errors $e_h = y - y_h$ for Example 3.80 with $\omega = 3\pi/4$ and $\mu = 0.83$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	6.04e-02		2.43e-01		1.22e-01	
0.353553	2.34e-02	1.37	1.58e-01	0.63	5.68e-02	1.10
0.176777	9.05e-03	1.37	1.01e-01	0.64	2.67e-02	1.09
0.088388	3.50e-03	1.37	6.45e-02	0.65	1.24e-02	1.11
0.044194	1.36e-03	1.37	4.10e-02	0.65	5.67e-03	1.12
0.022097	5.27e-04	1.36	2.60e-02	0.66	2.58e-03	1.14
0.011049	2.06e-04	1.36	1.64e-02	0.66	1.17e-03	1.14
0.005524	8.06e-05	1.35	1.04e-02	0.66	5.26e-04	1.15
0.002762	3.17e-05	1.35	6.54e-03	0.66	2.36e-04	1.15

Table 3.16: Discretization errors $e_h = y - y_h$ for Example 3.80 with $\omega = 3\pi/2$ and $\mu = 1$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	6.04e-02		2.43e-01		1.22e-01	
0.403914	2.44e-02	1.62	1.65e-01	0.69	5.37e-02	1.46
0.233893	8.53e-03	1.92	1.01e-01	0.91	2.01e-02	1.80
0.135498	2.61e-03	2.17	5.75e-02	1.03	6.64e-03	2.03
0.070628	7.42e-04	1.93	3.14e-02	0.93	2.06e-03	1.79
0.036008	2.04e-04	1.92	1.67e-02	0.93	6.17e-04	1.79
0.018176	5.48e-05	1.92	8.77e-03	0.94	1.80e-04	1.80
0.009131	1.45e-05	1.93	4.56e-03	0.95	5.14e-05	1.82
0.004587	3.80e-06	1.95	2.35e-03	0.96	1.45e-05	1.84

Table 3.17: Discretization errors $e_h = y - y_h$ for Example 3.80 with $\omega = 3\pi/2$ and $\mu = 0.6$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	6.04e-02		2.43e-01		1.22e-01	
0.425046	2.58e-02	1.67	1.70e-01	0.70	5.46e-02	1.58
0.258029	9.66e-03	1.97	1.06e-01	0.95	2.09e-02	1.92
0.156360	3.01e-03	2.33	6.09e-02	1.11	6.65e-03	2.29
0.083008	8.33e-04	2.03	3.26e-02	0.99	1.90e-03	1.98
0.042742	2.20e-04	2.00	1.69e-02	0.99	5.18e-04	1.96
0.021687	5.69e-05	2.00	8.64e-03	0.99	1.37e-04	1.96
0.010923	1.45e-05	1.99	4.38e-03	0.99	3.56e-05	1.97
0.005496	3.66e-06	2.00	2.21e-03	1.00	9.13e-06	1.98

Table 3.18: Discretization errors $e_h = y - y_h$ for Example 3.80 with $\omega = 3\pi/2$ and $\mu = 0.5$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	8.60e-03		4.69e-02		1.69e-02	
0.250000	2.30e-03	1.90	2.60e-02	0.85	4.58e-03	1.88
0.125000	5.91e-04	1.96	1.36e-02	0.94	1.22e-03	1.90
0.062500	1.49e-04	1.98	6.92e-03	0.97	3.24e-04	1.92
0.031250	3.74e-05	2.00	3.49e-03	0.99	8.50e-05	1.93
0.015625	9.23e-06	2.02	1.74e-03	1.00	2.20e-05	1.95
0.007812	2.19e-06	2.07	8.60e-04	1.02	5.54e-06	1.99

Table 3.19: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.81 with $\omega = 2\pi/3$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.18e-02		6.62e-02		2.80e-02	
0.250000	3.06e-03	1.95	3.45e-02	0.94	7.43e-03	1.92
0.125000	7.86e-04	1.96	1.78e-02	0.95	2.00e-03	1.90
0.062500	2.00e-04	1.98	9.09e-03	0.97	5.41e-04	1.89
0.031250	5.02e-05	1.99	4.60e-03	0.98	1.47e-04	1.88
0.015625	1.24e-05	2.01	2.31e-03	1.00	3.99e-05	1.88
0.007812	2.94e-06	2.08	1.14e-03	1.02	1.07e-05	1.90

Table 3.20: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.81 with $\omega = 3\pi/4$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.79e-02		6.77e-02		3.92e-02	
0.250000	4.29e-03	2.06	3.36e-02	1.01	9.70e-03	2.02
0.125000	1.36e-03	1.66	1.67e-02	1.01	2.96e-03	1.71
0.062500	3.24e-04	2.07	8.34e-03	1.00	7.18e-04	2.05
0.031250	8.10e-05	2.00	4.18e-03	0.99	1.81e-04	1.99
0.015625	2.25e-05	1.85	2.08e-03	1.01	4.91e-05	1.88
0.007812	5.27e-06	2.09	1.02e-03	1.03	1.16e-05	2.09

Table 3.21: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.81 with $\omega = 3\pi/4$, $\mu = 0.83$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	3.06e-02		1.81e-01		6.17e-02	
0.250000	1.07e-02	1.51	1.07e-01	0.76	2.60e-02	1.24
0.125000	3.93e-03	1.45	6.46e-02	0.72	1.14e-02	1.19
0.062500	1.49e-03	1.40	3.97e-02	0.70	5.08e-03	1.17
0.031250	5.72e-04	1.38	2.46e-02	0.69	2.28e-03	1.16
0.015625	2.23e-04	1.36	1.53e-02	0.68	1.02e-03	1.16
0.007812	8.72e-05	1.35	9.59e-03	0.68	4.57e-04	1.16

Table 3.22: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.81 with $\omega = 3\pi/2$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	2.33e-02		1.56e-01		4.61e-02	
0.250000	5.94e-03	1.97	7.64e-02	1.03	1.22e-02	1.92
0.125000	1.56e-03	1.93	3.96e-02	0.95	3.50e-03	1.80
0.062500	4.29e-04	1.86	2.10e-02	0.91	1.04e-03	1.76
0.031250	1.08e-04	1.99	1.05e-02	1.00	2.65e-04	1.97
0.015625	2.77e-05	1.96	5.35e-03	0.97	7.43e-05	1.84
0.007812	7.09e-06	1.96	2.73e-03	0.97	2.11e-05	1.82

Table 3.23: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.81 with $\omega = 3\pi/2$, $\mu = 0.6$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|e_h\|_{L^2(\Omega_\omega)}$	eoc	$\|e_h\|_{H^1(\Omega_\omega)}$	eoc	$\|e_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	2.10e-02		1.41e-01		4.09e-02	
0.250000	5.46e-03	1.94	7.12e-02	0.98	1.07e-02	1.93
0.125000	1.41e-03	1.96	3.61e-02	0.98	2.76e-03	1.96
0.062500	3.57e-04	1.98	1.82e-02	0.98	7.09e-04	1.96
0.031250	8.96e-05	1.99	9.16e-03	0.99	1.81e-04	1.97
0.015625	2.23e-05	2.00	4.58e-03	1.00	4.53e-05	1.99
0.007812	5.39e-06	2.05	2.25e-03	1.02	1.10e-05	2.04

Table 3.24: Discretization errors $e_h = y_{ref} - y_h$ for Example 3.81 with $\omega = 3\pi/2$, $\mu = 0.5$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

CHAPTER 4

Neumann boundary control problems

In this chapter we investigate control constrained Neumann boundary control problems governed by linear and semilinear elliptic partial differential equations in polygonal domains. We focus on the derivation of error estimates for the concept of variational discretization and the postprocessing approach, each applied to linear and to semilinear problems. We start with the consideration of linear problems in Section 4.1. There, we introduce first order necessary optimality conditions, which are also sufficient for such problems, and regularity results in weighted Sobolev spaces for the solution. Afterwards, we prove in Section 4.2 quasi-optimal discretization error estimates on quasi-uniform and graded triangulations for both discretization strategies. Numerical experiments for the postprocessing approach, which confirm our theoretical findings, are presented in Section 4.2.3. Next, in Section 4.3, we discuss semilinear problems. As we will see, semilinear problems do in general not possess a unique global solution any longer in contrast to the linear ones. Furthermore, we are faced with locally optimal solutions. Therefore, we introduce in Section 4.3 not only first order necessary optimality conditions but also second order sufficient optimality conditions. Moreover, we prove for locally optimal solutions regularity results in weighted Sobolev spaces as in the linear case. The error analysis for the variational discretization concept and the postprocessing approach can be found in Section 4.4. For each approach we show in a preliminary step, that there is a certain mesh size, such that for every local solution of the continuous problem, which satisfies the second order sufficient optimality condition, there is a local solution of the respective discrete problem, which converges to the continuous solution. Based on this, we show afterwards that the concept of variational discretization and the postprocessing approach admit for semilinear problems the quasi-optimal convergence rates as well. Numerical experiments for the postprocessing can be found in Section 4.4.3. Let us remark that we require for the postprocessing approach in the semilinear case a slightly stronger structural assumption on the optimal control compared to the linear one. More precisely, for linear problems we will assume that the union of all elements, where the optimal control has kinks with the control constraints, is of order h. This assumption does not suffice for semilinear problems. In addition, we will need that the number of elements, where the locally optimal control intersects smoothly the control constraints, is finite, too.

Finally, we emphasize that the constant c denotes again a positive generic constant in the sequel which is independent of the mesh parameter. In contrast to the previous chapter, we do not track all dependencies of the constants on the data of the optimal control problems. More precisely, when discussing linear Neumann boundary control problems in Section 4.1, we only separate the desired state y_d and the optimal control $\bar{u}$ from the constants, whereas in Section 4.4 about semilinear Neumann boundary control problems, the constants may depend on all data of the problem. The main reason for this is to improve the readability, especially when discussing the semilinear problems. But, whenever it is necessary to know certain dependencies of the constants in order to ensure that they are independent of the mesh parameter, we will state them in detail. To understand the notation in that case, we refer to Remark 3.21.

4.1 Linear problems

In this section we analyze the following linear elliptic Neumann boundary control problem:

$$\begin{aligned} \text{Minimize} \quad & F(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \\ \text{subject to} \quad & u \in U_{ad} := \{u \in L^2(\Gamma) : u_a \le u \le u_b \text{ a.e. on } \Gamma\}, \\ & -\Delta y + \alpha y = 0 \quad \text{in } \Omega, \\ & \partial_n y = u \quad \text{on } \Gamma_j, \quad j = 1, \dots, m. \end{aligned} \tag{4.1}$$

We denote this optimal control problem by (P_l). The functional F is called cost functional, y_d denotes the desired state and y the state which is associated with the control u by the state equation (4.1). Furthermore, we call U_{ad} set of admissible controls or admissible set.

The precise conditions on the given quantities of problem (P_l) are collected in the following assumption, which we require to hold throughout Sections 4.1–4.2.

Assumption 4.1.

(A1) The domain Ω is a polygonal domain according to Definition 2.17 with m corner points and boundary $\Gamma = \bigcup_{j=1}^m \bar{\Gamma}_j$.

(A2) The function $y_d \in C^{0,\sigma}(\bar{\Omega})$ is given for some $\sigma > 0$.

(A3) The regularization parameter $\nu > 0$ and the control bounds $u_a < u_b$ are fixed real numbers.

(A4) The function α satisfies Assumption 3.1 (A2).

As usual in the context of PDE constrained optimization, we can also consider a reduced formulation of problem (P_l) by introducing the so-called control-to-state operator. Using Lemma 3.4 we can define a linear and continuous operator $G : L^2(\Gamma) \to H^1(\Omega)$ that associates an element $u \in L^2(\Gamma)$ with the unique weak solution $y \in H^1(\Omega)$ of (4.1). According to Theorem 2.7 there is a linear and continuous operator E_2 that maps a function $y \in H^1(\Omega)$ to the same function in $L^2(\Omega)$. Thus, we can introduce the linear and continuous operator $S : L^2(\Gamma) \to L^2(\Omega)$ by $S = E_2 G$. The reduced formulation of problem (P_l) is now given by

$$\min_{u \in U_{ad}} J(u) := F(Su, u) = \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \tag{4.2}$$

where J is called reduced cost functional. A control $\bar{u} \in U_{ad}$ is called optimal or solution of problem (P_l) with associated optimal state $\bar{y} := S\bar{u}$ if

$$J(\bar{u}) \le J(u) \quad \forall u \in U_{ad}.$$

Next, let us state the Fréchet derivative of the functional J given in (4.2). We will need this for the derivation of the optimality system below.

Lemma 4.2. *The functional $J : L^2(\Gamma) \to \mathbb{R}$ from* (4.2) *is Fréchet differentiable. Its derivative at $u \in L^2(\Gamma)$ in the direction $v \in L^2(\Gamma)$ is given by*

$$J'(u)v = (Su - y_d, Sv)_{L^2(\Omega)} + \nu(u, v)_{L^2(\Gamma)} = (S^*(Su - y_d) + \nu u, v)_{L^2(\Gamma)},$$

where $S^* : L^2(\Omega) \to L^2(\Gamma)$ denotes the adjoint operator of S.

Proof. The operator S is linear and continuous. Thus, the Fréchet derivative of the operator S at $u \in L^2(\Gamma)$ is simply S. Applying the chain rule yields the first equality of the assertion, cf. [107, Section 2.6]. The second one is obtained by the definition of adjoint operators. □

So far, we do not know how to implement the adjoint operator S^*. For that reason we introduce the partial differential equation

$$\begin{aligned} -\Delta p + \alpha p &= z \quad \text{in } \Omega, \\ \partial_n p &= 0 \quad \text{on } \Gamma_j, \quad j = 1, \dots, m. \end{aligned} \tag{4.3}$$

This problem is uniquely solvable in $H^1(\Omega)$ for every $z \in L^2(\Omega)$, cf. Lemma 3.4. Thus, we can define a linear and continuous operator $P : L^2(\Omega) \to H^1(\Omega)$ by $Pz := p$ where $p \in H^1(\Omega)$ is the weak solution of (4.3) associated with the right hand side $z \in L^2(\Omega)$. The next lemma relates the operator S^* to the operator P.

Lemma 4.3. *The adjoint operator S^* of S and the operator P satisfy*

$$(S^* z, u)_{L^2(\Gamma)} = (Pz, u)_{L^2(\Gamma)} \quad \forall z \in L^2(\Omega),\ \forall u \in L^2(\Gamma).$$

As a consequence there holds $(Pz)_{|\Gamma} = S^* z$.

Proof. Since Su and Pz are the weak solutions y and p of (4.1) and (4.3), respectively, we can conclude

$$(S^* z, u)_{L^2(\Gamma)} = (z, Su)_{L^2(\Omega)} = (z, y)_{L^2(\Omega)} = a(y, p) = (p, u)_{L^2(\Gamma)} = (Pz, u)_{L^2(\Gamma)}$$

for all $z \in L^2(\Omega)$ and $u \in L^2(\Gamma)$ and consequently $(Pz)_{|\Gamma} = S^* z$. □

Before we state the main theorem of this section let us define the adjoint state of a control $u \in L^2(\Gamma)$ by $P(Su - y_d)$ and the projection operator $\Pi_{[u_a,u_b]}\colon \mathbb{R} \to [u_a, u_b]$ by

$$\Pi_{[u_a,u_b]} f := \max(u_a, \min(u_b, f)).$$

Theorem 4.4. *The optimal control problem (P_l) has a unique solution $\bar{u} \in U_{ad}$. Let $\bar{y} = S\bar{u}$ and $\bar{p} = P(S\bar{u} - y_d)$ be the state and adjoint state, respectively, associated with $\bar{u}$. Then the variational inequality*

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad} \tag{4.4}$$

is satisfied, which can be expressed equivalently by

$$\bar{u}(x) = \Pi_{[u_a,u_b]} \left(-\frac{1}{\nu} \bar{p}(x) \right) \quad \text{for a.a. } x \in \Gamma. \tag{4.5}$$

Moreover, let β_j, γ_j, τ_j and κ_j satisfy the conditions

$$1 > \beta_j > \max(0, 1 - \lambda_j) \quad \text{or} \quad \beta_j = 0 \text{ and } 1 - \lambda_j < 0, \tag{4.6}$$
$$2 > \gamma_j > \max(0, 2 - \lambda_j) \quad \text{or} \quad \gamma_j = 0 \text{ and } 2 - \lambda_j < 0, \tag{4.7}$$
$$1 > \tau_j > \max(0, 1 - \lambda_j) \quad \text{or} \quad \tau_j = 0 \text{ and } 1 - \lambda_j < 0, \tag{4.8}$$
$$3/2 > \kappa_j > \max(-1/2, 3/2 - \lambda_j) \tag{4.9}$$

for each $j \in \{1, \ldots, m\}$. Then $\bar{y}$ and $\bar{p}$ fulfill the a priori estimates

$$\begin{aligned} &\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} + \|\bar{p}\|_{W^{1,\infty}_{\vec{\tau}}(\Gamma)} + \|\bar{p}\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \\ &\leq c \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right). \end{aligned}$$

Proof. First, we observe that the admissible set $U_{ad} \subset L^2(\Gamma)$ is non-empty, bounded, closed and convex. Furthermore, the operator S is linear and continuous from $L^2(\Gamma)$ to $L^2(\Omega)$. Thus, the existence and uniqueness of a solution $\bar{u} \in U_{ad}$ of problem (P_l) can be deduced from e.g. [107, Theorem 2.14]. According to Lemma 4.2 the functional J is Fréchet differentiable in an open subset of $L^2(\Gamma)$ which includes the convex set U_{ad}. Therefore, the variational inequality (4.4) represents the necessary optimality condition which is also sufficient due to the strict convexity of J, see e.g. Lemma 2.21 of [107]. The equivalence of the variational inequality (4.4) and the projection formula (4.5) can be found e.g. in [27] or [107, Section 2.8.4]. To prove the assertion on the regularity and the a priori estimates we start with the optimal control $\bar{u}$ in $L^2(\Gamma)$ which implies according to Corollary 3.6 that there is a $t \in (1, 3/2)$ such that $\bar{y} \in H^t(\Omega)$. Furthermore the a priori estimate

$$\|\bar{y}\|_{H^t(\Omega)} \leq c\|\bar{u}\|_{L^2(\Gamma)} \tag{4.10}$$

is valid. For every $\vec{\beta}$ satisfying (4.6) there holds $H^t(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{0,2}_{\vec{\beta}}(\Omega)$ and consequently we can conclude $\bar{y} - y_d \in W^{0,2}_{\vec{\beta}}(\Omega)$. Therefore, we obtain from Lemma 2.29 and Lemma 3.11 $\bar{p} = P(\bar{y} - y_d) \in W^{2,2}_{\vec{\beta}}(\Omega) \hookrightarrow H^1(\Omega)$ and

$$\|\bar{p}\|_{H^1(\Omega)} \leq c\|\bar{p}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c\|\bar{y} - y_d\|_{W^{0,2}_{\vec{\beta}}(\Omega)} \leq c \left(\|\bar{y}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} \right). \tag{4.11}$$

Furthermore, Theorem 2.12 implies $\bar{p}|_\Gamma \in H^{1/2}(\Gamma)$ and

$$\|\bar{p}\|_{H^{1/2}(\Gamma)} \leq c\|\bar{p}\|_{H^1(\Omega)}. \tag{4.12}$$

Since the optimal control $\bar{u}$ is related to the optimal adjoint state $\bar{p}$ via the projection formula (4.5), we obtain

$$\begin{aligned}
\|\bar{u}\|_{H^{1/2}(\Gamma)} &\leq \|\bar{u}\|_{L^2(\Gamma)} + \left(\int_\Gamma\int_\Gamma \frac{|\bar{u}(x_1)-\bar{u}(x_2)|^2}{|x_1-x_2|^2}\,\mathrm{d}s_{x_1}\,\mathrm{d}s_{x_2}\right)^{1/2} \\
&= \|\bar{u}\|_{L^2(\Gamma)} + \left(\int_\Gamma\int_\Gamma \frac{|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}(x_1)\right) - \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}(x_2)\right)|^2}{|x_1-x_2|^2}\,\mathrm{d}s_{x_1}\,\mathrm{d}s_{x_2}\right)^{1/2} \\
&\leq \|\bar{u}\|_{L^2(\Gamma)} + c\left(\int_\Gamma\int_\Gamma \frac{|\bar{p}(x_1)-\bar{p}(x_2)|^2}{|x_1-x_2|^2}\,\mathrm{d}s_{x_1}\,\mathrm{d}s_{x_2}\right)^{1/2}.
\end{aligned}$$

The last step can easily be verified, if one distinguishes the nine cases $-\bar{p}(x_1)/\nu < u_a \ \wedge\ -\bar{p}(x_2)/\nu < u_a$, $-\bar{p}(x_1)/\nu < u_a \ \wedge\ u_a \leq -\bar{p}(x_2)/\nu \leq u_b$, $-\bar{p}(x_1)/\nu < u_a \ \wedge\ -\bar{p}(x_2)/\nu > u_b$, $u_a \leq -\bar{p}(x_1)/\nu \leq u_b \ \wedge\ -\bar{p}(x_2)/\nu < u_a$, etc. Thus, we have

$$\|\bar{u}\|_{H^{1/2}(\Gamma)} \leq \|\bar{u}\|_{L^2(\Gamma)} + c\|\bar{p}\|_{H^{1/2}(\Gamma)}. \tag{4.13}$$

The embedding $H^{1/2}(\Gamma) \hookrightarrow W^{1/2,2}_{\vec{\beta}}(\Gamma)$, which is definitely valid for $\vec{\beta}$ satisfying (4.6), yields together with Lemma 3.11 that $\bar{y} \in W^{2,2}_{\vec{\beta}}(\Omega)$ and

$$\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c\|\bar{u}\|_{W^{1/2,2}_{\vec{\beta}}(\Gamma)} \leq c\|\bar{u}\|_{H^{1/2}(\Gamma)}. \tag{4.14}$$

Furthermore, we know according to Theorem 2.7 and Lemma 2.36

$$H^t(\Omega) \hookrightarrow C^{0,t-1}(\bar{\Omega}) \hookrightarrow N^{0,t-1}_{\vec{t}-\vec{1}}(\Omega) \tag{4.15}$$

and

$$C^{0,\sigma}(\bar{\Omega}) \hookrightarrow N^{0,\sigma}_{\vec{\sigma}}(\Omega).$$

Thus, Lemma 3.13, Corollary 3.14 and Corollary 3.15 imply

$$\begin{aligned}
&\|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} + \|\bar{p}\|_{W^{1,\infty}_{\vec{\tau}}(\Gamma)} + \|\bar{p}\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \\
&\qquad \leq c\|\bar{y}-y_d\|_{C^{0,\min(t-1,\sigma)}(\bar{\Omega})} \leq c\left(\|\bar{y}\|_{C^{0,t-1}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)
\end{aligned} \tag{4.16}$$

for $\vec{\gamma}$, $\vec{\tau}$ and $\vec{\kappa}$ satisfying (4.7), (4.8) and (4.9), respectively. Finally, (4.10)–(4.16) and Theorem 2.7 yield the desired result. □

4.2 Discretization and error estimates for linear problems

In this section we will consider the concept of variational discretization and the postprocessing approach. But first, let us introduce the discrete versions of (4.1) and (4.3), and based on this the discrete version of the reduced cost functional J.

In Section 3.2.1 we have already introduced the space V_h as the space consisting of piecewise linear and continuous functions. A discrete solution of the state equation (4.1) is an element $y_h \in V_h$ that satisfies

$$a(y_h, v_h) = (u, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h \tag{4.17}$$

with some $u \in L^2(\Gamma)$ and the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ from (3.3). According to Lemma 3.40 the discrete state equation possesses a unique solution in $V_h \subset H^1(\Omega)$ for every $u \in L^2(\Gamma)$. Therefore, we can introduce the linear and continuous discrete control-to-state operator $G_h : L^2(\Gamma) \to H^1(\Omega)$ which maps a control $u \in L^2(\Gamma)$ to $G_h u := y_h$ via (4.17). Moreover, we define the discrete analogon $S_h : L^2(\Gamma) \to L^2(\Omega)$ of the operator S by $S_h = E_2 G_h$. The discrete reduced cost functional $J_h : L^2(\Gamma) \to \mathbb{R}$ is now given by

$$J_h(u) := F(S_h u, u) = \frac{1}{2}\|S_h u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|_{L^2(\Gamma)}. \tag{4.18}$$

We also introduce the discrete solution of (4.3) as the unique element $p_h \in V_h$ that fulfills

$$a(v_h, p_h) = (z, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h \tag{4.19}$$

with some $z \in L^2(\Omega)$. Arguing as for the state we can define the discrete version $P_h : L^2(\Omega) \to H^1(\Omega)$ of the solution operator P by $P_h z := p_h$. The discrete adjoint state of a control $u \in L^2(\Gamma)$ is the unique element $P_h(S_h u - y_d) \in V_h$. Analogously to the continuous case we can show the following two lemmas.

Lemma 4.5. *The functional $J_h : L^2(\Gamma) \to \mathbb{R}$ from (4.18) is Fréchet differentiable. Its derivative at $u \in L^2(\Gamma)$ in the direction $v \in L^2(\Gamma)$ is given by*

$$J_h'(u)v = (S_h u - y_d, S_h v)_{L^2(\Omega)} + \nu(u, v)_{L^2(\Gamma)} = (S_h^*(S_h u - y_d) + \nu u, v)_{L^2(\Gamma)},$$

where $S_h^* : L^2(\Omega) \to L^2(\Gamma)$ denotes the adjoint operator of S_h.

Lemma 4.6. *The adjoint operator S_h^* of S_h and the operator P_h satisfy*

$$(S_h^* z, u)_{L^2(\Gamma)} = (P_h z, u)_{L^2(\Gamma)} \quad \forall z \in L^2(\Omega),\ \forall u \in L^2(\Gamma)$$

and consequently $(P_h z)_{|\Gamma} = S_h^* z$.

The following lemma will simplify the discussions in the sequel.

Lemma 4.7. *Let $v \in L^2(\Gamma)$ and $z \in L^2(\Omega)$. The discrete solution operators S_h, P_h and S_h^* admit for $\vec{0} < \vec{\mu} \le \vec{1}$ the estimates*

$$\begin{aligned}
\|S_h v\|_{L^2(\Omega)} &\le c\|v\|_{L^2(\Gamma)}, \\
\|P_h z\|_{L^2(\Omega)} &\le c\|z\|_{L^2(\Omega)}, \\
\|S_h^* z\|_{L^2(\Gamma)} &\le c\|z\|_{L^2(\Omega)}.
\end{aligned}$$

Proof. One obtains the validity of the first and second inequality from the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and Lemma 3.40. The third one is a consequence of Lemma 4.6, the trace Theorem 2.8 and Lemma 3.40, i.e.,

$$\|S_h^* z\|_{L^2(\Gamma)} = \|P_h z\|_{L^2(\Gamma)} \le c\|P_h z\|_{H^1(\Omega)} \le c\|z\|_{L^2(\Omega)}.$$

□

4.2.1 Error estimates for the concept of variational discretization

The concept of variational discretization was first presented in [60] for distributed control problems and in [26] for Neumann boundary control problems, cf. also [77] and [61]. This discretization concept is based on a discretization of the state according to (4.17), whereas the control is considered as a general function in the continuous admissible set U_{ad}. The discretized optimal control problem reads in reduced form as follows:

$$\min_{u \in U_{ad}} J_h(u). \tag{4.20}$$

Using Lemma 4.5 and Lemma 4.6 we can show the following assertion as in the continuous case.

Lemma 4.8. *The discrete optimal control problem* (4.20) *has a unique solution $\bar{u}_h \in U_{ad}$. Let $\bar{y}_h = S_h\bar{u}_h$ and $\bar{p}_h = P_h(S_h\bar{u}_h - y_d)$ be the discrete state and discrete adjoint state, respectively, associated with $\bar{u}_h$. Then the variational inequality*

$$(\bar{p}_h + \nu\bar{u}_h, u - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad} \tag{4.21}$$

is satisfied.

Next, we are going to derive error estimates for the concept of variational discretization. The following lemma provides a general error estimate for this concept.

Lemma 4.9. *The estimate*

$$\nu\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq \|(S^* - S_h^*)(S\bar{u} - y_d)\|_{L^2(\Gamma)} + c\|(S - S_h)\bar{u}\|_{L^2(\Omega)} \tag{4.22}$$

is valid.

Proof. The proof is given in e.g. Section 7 of [77]. It also holds for graded meshes. We state it for the sake of completeness. On the one hand we can test the continuous optimality condition (4.4) with $\bar{u}_h \in U_{ad}$, on the other hand we can test the semi-discrete optimality condition (4.21) with $\bar{u} \in U_{ad}$. Adding these two inequalities we obtain

$$(\bar{p} - \bar{p}_h + \nu(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u})_{L^2(\Gamma)} \geq 0.$$

Thus, we get

$$\nu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (\bar{p} - \bar{p}_h, \bar{u}_h - \bar{u})_{L^2(\Gamma)}.$$

After inserting $S_h^*(S_h\bar{u} - y_d)$ as intermediate function we can continue with

$$\nu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (\bar{p} - S_h^*(S_h\bar{u} - y_d), \bar{u}_h - \bar{u})_{L^2(\Gamma)} + (S_h^*(S_h\bar{u} - y_d) - \bar{p}_h, \bar{u}_h - \bar{u})_{L^2(\Gamma)}. \tag{4.23}$$

Now we estimate both terms separately. According to Lemma 4.6 and the definition of adjoint operators, we get for the second one

$$\begin{aligned} (S_h^*(S_h\bar{u} - y_d) - \bar{p}_h, \bar{u}_h - \bar{u})_{L^2(\Gamma)} &= (S_h^* S_h(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u})_{L^2(\Gamma)} \\ &= (S_h(\bar{u} - \bar{u}_h), S_h(\bar{u}_h - \bar{u}))_{L^2(\Omega)} \\ &= -\|S_h(\bar{u} - \bar{u}_h)\|_{L^2(\Omega)} \leq 0. \end{aligned} \tag{4.24}$$

To estimate the first term in equation (4.23) we apply the Cauchy-Schwarz inequality to obtain

$$(\bar p - S_h^*(S_h\bar u - y_d), \bar u_h - \bar u)_{L^2(\Gamma)} \le \|\bar p - S_h^*(S_h\bar u - y_d)\|_{L^2(\Gamma)}\|\bar u_h - \bar u\|_{L^2(\Gamma)}. \quad (4.25)$$

Merging (4.23), (4.24) and (4.25) together and introducing the intermediate function $S_h^*(S\bar u - y_d)$ we get

$$\begin{aligned}\nu\|\bar u - \bar u_h\|_{L^2(\Gamma)} &\le \|\bar p - S_h^*(S_h\bar u - y_d)\|_{L^2(\Gamma)} = \|S^*(S\bar u - y_d) - S_h^*(S_h\bar u - y_d)\|_{L^2(\Gamma)}\\ &\le \|(S^* - S_h^*)(S\bar u - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)\bar u\|_{L^2(\Gamma)}. \quad (4.26)\end{aligned}$$

Using the continuity of the operator S_h^* according to Lemma 4.7 we can continue with

$$\|S_h^*(S - S_h)\bar u\|_{L^2(\Gamma)} \le c\|(S - S_h)\bar u\|_{L^2(\Omega)}. \quad (4.27)$$

Finally, (4.26) and (4.27) yield the assertion. □

Let us state the main result of this section, the quasi-optimal convergence rates for the concept of variational discretization.

Theorem 4.10. *Let the mesh grading parameters $\vec\mu$ are chosen such that $\vec 1/4 < \vec\mu < \vec 1/4 + \vec\lambda/2$. Then the discretization error estimates*

$$\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right), \quad (4.28)$$

$$\|\bar y - \bar y_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right), \quad (4.29)$$

$$\|\bar p - \bar p_h\|_{L^2(\Gamma)} + \|\bar p - \bar p_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right) \quad (4.30)$$

hold.

Proof. We get from (4.22)

$$\nu\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le \|(S^* - S_h^*)(S\bar u - y_d)\|_{L^2(\Gamma)} + c\|(S - S_h)\bar u\|_{L^2(\Omega)}. \quad (4.31)$$

Having regard to Lemma 4.6 we can observe that the first term in (4.31) is nothing else than the finite element error of the adjoint state in $L^2(\Gamma)$. We get with Theorem 3.48 and the regularity results of Theorem 4.4

$$\|(S^* - S_h^*)(S\bar u - y_d)\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\|\bar p\|_{W^{2,\infty}_{\vec 5/2 - 2\vec\mu}(\Omega)} \le ch^2|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right), \quad (4.32)$$

provided that $\vec 1/4 < \vec\mu < \vec 1/4 + \vec\lambda/2$. The second term in (4.31) is the finite element error of the state in $L^2(\Omega)$. Using Lemma 3.41 and Theorem 4.4 we can conclude with $\vec\mu < \vec\lambda$ that

$$\|(S - S_h)\bar u\|_{L^2(\Omega)} \le ch^2\|\bar y\|_{W^{2,2}_{\vec 1 - \vec\mu}(\Omega)} \le ch^2\left(\|\bar u\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right). \quad (4.33)$$

The estimates (4.31), (4.32) and (4.33) yield the desired estimate for the control. To get the estimate for the state, we introduce the intermediate function $S_h\bar u$, which yields

$$\begin{aligned}\|\bar y - \bar y_h\|_{L^2(\Omega)} &= \|S\bar u - S_h\bar u_h\|_{L^2(\Omega)} \le \|(S - S_h)\bar u\|_{L^2(\Omega)} + \|S_h(\bar u - \bar u_h)\|_{L^2(\Omega)}\\ &\le \|(S - S_h)\bar u\|_{L^2(\Omega)} + c\|\bar u - \bar u_h\|_{L^2(\Gamma)}, \quad (4.34)\end{aligned}$$

where we used the continuity of the operator S_h in the last step, cf. Lemma 4.7. We get the validity of the second assertion by inserting the estimates (4.33) and (4.28) into (4.34). Finally, we consider the error of the adjoint state on the boundary and in the domain. By introducing appropriate intermediate functions we can conclude

$$\begin{aligned}\|\bar p-\bar p_h\|_{L^2(\Gamma)}&+\|\bar p-\bar p_h\|_{L^2(\Omega)}\\&\le\|(S^*-S_h^*)(\bar y-y_d)\|_{L^2(\Gamma)}+\|S_h^*(\bar y-\bar y_h)\|_{L^2(\Gamma)}\\&\quad+\|(P-P_h)(\bar y-y_d)\|_{L^2(\Omega)}+\|P_h(\bar y-\bar y_h)\|_{L^2(\Omega)}\\&\le\|(S^*-S_h^*)(\bar y-y_d)\|_{L^2(\Gamma)}+\|(P-P_h)(\bar y-y_d)\|_{L^2(\Omega)}+c\|\bar y-\bar y_h\|_{L^2(\Omega)},\end{aligned}\tag{4.35}$$

where we used the continuity of the operators S_h^* and P_h according to Lemma 4.7. The first and the third term on the right hand side of (4.35) have already been estimated in (4.32) and (4.29). We can argue for the second term as for (4.33). This yields for $\vec\mu<\vec\lambda$

$$\|(P-P_h)(\bar y-y_d)\|_{L^2(\Omega)}\le ch^2\|\bar p\|_{W^{2,2}_{\vec 1-\vec\mu}(\Omega)}\le ch^2\left(\|\bar u\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right),$$

which ends the proof. □

Using Corollary 3.42 and Corollary 3.49 instead of Lemma 3.41 and Theorem 3.48, respectively, in the proof of Theorem 4.10, we get the following corollary.

Corollary 4.11. *Let $\vec\mu=\vec 1$ (quasi-uniform mesh), $\vec 0<\vec\epsilon<\vec\lambda$, and $\rho=\min(2,\min(\vec 1/2+\vec\lambda-\vec\epsilon))$. Then the discretization error estimates*

$$\begin{aligned}\|\bar u-\bar u_h\|_{L^2(\Gamma)}&\le ch^\rho|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right),\\\|\bar y-\bar y_h\|_{L^2(\Omega)}&\le ch^\rho|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right),\\\|\bar p-\bar p_h\|_{L^2(\Gamma)}+\|\bar p-\bar p_h\|_{L^2(\Omega)}&\le ch^\rho|\ln h|^{3/2}\left(\|\bar u\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right)\end{aligned}$$

are valid.

4.2.2 Error estimates for the postprocessing approach

This section is devoted to the postprocessing approach, which was first introduced in [87] for distributed control problems and in [77] for Neumann boundary control problems. This approach relies on a full discretization of the optimal control problem (P_l). Only in a post-processing step a new control is computed which possesses superconvergence properties. As for the concept of variational discretization we will approximate the state according to (4.17). The control will be approximated by piecewise constant functions, i.e., the discrete controls u_h will belong to U_h in general, see Section 3.2.1 for the definition of the space U_h. Furthermore, we want that the discrete controls fulfill the pointwise inequality constraints. For that reason, we introduce the discrete admissible set U_h^{ad} by

$$U_h^{ad}:=U_h\cap U_{ad}.$$

The fully discretized version of the optimal control problem (P_l) can now be stated as

$$\min_{u_h \in U_h^{ad}} J_h(u_h). \tag{4.36}$$

Using Lemma 4.5 and Lemma 4.6 we can argue as in the continuous case to conclude the following assertion with regard to existence and uniqueness of a solution of (4.36). Actually, one can also use arguments from finite dimensional optimization, since problem (4.36) is completely finite dimensional.

Lemma 4.12. *The discrete optimal control problem* (4.36) *admits a unique solution* $\bar{u}_h \in U_h^{ad}$*. Let* $\bar{y}_h = S_h\bar{u}_h$ *and* $\bar{p}_h = P_h(S_h\bar{u}_h - y_d)$ *be the discrete state and discrete adjoint state, respectively, associated with* $\bar{u}_h$*. Then the discrete variational inequality*

$$(\bar{p}_h + \nu\bar{u}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \qquad \forall u_h \in U_h^{ad} \tag{4.37}$$

is fulfilled.

Based on the results of Section 3.2 we first analyze the fully discrete optimal control problem (4.36) with respect to its discretization error. Afterwards on page 116 we construct in a postprocessing step a new control and prove its superconvergence properties. However, the better approximation properties rely on a structural assumption for the optimal control $\bar{u}$ of problem (P_l), which we are going to state first. Let

$$K_1 := \bigcup_{E\in\mathcal{E}_h : \bar{u}\notin W^{2,2}_{2(\vec{1}-\vec{\mu})}(E)} E, \qquad K_2 := \bigcup_{E\in\mathcal{E}_h : \bar{u}\in W^{2,2}_{2(\vec{1}-\vec{\mu})}(E)} E.$$

Assumption 4.13. Let $|K_1| \leq ch$ with a positive constant c independent of h.

Remark 4.14. This assumption is satisfied in many practical applications. For example it is fulfilled if the optimal control $\bar{u}$ has only a finite number of kinks due to the projection on the interval $[u_a, u_b]$. See Section 4 in [77] for a more sophisticated discussion on its validity.

Next, we are going to prove some auxiliary estimates, which are needed for the main result of this section on page 116. The operator R_h, which appears in the sequel, has already been defined on page 55.

Lemma 4.15. *Let Assumption 4.13 be satisfied. Then the estimate*

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

is valid, provided that the mesh parameters $\vec{\mu}$ *are chosen such that* $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$*.*

Proof. First, we introduce the function $S(\bar{u} - R_h\bar{u})$ and apply the triangle inequality. This yields

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \leq \|(S_h - S)(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)}. \tag{4.38}$$

Using the finite element error estimates in the domain from Lemma 3.41, the continuity of the operators S and S_h from $L^2(\Gamma)$ to $H^1(\Omega)$ according to Lemma 3.4 and Lemma 3.40, respectively, we can conclude for the first term in (4.38) with $\vec{\mu} < \vec{\lambda}$

$$\begin{aligned}\|(S - S_h)(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} &\leq ch\|(S - S_h)(\bar{u} - R_h\bar{u})\|_{H^1(\Omega)} \\ &\leq ch\|\bar{u} - R_h\bar{u}\|_{L^2(\Gamma)} \leq ch^2|\bar{u}|_{H^1(\Gamma)}, \end{aligned} \tag{4.39}$$

where we additionally used Corollary 3.35 and $h_E \leq ch$ in the last step, cf. (3.70). Next, let $z = S(\bar{u} - R_h\bar{u})$. Then we get for the second term in (4.38) using Lemma 4.3

$$\begin{aligned}\|S(\bar{u} - R_h\bar{u})\|^2_{L^2(\Omega)} &= (S(\bar{u} - R_h\bar{u}), z)_{L^2(\Omega)} = (\bar{u} - R_h\bar{u}, S^*z)_{L^2(\Gamma)} = (\bar{u} - R_h\bar{u}, Pz)_{L^2(\Gamma)} \\ &= (\bar{u} - Q_h\bar{u}, Pz)_{L^2(\Gamma)} + (Q_h\bar{u} - R_h\bar{u}, Pz)_{L^2(\Gamma)}, \end{aligned} \tag{4.40}$$

where we introduced the intermediate function $Q_h\bar{u}$. Again, we estimate both terms in (4.40) separately. One obtains for the first term with Corollary 3.38 and $h_E \leq ch$

$$(\bar{u} - Q_h\bar{u}, Pz)_{L^2(\Gamma)} = \sum_{E\in\mathcal{E}_h} (\bar{u} - Q_h\bar{u}, Pz)_{L^2(E)} \leq c \sum_{E\in\mathcal{E}_h} h^2|\bar{u}|_{H^1(E)}|Pz|_{H^1(E)}.$$

Next we apply the discrete Cauchy-Schwarz inequality, Theorem 2.16, Lemma 2.29 and the a priori estimate from Lemma 3.11. This yields

$$\begin{aligned}(\bar{u} - Q_h\bar{u}, Pz)_{L^2(\Gamma)} &\leq ch^2|\bar{u}|_{H^1(\Gamma)}|Pz|_{H^1(\Gamma)} \leq ch^2|\bar{u}|_{H^1(\Gamma)}\|Pz\|_{W^{2,4/3}(\Omega)} \\ &\leq ch^2|\bar{u}|_{H^1(\Gamma)}\|Pz\|_{W^{2,2}_{\vec{1}/2-\vec{\epsilon}}(\Omega)} \leq ch^2|\bar{u}|_{H^1(\Gamma)}\|z\|_{W^{0,2}_{\vec{1}/2-\vec{\epsilon}}(\Omega)} \\ &\leq ch^2|\bar{u}|_{H^1(\Gamma)}\|z\|_{L^2(\Omega)}, \end{aligned} \tag{4.41}$$

which holds for $\vec{0} < \vec{\epsilon} < \vec{1}/2 - \max(0, \vec{1} - \vec{\lambda})$. For the second term in (4.40) we get with the Hölder inequality

$$(Q_h\bar{u} - R_h\bar{u}, Pz)_{L^2(\Gamma)} \leq \|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)}\|Pz\|_{L^\infty(\Gamma)} \leq c\|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)}\|z\|_{L^2(\Omega)}, \tag{4.42}$$

where we additionally used the embedding $H^1(\Gamma) \hookrightarrow L^\infty(\Gamma)$ according to Theorem 2.7 and $\|Pz\|_{H^1(\Gamma)} \leq c\|z\|_{L^2(\Omega)}$ as in (4.41). Since $R_h\bar{u}$ is constant on every element E we can continue with

$$\begin{aligned}\|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)} &= \|Q_h\left(\bar{u} - R_h\bar{u}\right)\|_{L^1(\Gamma)} = \sum_{E\in\mathcal{E}_h} \left|\int_E (\bar{u} - R_h\bar{u})\right| \\ &= \sum_{j=0}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} \left|\int_E (\bar{u} - R_h\bar{u})\right| + \sum_{j=0}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} \left|\int_E (\bar{u} - R_h\bar{u})\right|. \end{aligned} \tag{4.43}$$

Using Corollary 3.35 we get for $\mu_j > 1/4$

$$\begin{aligned}
\|Q_h\bar u - R_h\bar u\|_{L^1(\Gamma)} &\le c\Bigg(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_1}} h|E||\bar u|_{W^{1,\infty}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} h|E||\bar u|_{W^{1,\infty}_{1-\mu_j}(E)} \\
&\quad + \sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_2}} h^2|E|^{1/2}|\bar u|_{W^{2,2}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} h^2|E|^{1/2}|\bar u|_{W^{2,2}_{2(1-\mu_j)}(E)}\Bigg) \\
&\le ch|K_1|\left(|\bar u|_{W^{1,\infty}(K_1\cap\Gamma^0)} + \sum_{j=1}^m |\bar u|_{W^{1,\infty}_{1-\mu_j}(K_1\cap\Gamma_j^\pm)}\right) \\
&\quad + ch^2|K_2|^{1/2}\left(|\bar u|_{W^{2,2}(K_2\cap\Gamma^0)} + \sum_{j=1}^m |\bar u|_{W^{2,2}_{2(1-\mu_j)}(K_2\cap\Gamma_j^\pm)}\right) \\
&\le ch^2\left(|\bar u|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1)} + |\bar u|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2)}\right), \qquad (4.44)
\end{aligned}$$

where we used the discrete Cauchy-Schwarz inequality and Assumption 4.13. Collecting the results from the inequalities (4.38), (4.39), (4.40), (4.41), (4.42) and (4.44) yields

$$\|S_h(\bar u - R_h\bar u)\|_{L^2(\Omega)} \le ch^2\left(|\bar u|_{H^1(\Gamma)} + |\bar u|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1)} + |\bar u|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2)}\right). \qquad (4.45)$$

Next, we take into account that $\bar u$ is given by the projection formula (4.5). We divide the boundary Γ into the boundary parts $\mathcal{I}$, where $\bar u = -\bar p/\nu$, and $\mathcal{A}$, where $\bar u = u_a$ or $\bar u = u_b$. We obtain

$$\begin{aligned}
|\bar u|_{H^1(\Gamma)} &= \left|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar p\right)\right|_{H^1(\Gamma)} \\
&\le \left|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar p\right)\right|_{H^1(\mathcal{I})} + \left|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar p\right)\right|_{H^1(\mathcal{A})} \\
&\le c|\bar p|_{H^1(\mathcal{I})} \le c|\bar p|_{W^{1,2}_{\vec 0}(\Gamma)} \le c\|\bar p\|_{W^{2,2}_{\vec 1}(\Gamma)}, \qquad (4.46)
\end{aligned}$$

see also [69, Theorem A.1]. The last step holds due to the embedding $W^{2,2}_{\vec 1}(\Gamma) \hookrightarrow W^{1,2}_{\vec 0}(\Gamma)$, cf. Lemma 2.29. Analogously we get

$$\begin{aligned}
|\bar u|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1)} + |\bar u|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2)} &\le c\left(|\bar p|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1\cap\mathcal{I})} + |\bar p|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2\cap\mathcal{I})}\right) \\
&\le c\left(|\bar p|_{W^{1,\infty}_{\vec 1-\vec\mu}(\Gamma)} + |\bar p|_{W^{2,2}_{2(\vec 1-\vec\mu)}(\Gamma)}\right). \qquad (4.47)
\end{aligned}$$

In summary one obtains from the inequalities (4.45), (4.46) and (4.47)

$$\|S_h(\bar u - R_h\bar u)\|_{L^2(\Omega)} \le ch^2\left(\|\bar p\|_{W^{2,2}_{\vec 1}(\Gamma)} + |\bar p|_{W^{1,\infty}_{\vec 1-\vec\mu}(\Gamma)} + |\bar p|_{W^{2,2}_{2(\vec 1-\vec\mu)}(\Gamma)}\right).$$

Finally, the regularity results of Theorem 4.4 imply for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \le ch^2 \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right).$$

□

Corollary 4.16. *Let Assumption 4.13 be satisfied. Furthermore, let $\vec{\mu} = \vec{1}$ (quasi-uniform mesh) and $\vec{0} < \vec{\epsilon} < \vec{\lambda}$. Then the estimate*

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \le ch^\rho \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

is valid with $\rho = \min(2, \min(\vec{1}/2 + \vec{\lambda} - \vec{\epsilon}))$.

Proof. The proof follows the same steps as the proof of Lemma 4.15. We point out the differences only. Instead of (4.39) we conclude using the finite element error estimates in the domain from Corollary 3.42 that

$$\begin{aligned}\|(S - S_h)(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} &\le ch^\lambda \|(S - S_h)(\bar{u} - R_h\bar{u})\|_{H^1(\Omega)} \\ &\le ch^\lambda \|\bar{u} - R_h\bar{u}\|_{L^2(\Gamma)} \le ch^{1+\lambda}|\bar{u}|_{H^1(\Gamma)}\end{aligned}$$

with $\lambda = \min(1, \min(\vec{\lambda} - \vec{\epsilon}))$. Next, let $\vec{\kappa} = \vec{3}/2 - \vec{\lambda} + \vec{\epsilon}$ and $\vec{\kappa} \ge \vec{0}$ and let $\vec{\tau} = \vec{1} - \vec{\lambda} + \vec{\epsilon}$ and $\vec{\tau} \ge \vec{0}$. Then using Corollary 3.35 with $\vec{\mu} = \vec{1}$ the inequality (4.44) can be replaced by

$$\begin{aligned}\|Q_h\bar{u} - R_h\bar{u}\|_{L^1(\Gamma)} \le c \Bigg(&\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_1}} h|E||\bar{u}|_{W^{1,\infty}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} h^{\min(1,\lambda_j-\epsilon)}|E||\bar{u}|_{W^{1,\infty}_{\tau_j}(E)} \\ &+ \sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_2}} h^2|E|^{1/2}|\bar{u}|_{W^{2,2}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} h^{\min(2,1/2+\lambda_j-\epsilon))}|E|^{1/2}|\bar{u}|_{W^{2,2}_{\kappa_j}(E)} \Bigg) \\ &\le ch^{\min(1,\min(\vec{\lambda}-\vec{\epsilon}))}|K_1| \left(|\bar{u}|_{W^{1,\infty}(K_1\cap\Gamma^0)} + \sum_{j=1}^m |\bar{u}|_{W^{1,\infty}_{\tau_j}(K_1\cap\Gamma_j^\pm)}\right) \\ &+ ch^{\min(2,\min(\vec{1}/2+\vec{\lambda}-\vec{\epsilon}))}|K_2|^{1/2} \left(|\bar{u}|_{W^{2,2}(K_2\cap\Gamma^0)} + \sum_{j=1}^m |\bar{u}|_{W^{2,2}_{\kappa_j}(K_2\cap\Gamma_j^\pm)}\right) \\ &\le ch^\rho \left(|\bar{u}|_{W^{1,\infty}_{\vec{\tau}}(K_1)} + |\bar{u}|_{W^{2,2}_{\vec{\kappa}}(K_2)}\right),\end{aligned} \tag{4.48}$$

where we used the same arguments as for (4.44). All other steps remain unchanged. □

The following lemma can be interpreted as counterpart of Lemma 4.9. The concept of variational discretization benefits from the fact, that the admissible sets coincide for the continuous and discrete problem. In the context of the postprocessing approach this does not hold. We have to deal with the discrepancy between the continuous admissible set U_{ad} and the discrete admissible set U_h^{ad}. For that reason the piecewise constant function $R_h\bar{u}$ appears, which belongs to the discrete admissible set U_h^{ad}.

Lemma 4.17. *The inequality*

$$\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \tag{4.49}$$

is valid.

Proof. The proof of this lemma is nested in Proposition 4.5 of [77]. In Section 4.4.2 we will modify the assertion of this lemma. For that reason we recall the proof in detail to highlight the differences. According to [107, Section 2.8.4] the optimality condition (4.4) also holds pointwise, i.e.,

$$(\bar{p}(x) + \nu\bar{u}(x))(u - \bar{u}(x)) \ge 0 \qquad \forall u \in [u_a, u_b] \text{ and for a.a. } x \in \Gamma.$$

For every $E \in \mathcal{E}_h$ we choose x as the midpoint S_E of E. Integrating over E and summing up over all elements yields

$$(R_h\bar{p} + \nu R_h\bar{u}, u - R_h\bar{u})_{L^2(\Gamma)} \ge 0 \qquad \forall u \in U_h^{ad} \tag{4.50}$$

having regard to the definition of R_h. If we test the variational inequalities (4.50) and (4.37) with $\bar{u}_h \in U_h^{ad}$ and $R_h\bar{u} \in U_h^{ad}$, respectively, we can conclude by adding both inequalities

$$(R_h\bar{p} - \bar{p}_h + \nu(R_h\bar{u} - \bar{u}_h), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \ge 0.$$

Rearranging terms yields the assertion. □

Next, we show that $R_h\bar{u}$ is closer to $\bar{u}_h$ than to $\bar{u}$ in general.

Lemma 4.18 (Supercloseness)**.** *Let Assumption 4.13 and the condition $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$ be fulfilled. Then the estimate*

$$\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

holds true.

Proof. The proof relies on Lemma 4.17. Inserting appropriate intermediate functions into (4.49) yields

$$\begin{aligned}\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le{}& (R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} + (\bar{p} - S_h^*(S_hR_h\bar{u} - y_d), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)}\\ &+ (S_h^*(S_hR_h\bar{u} - y_d) - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)}.\end{aligned} \tag{4.51}$$

We are going to estimate each term on the right hand side of (4.51) separately. Since $\bar{u}_h - R_h\bar{u}$ is constant on every boundary element E we obtain for the first term

$$\begin{aligned}(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} &= \sum_{E\in\mathcal{E}_h}\int_E (R_h\bar{p} - \bar{p})(\bar{u}_h - R_h\bar{u})\\ &= \sum_{j=0}^{m}\sum_{E\in\mathcal{E}_{h,j}} (\bar{u}_h - R_h\bar{u})_{|E}\int_E (R_h\bar{p} - \bar{p}).\end{aligned}$$

Using Corollary 3.35 we can conclude as in the proof of Lemma 4.15 for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\begin{aligned}
&(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\
&\le ch^2 \left(\sum_{E\in\mathcal{E}_{h,0}} |E|^{1/2} \left|(\bar{u}_h - R_h\bar{u})_{|E}\right| |\bar{p}|_{W^{2,2}(E)} + \sum_{j=1}^{m} \sum_{E\in\mathcal{E}_{h,j}} |E|^{1/2} \left|(\bar{u}_h - R_h\bar{u})_{|E}\right| |\bar{p}|_{W^{2,2}_{2(1-\mu_j)}(E)} \right) \\
&= ch^2 \left(\sum_{E\in\mathcal{E}_{h,0}} \|\bar{u}_h - R_h\bar{u}\|_{L^2(E)} |\bar{p}|_{W^{2,2}(E)} + \sum_{j=1}^{m} \sum_{E\in\mathcal{E}_{h,j}} \|\bar{u}_h - R_h\bar{u}\|_{L^2(E)} |\bar{p}|_{W^{2,2}_{2(1-\mu_j)}(E)} \right) \\
&\le ch^2 \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} |\bar{p}|_{W^{2,2}_{2(\vec{1}-\vec{\mu})}(\Gamma)} \le ch^2 \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right). \qquad (4.52)
\end{aligned}$$

The last two inequalities hold with respect to the discrete Cauchy-Schwarz inequality and Theorem 4.4. For the second term in (4.51) we get with the Cauchy-Schwarz inequality

$$(\bar{p} - S_h^*(S_hR_h\bar{u} - y_d), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \le \|\bar{p} - S_h^*(S_hR_h\bar{u} - y_d)\|_{L^2(\Gamma)} \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)}.$$

We again introduce intermediate functions, apply the triangle inequality and use the continuity of the operator S_h^* from $L^2(\Omega)$ to $L^2(\Gamma)$ according to Lemma 4.7. By this we get

$$\begin{aligned}
&\|\bar{p} - S_h^*(S_hR_h\bar{u} - y_d)\|_{L^2(\Gamma)} = \|S^*(\bar{y} - y_d) - S_h^*(S_hR_h\bar{u} - y_d))\|_{L^2(\Gamma)} \\
&\quad \le \|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)\bar{u}\|_{L^2(\Gamma)} + \|S_h^*S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Gamma)} \\
&\quad \le \|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Gamma)} + c\|(S - S_h)\bar{u}\|_{L^2(\Omega)} + c\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)}.
\end{aligned}$$

Having regard to Lemma 4.6, we observe that the first term is a finite element error on the boundary for the adjoint sates. The second one represents a finite element error in the domain for the sates. Thus, we can use Theorem 3.48, Lemma 3.41 and Lemma 4.15. This yields

$$\begin{aligned}
&(\bar{p} - S_h^*(S_hR_h\bar{u} - y_d), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\
&\le ch^2 |\ln h|^{3/2} \left(\|\bar{p}\|_{W^{2,\infty}_{\vec{5}/2-2\vec{\mu}}(\Omega)} + \|\bar{y}\|_{W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} \\
&\le ch^2 |\ln h|^{3/2} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} \qquad (4.53)
\end{aligned}$$

for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$, where we used the regularity results of Theorem 4.4 in the last step. Having in mind the definition of $\bar{p}_h$ and S_h^* we get for the third term in (4.51)

$$\begin{aligned}
&(S_h^*(S_hR_h\bar{u} - y_d) - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} = (S_h^*(S_h(R_h\bar{u} - \bar{u}_h)), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\
&\quad = (S_h(R_h\bar{u} - \bar{u}_h), S_h(\bar{u}_h - R_h\bar{u}))_{L^2(\Omega)} = -\|S_h(R_h\bar{u} - \bar{u}_h)\|^2_{L^2(\Omega)} \le 0. \qquad (4.54)
\end{aligned}$$

Finally the inequalities (4.51)–(4.54) imply the desired result. □

Corollary 4.19 (Supercloseness)**.** *Let Assumption 4.13 be fulfilled. Moreover, let* $\vec{\mu} = \vec{1}$ *(quasi-uniform mesh),* $\vec{0} < \vec{\epsilon} < \vec{\lambda}$ *and* $\rho = \min(2, \min(\vec{1}/2 + \vec{\lambda} - \vec{\epsilon}))$*. Then the estimate*

$$\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le ch^\rho |\ln h|^{3/2} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

is valid.

Proof. The proof is a word-by-word repetition of the proof of Lemma 4.18, if one uses Corollary 3.35 with $\vec{\mu} = \vec{1}$, Corollary 3.49 instead of Theorem 3.48, Corollary 3.42 instead of Lemma 3.41, and Corollary 4.16 instead of Lemma 4.15, i.e., we get instead of (4.52)

$$\begin{aligned}(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} &\leq ch^\rho \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} |\bar{p}|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \\ &\leq ch^\rho \|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} \left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\end{aligned}$$

with $\vec{\kappa} = \vec{3}/2 - \vec{\lambda} + \vec{\epsilon}$, $\vec{\kappa} \geq \vec{0}$ (see also the proof of Corollary 4.16) and instead of (4.53)

$$\begin{aligned}&(\bar{p} - S_h^*(S_hR_h\bar{u} - y_d), \bar{u}_h - R_h\bar{u})_{L^2(\Gamma)} \\ &\leq c\left(h^\rho|\ln h|^{3/2}\|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + h^\lambda\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + h^\rho(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})})\right)\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)} \\ &\leq ch^\rho|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Gamma)}\end{aligned}$$

with $\vec{\gamma} = \vec{2} - \vec{\lambda} + \vec{\epsilon}$, $\vec{\gamma} \geq \vec{0}$, $\vec{\beta} = \vec{1} - \vec{\lambda} + \vec{\epsilon}$, $\vec{\beta} \geq \vec{0}$ and $\lambda = \min(2, \min(2(\vec{\lambda} - \vec{\epsilon})))$. □

Now, let us state the main result of this section. We define the projection $\tilde{u}_h$ of $\bar{p}_h$ by

$$\tilde{u}_h := \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h\right).$$

Note that this projection is piecewise linear and continuous, but this postprocessed control does not belong to the discrete admissible set and even not to V_h^∂ in general.

Theorem 4.20. *Let Assumption 4.13 be satisfied. Then the discretization error estimates*

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right), \\ \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &\leq ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right), \\ \|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} &\leq ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\end{aligned}$$

hold, provided that the grading parameters $\vec{\mu}$ fulfill the condition $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$.

Proof. Introducing intermediate functions, applying the triangle inequality and using the continuity of S_h from $L^2(\Gamma)$ to $L^2(\Omega)$ according to Lemma 4.7 yields

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S_h(R_h\bar{u} - \bar{u}_h)\|_{L^2(\Omega)} \\ &\leq \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + c\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}.\end{aligned}$$

If we apply the finite element error estimates in the domain from Lemma 3.41, Lemma 4.15, Lemma 4.18, and the regularity results of Theorem 4.4, we obtain for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$ that

$$\begin{aligned}\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\leq ch^2|\ln h|^{3/2}\left(\|\bar{y}\|_{W^{2,2}_{\vec{1}-\vec{\mu}}(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \\ &\leq ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right). \qquad (4.55)\end{aligned}$$

As in (4.35) the error of the adjoint state on the boundary and in the domain can be estimated by

$$\begin{aligned}\|\bar{p}-\bar{p}_h\|_{L^2(\Gamma)}+\|\bar{p}-\bar{p}_h\|_{L^2(\Omega)} \le \|(S^*-S_h^*)(\bar{y}-y_d)\|_{L^2(\Gamma)}+\|(P-P_h)(\bar{y}-y_d)\|_{L^2(\Omega)}\\ + c\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}.\end{aligned}$$

Moreover, by employing the finite element error estimates of Lemma 3.41 and Theorem 3.48, together with Lemma 4.6, and (4.55), we get

$$\|\bar{p}-\bar{p}_h\|_{L^2(\Gamma)}+\|\bar{p}-\bar{p}_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \tag{4.56}$$

if $\vec{1}/4<\vec{\mu}<\vec{1}/4+\vec{\lambda}/2$. Finally, we observe that the projection operator $\Pi_{[u_a,u_b]}$ is Lipschitz continuous (cf. also the proof of Theorem 4.4). This implies together with (4.56)

$$\begin{aligned}\|\bar{u}-\tilde{u}_h\|_{L^2(\Gamma)} &= \left\|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}\right)-\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h\right)\right\|_{L^2(\Gamma)} \le c\|\bar{p}-\bar{p}_h\|_{L^2(\Gamma)}\\ &\le ch^2|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right).\end{aligned}$$

This ends the proof. □

Using Corollary 3.42, Corollary 4.16, Corollary 4.19 and Corollary 3.49 instead of Lemma 3.41, Lemma 4.15, Lemma 4.18 and Theorem 3.48, respectively, in the proof of Theorem 4.20, we get the following assertion for quasi-uniform meshes.

Corollary 4.21. *Let Assumption 4.13 be satisfied. Furthermore, let $\vec{\mu}=\vec{1}$ (quasi-uniform mesh), $\vec{0}<\vec{\epsilon}<\vec{\lambda}$, and $\rho=\min(2,\min(\vec{1}/2+\vec{\lambda}-\vec{\epsilon}))$. Then the discretization error estimates*

$$\begin{aligned}\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)} &\le ch^\rho|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right),\\ \|\bar{p}-\bar{p}_h\|_{L^2(\Gamma)}+\|\bar{p}-\bar{p}_h\|_{L^2(\Omega)} &\le ch^\rho|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right),\\ \|\bar{u}-\tilde{u}_h\|_{L^2(\Gamma)} &\le ch^\rho|\ln h|^{3/2}\left(\|\bar{u}\|_{L^2(\Gamma)}+\|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\end{aligned}$$

are valid.

4.2.3 Numerical example for the postprocessing approach

Now, let us present two numerical examples which illustrate the theoretical estimates of the previous section for the postprocessing approach. Let the computational domain Ω_ω be that of (3.142). In both examples we numerically solve the Neumann boundary control problem (P_{ex})

$$\begin{aligned}\text{Minimize}\quad & \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega_\omega)}+\frac{\nu}{2}\|u\|^2_{L^2(\Gamma_\omega)}+\int_{\Gamma_\omega} g_1 y,\\ \text{subject to}\quad & u\in U_{ad}:=\{u\in L^2(\Gamma_\omega): u_a\le u\le u_b \text{ a.e. on } \Gamma_\omega\},\\ & -\Delta y+y=f \quad \text{in } \Omega_\omega,\\ & \partial_n y = u+g_2 \quad \text{on } \Gamma_j,\quad j=1,\dots,m.\end{aligned}$$

This control problem differs from (P_l) in the additional term $\int_{\Gamma_\omega} g_1 y$ in the cost functional and the additional functions f and g_2 on the right hand side of the state equation. But one can analyze this problem in an analogous way. The optimality system of problem (P_{ex}) can be written as

$$\begin{aligned} -\Delta y + y &= f && \text{in } \Omega_\omega, \\ \partial_n y &= u + g_2 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \\ -\Delta p + p &= y - y_d && \text{in } \Omega_\omega, \\ \partial_n p &= g_1 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \\ u &= \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}p\right) && \text{on } \Gamma_j, \quad j = 1, \dots, m. \end{aligned}$$

As we will see in the first example, we benefit from the additional terms. These enable us to choose the data such that we can state an exact solution, which has exactly the proven regularity. In the second example we set the additional terms to zero and the remaining data as in Assumption 4.1. Thus, we are in the framework of the previous section. Then we use a reference solution on a finer mesh for the purpose of comparison.

If we discretize the state equation by linear finite elements and the control by piecewise constant functions as described in the previous section and if we further use the notation introduced in the beginning of Section 3.2, we end up with the discrete optimal control problem

$$\begin{aligned} &\text{Minimize} \quad \frac{1}{2}\| \sum_{k\in I_X} y_k\phi_k - y_d\|^2_{L^2(\Omega_\omega)} + \frac{\nu}{2}\sum_{k\in I_E} u_k^2 \|e_k\|^2_{L^2(\Gamma_\omega)} + \sum_{k\in I_X} y_k \int_{\Gamma_\omega} g_1\phi_k, \\ &\text{subject to} \quad u_k \in [u_a,u_b],\ k\in I_E, \\ &\qquad \sum_{k\in I_X} y_k \int_{\Omega_\omega} (\nabla\phi_k\cdot\nabla\phi_i + \phi_k\phi_i) = \int_{\Omega_\omega} f\phi_i + \int_{\Gamma_\omega} g_2\phi_i + \sum_{k\in I_E} u_k \int_{\Gamma_\omega} e_k\phi_i, \quad \forall i\in I_X. \end{aligned}$$

Its discrete optimality system can be formulated as

$$\begin{aligned} &\sum_{k\in I_X} y_k \int_{\Omega_\omega} (\nabla\phi_k\cdot\nabla\phi_i + \phi_k\phi_i) - \sum_{k\in I_E} u_k \int_{\Gamma_\omega} e_k\phi_i = \int_{\Omega_\omega} f\phi_i + \int_{\Gamma_\omega} g_2\phi_i, \quad \forall i\in I_X, \\ &\sum_{k\in I_X} p_k \int_{\Omega_\omega} (\nabla\phi_k\cdot\nabla\phi_i + \phi_k\phi_i) - \sum_{k\in I_X} y_k \int_{\Omega_\omega} \phi_k\phi_i = -\int_{\Omega_\omega} y_d\phi_i + \int_{\Gamma_\omega} g_1\phi_i, \quad \forall i\in I_X, \\ &\qquad \left(\sum_{k\in I_X} p_k \int_{\Gamma_\omega} \phi_k e_i + \nu u_i \int_{\Gamma_\omega} e_i^2\right)(v_i - u_i) \geq 0, \quad \forall v_i\in[u_a,u_b], \forall i\in I_E. \end{aligned}$$

The implementation is accomplished as in Section 3.2.5. But note that one has to extend the algorithms such that one can calculate

$$\int_{\Gamma_\omega} e_k\phi_i \quad \forall k\in I_E, \forall i\in I_X \tag{4.57}$$

and

$$\int_{\Gamma_\omega} e_i^2 \quad \forall i\in I_E. \tag{4.58}$$

In order to solve the discrete optimality system we use a primal-dual active set strategy as described in [107, Section 2.12.4]. We also mention [19], [66] and [75].

Now let us present the specific numerical examples.

Example 4.22. Let us set $\nu = 1$, $u_a = -0.5$ and $u_b = 0.5$. Moreover, the data f, y_d, g_1 and g_2 are chosen in the following way

$$\begin{aligned} f &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ y_d &= 2r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g_1 &= -\partial_n \left(r^\lambda \cos(\lambda\varphi)\right) && \text{on } \Gamma_j, \quad j = 1, \dots, m, \\ g_2 &= \partial_n \left(r^\lambda \cos(\lambda\varphi)\right) - \Pi_{[u_a,u_b]} \left(r^\lambda \cos(\lambda\varphi)\right) && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

with $\lambda = \pi/\omega$. Then the unique solution of this problem is given by

$$\begin{aligned} \bar{y} &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ \bar{p} &= -r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ \bar{u} &= \Pi_{[u_a,u_b]} \left(r^\lambda \cos(\lambda\varphi)\right) && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

which has exactly the singular behavior we have proven. One can find the discrete state $\bar{y}_h$, the discrete adjoint state $\bar{p}_h$ and the postprocessed control $\tilde{u}_h$ in Figures 4.1–4.3 for $\omega = 3\pi/2$ plotted on a graded mesh with $\mu = 0.5$ and $R = 0.4$ which was generated by a transformation of the nodes. We calculated the discretization errors for the state and the adjoint state in $L^2(\Omega_\omega)$ and for the postprocessed control in $L^2(\Gamma_\omega)$ for the angles $w \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes and different mesh grading parameters. The grading is established by a transformation of the nodes, see Section 3.2.5 for details. Moreover, we determined the corresponding experimental orders of convergence as in Example 3.66. The results are given in Tables 4.1–4.5. For an interior angle of $2\pi/3$ we observe on quasi-uniform meshes a convergence rate equal to 2 or close to 2 in all three variables as expected according to Theorem 4.20. In case of an interior angle of $3\pi/4$ and $\mu = 1$ the error of the postprocessed control behaves as proven in Corollary 4.21, i.e., the experimental order of convergence is about 1.82. However, the state and the adjoint state are approximated with a rate of 2 which is better than expected. Such an effect has already been analyzed for unconstrained Dirichlet boundary control problems in [78]. In that paper, improved estimates in $L^2(\Omega_\omega)$ for the state and adjoint state are shown by means of better estimates in weaker norms for the control and the state, respectively. But to the best of our knowledge there are no references, where this is done for control constrained Neumann boundary control problems. Next, if we choose $\mu = 0.83 < 0.92 \approx 1/4 + \lambda/2$ we retain the full order of convergence for the postprocessed control as shown in Theorem 4.20. For the domain $\Omega_{3\pi/2}$ we see on a quasi-uniform mesh an approximation rate of about 1.15 for the postprocessed control which fits to the theoretical results of Corollary 4.21. For the state and the adjoint state we observe a convergence order of about 1.35 which is again higher than expected. Thus, one can conjecture that results comparable to those of [78] also hold for control constrained Neumann boundary control problems. Finally, if we choose $\mu = 0.5 < 0.58 \approx 1/4 + \lambda/2$ we achieve the full convergence order of about 2 in all variables, which we have shown in Theorem 4.20.

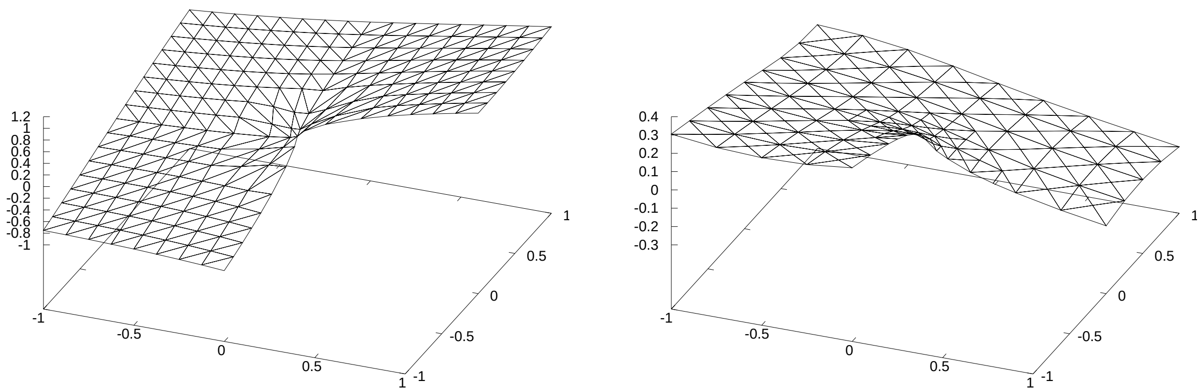

Figure 4.1: Solution $\bar{y}_h$ of Example 4.22 (left) and solution $\bar{y}_h$ of Example 4.23 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

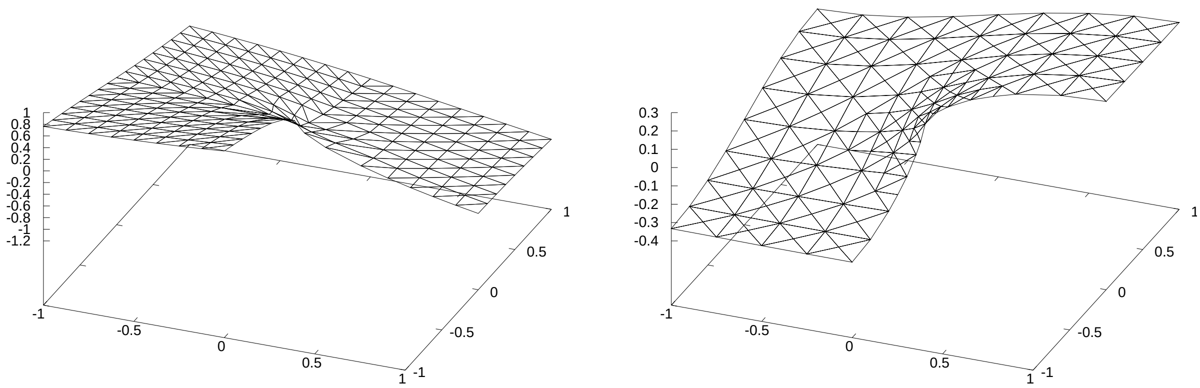

Figure 4.2: Solution $\bar{p}_h$ of Example 4.22 (left) and solution $\bar{p}_h$ of Example 4.23 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

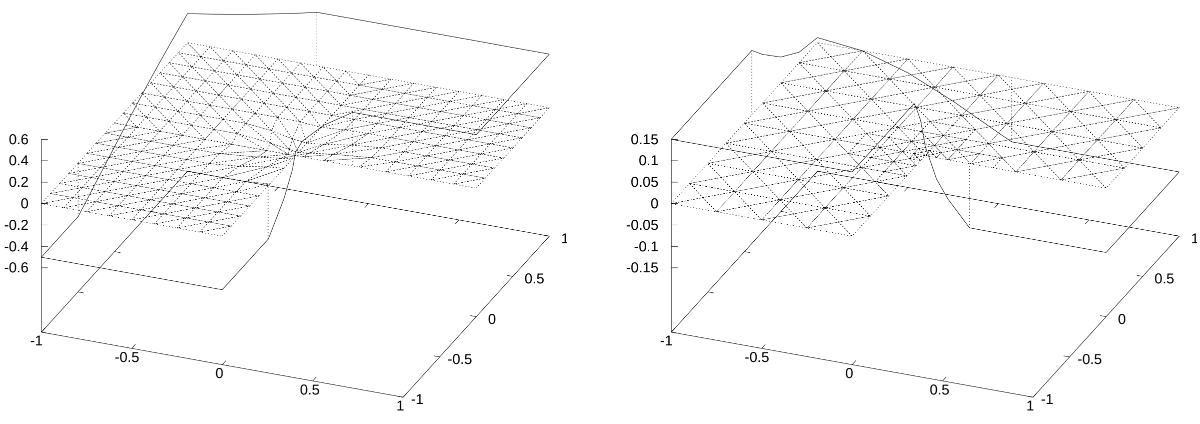

Figure 4.3: Solution $\tilde{u}_h$ of Example 4.22 (left) and solution $\tilde{u}_h$ of Example 4.23 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

Example 4.23. We set $\nu = 1$, $u_a = -0.15$, $u_b = 0.15$ and $\omega > \pi/4$. We define

$$b(x) := \left(\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2\right)^{1/2}, \quad x = (x_1, x_2) \in \Omega_\omega.$$

Furthermore, the data f, y_d, g_1 and g_2 are chosen as follows

$$\begin{aligned} f &= 0 && \text{in } \Omega_\omega, \\ y_d &= -b^{1/10}\cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g_1 &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \\ g_2 &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned}$$

with $\lambda = \pi/\omega$. In Figures 4.1–4.3 the discrete state $\bar{y}_h$, the discrete adjoint state $\bar{p}_h$ and the postprocessed control $\tilde{u}_h$ are illustrated for $\omega = 3\pi/2$ on a graded mesh with $\mu = 0.5$ and $R = 0.4$ generated by a bisection algorithm as described in Section 3.2.5. Tables 4.6–4.10 contain the discretization errors for the state and adjoint state in $L^2(\Omega_\omega)$ and for the postprocessed control in $L^2(\Gamma_\omega)$ for different interior angles $w \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes and different mesh grading parameters, where the grading is implemented by a bisection algorithm as in Section 3.2.5. Since we do not know the exact solution of this problem we used, for the purpose of comparison, reference solutions on meshes with mesh size h_{ref} and mesh grading parameter μ_{ref} as indicated in the different tables. In each case study the approximate experimental orders of convergence are determined as in Example 3.67. The observations do not differ significantly from those in the previous example. However, the usage of a reference solution instead of an exact solution does not produce results, which fit such as perfectly to the theory as in the foregoing example. But the essential effects of the corner singularities are apparent.

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.577350	1.85e-02		9.78e-03		4.40e-02	
0.288675	4.45e-03	2.06	3.11e-03	1.65	1.31e-02	1.75
0.144338	1.18e-03	1.92	7.73e-04	2.01	3.66e-03	1.84
0.072169	2.99e-04	1.98	1.93e-04	2.00	9.98e-04	1.87
0.036084	7.46e-05	2.00	4.90e-05	1.98	2.68e-04	1.90
0.018042	1.88e-05	1.99	1.22e-05	2.01	7.14e-05	1.91
0.009021	4.72e-06	2.00	3.03e-06	2.00	1.89e-05	1.92
0.004511	1.18e-06	2.00	7.57e-07	2.00	4.96e-06	1.93
0.002255	2.96e-07	2.00	1.89e-07	2.00	1.30e-06	1.93

Table 4.1: Discretization errors for Example 4.22 with $\omega = 2\pi/3$ and $\mu = 1$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	3.89e-02		1.56e-02		5.92e-02	
0.353553	9.76e-03	1.99	4.29e-03	1.86	1.91e-02	1.63
0.176777	2.27e-03	2.10	1.14e-03	1.91	5.81e-03	1.72
0.088388	5.84e-04	1.96	2.67e-04	2.09	1.71e-03	1.76
0.044194	1.44e-04	2.02	6.53e-05	2.03	4.94e-04	1.79
0.022097	3.63e-05	1.99	1.58e-05	2.04	1.42e-04	1.80
0.011049	9.07e-06	2.00	3.89e-06	2.02	4.04e-05	1.81
0.005524	2.27e-06	2.00	9.59e-07	2.02	1.15e-05	1.82
0.002762	5.67e-07	2.00	2.38e-07	2.01	3.25e-06	1.82

Table 4.2: Discretization errors for Example 4.22 with $\omega = 3\pi/4$ and $\mu = 1$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	3.89e-02		1.56e-02		5.92e-02	
0.370133	1.00e-02	2.09	4.48e-03	1.93	1.89e-02	1.77
0.195646	2.43e-03	2.23	1.24e-03	2.01	5.35e-03	1.97
0.103664	6.30e-04	2.12	3.05e-04	2.21	1.43e-03	2.08
0.052560	1.57e-04	2.04	7.64e-05	2.04	3.71e-04	1.99
0.026439	3.96e-05	2.01	1.90e-05	2.03	9.53e-05	1.98
0.013258	9.91e-06	2.01	4.74e-06	2.01	2.42e-05	1.98
0.006639	2.48e-06	2.00	1.18e-06	2.01	6.14e-06	1.99
0.003324	6.19e-07	2.00	2.96e-07	2.00	1.55e-06	1.99

Table 4.3: Discretization errors Example 4.22 with $\omega = 3\pi/4$ and $\mu = 0.83$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	8.56e-02		6.16e-02		1.06e-01	
0.353553	3.26e-02	1.39	2.30e-02	1.42	4.85e-02	1.12
0.176777	1.31e-02	1.32	8.59e-03	1.42	2.50e-02	0.95
0.088388	5.07e-03	1.36	3.28e-03	1.39	1.19e-02	1.08
0.044194	1.95e-03	1.38	1.26e-03	1.37	5.50e-03	1.11
0.022097	7.62e-04	1.36	4.88e-04	1.37	2.53e-03	1.12
0.011049	2.97e-04	1.36	1.90e-04	1.36	1.15e-03	1.14
0.005524	1.17e-04	1.35	7.43e-05	1.36	5.20e-04	1.14
0.002762	4.59e-05	1.35	2.92e-05	1.35	2.34e-04	1.15

Table 4.4: Discretization errors for Example 4.22 with $\omega = 3\pi/2$ and $\mu = 1$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	8.56e-02		6.16e-02		1.06e-01	
0.425046	3.80e-02	1.59	2.47e-02	1.80	4.89e-02	1.51
0.258029	1.35e-02	2.08	9.19e-03	1.98	1.82e-02	1.98
0.156360	4.21e-03	2.32	2.83e-03	2.35	5.81e-03	2.28
0.083008	1.17e-03	2.03	7.80e-04	2.03	1.67e-03	1.97
0.042742	3.08e-04	2.00	2.06e-04	2.01	4.59e-04	1.95
0.021687	7.96e-05	2.00	5.32e-05	2.00	1.22e-04	1.95
0.010923	2.03e-05	1.99	1.36e-05	1.99	3.19e-05	1.96
0.005496	5.13e-06	2.00	3.42e-06	2.00	8.22e-06	1.97

Table 4.5: Discretization errors for Example 4.22 with $\omega = 3\pi/2$ and $\mu = 0.5$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	6.44e-03		5.32e-03		6.31e-03	
0.250000	1.47e-03	2.14	1.58e-03	1.76	1.27e-03	2.31
0.125000	3.53e-04	2.06	4.21e-04	1.91	3.43e-04	1.89
0.062500	8.92e-05	1.98	1.07e-04	1.97	9.78e-05	1.81
0.031250	2.21e-05	2.01	2.71e-05	1.99	2.74e-05	1.83
0.015625	5.48e-06	2.01	6.75e-06	2.01	7.42e-06	1.89
0.007812	1.28e-06	2.10	1.65e-06	2.03	2.00e-06	1.89

Table 4.6: Discretization errors for Example 4.23 with $\omega = 2\pi/3$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	6.80e-03		8.92e-03		7.13e-03	
0.250000	1.82e-03	1.90	2.37e-03	1.92	1.65e-03	2.11
0.125000	4.46e-04	2.03	6.21e-04	1.93	3.88e-04	2.09
0.062500	8.48e-05	2.40	1.71e-04	1.86	1.22e-04	1.67
0.031250	3.20e-05	1.41	3.87e-05	2.14	2.27e-05	2.42
0.015625	7.80e-06	2.03	9.65e-06	2.00	5.80e-06	1.97
0.007812	1.78e-06	2.13	2.39e-06	2.01	1.73e-06	1.74

Table 4.7: Discretization errors for Example 4.23 with $\omega = 3\pi/4$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.19e-02		9.19e-03		1.20e-02	
0.250000	2.77e-03	2.10	2.36e-03	1.96	2.92e-03	2.04
0.125000	9.33e-04	1.57	6.54e-04	1.85	8.95e-04	1.71
0.062500	1.96e-04	2.25	1.78e-04	1.88	2.40e-04	1.90
0.031250	5.89e-05	1.73	3.74e-05	2.25	5.06e-05	2.24
0.015625	1.68e-05	1.81	9.61e-06	1.96	1.40e-05	1.86
0.007812	3.76e-06	2.16	2.41e-06	1.99	3.56e-06	1.97

Table 4.8: Discretization errors for Example 4.23 with $\omega = 3\pi/4$, $\mu = 0.83$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	2.02e-02		2.40e-02		6.85e-03	
0.250000	8.18e-03	1.31	7.86e-03	1.61	2.87e-03	1.26
0.125000	3.07e-03	1.41	2.78e-03	1.50	1.71e-03	0.74
0.062500	1.14e-03	1.42	1.05e-03	1.40	1.23e-03	0.48
0.031250	4.27e-04	1.42	4.08e-04	1.37	6.85e-04	0.84
0.015625	1.66e-04	1.36	1.57e-04	1.37	3.52e-04	0.96
0.007812	6.35e-05	1.39	6.20e-05	1.34	1.71e-04	1.05

Table 4.9: Discretization errors for Example 4.23 with $\omega = 3\pi/2$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.26e-02		1.71e-02		9.46e-03	
0.250000	2.83e-03	2.15	4.67e-03	1.87	2.66e-03	1.83
0.125000	8.08e-04	1.81	1.09e-03	2.10	7.99e-04	1.73
0.062500	1.92e-04	2.08	2.97e-04	1.88	2.34e-04	1.77
0.031250	4.86e-05	1.98	7.12e-05	2.06	6.09e-05	1.94
0.015625	1.22e-05	1.99	1.83e-05	1.96	1.62e-05	1.91
0.007812	2.95e-06	2.05	4.59e-06	1.99	4.25e-06	1.93

Table 4.10: Discretization errors for Example 4.23 with $\omega = 3\pi/2$, $\mu = 0.5$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

4.3 Semilinear problems

In this section we analyze the following semilinear elliptic Neumann boundary control problem:

$$\begin{aligned} \text{Minimize} \quad & F(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \\ \text{subject to} \quad & u \in U_{ad} := \{u \in L^2(\Gamma) : u_a \le u \le u_b \text{ a.e. on } \Gamma\}, \\ & -\Delta y + d(\cdot, y) = 0 \quad \text{in } \Omega, \\ & \qquad \partial_n y = u \quad \text{on } \Gamma_j, \quad j = 1, \dots, m. \end{aligned} \tag{4.59}$$

In all what follows we denote this optimal control problem by (P_{sl}). The nomenclature of the given quantities in (P_{sl}) is as for linear Neumann boundary control problems. We call F cost functional, y_d desired state and U_{ad} set of admissible controls or admissible set. Furthermore, we denote by y the state and by u the control which are coupled by the state equation (4.59).

We suppose that the following conditions hold throughout Sections 4.3–4.4.

Assumption 4.24.

(A1) The domain Ω is a polygonal domain according to Definition 2.17 with m corner points and boundary $\Gamma = \bigcup_{j=1}^m \bar{\Gamma}_j$.

(A2) The function $y_d \in C^{0,\sigma}(\bar{\Omega})$ is given for some $\sigma > 0$.

(A3) The regularization parameter $\nu > 0$ and the control bounds $u_a < u_b$ are fixed real numbers.

(A4) The function $d = d(x, y) : \Omega \times \mathbb{R}$ is measurable with respect to $x \in \Omega$ for all fixed $y \in \mathbb{R}$, twice continuously differentiable with respect to y for almost all $x \in \Omega$, and

$$\frac{\partial d}{\partial y}(x,y) \ge 0 \quad \text{for a.a. } x \in \Omega \text{ and } y \in \mathbb{R}.$$

Let $d(\cdot, 0) \in L^2(\Omega)$ and $\frac{\partial^2 d}{\partial y^2}(\cdot, 0) \in L^\infty(\Omega)$. Moreover, for all $M > 0$ there exists a constant $L_{d,M} > 0$ such that d satisfies

$$\left|\frac{\partial^j d}{\partial y^j}(x,y_1) - \frac{\partial^j d}{\partial y^j}(x,y_2)\right| \le L_{d,M}|y_1 - y_2|, \quad j = 0, 2,$$

for a.a. $x \in \Omega$ and $y_1, y_2 \in \mathbb{R}$ with $|y_i| \le M$, $i = 1, 2$. Furthermore, we require for all $M > 0$ that there is a constant $L_{d,M} > 0$ such that

$$\left|\frac{\partial d}{\partial y}(x_1,y_1) - \frac{\partial d}{\partial y}(x_2,y_2)\right| \le L_{d,M}\left(|x_1 - x_2|^\sigma + |y_1 - y_2|\right)$$

for all $x_1, x_2 \in \Omega$ and $y_1, y_2 \in \mathbb{R}$ with $|y_i| \le M$, $i = 1, 2$, and σ from (A2).

(A5) There is a subset $E_\Omega \subset \Omega$ of positive measure and a constant $c_\Omega > 0$ such that $\frac{\partial d}{\partial y}(x,y) \ge c_\Omega$ in $E_\Omega \times \mathbb{R}$.

The following remark is with respect to the notation in the sequel.

Remark 4.25. To shorten the notation we will write d_y and d_{yy} instead of $\frac{\partial d}{\partial y}$ and $\frac{\partial^2 d}{\partial y^2}$, respectively.

Now, let us turn our attention to the analysis of the continuous optimal control problem. Based on Lemma 3.20 we can introduce the control-to-state mapping

$$G : L^2(\Gamma) \to H^1(\Omega) \cap C^0(\bar{\Omega}),\ G(u) = y, \tag{4.60}$$

that assigns to every control $u \in L^2(\Gamma)$ the unique weak solution $y \in H^1(\Omega) \cap C^0(\bar{\Omega})$ of the state equation (4.59). By this we can reformulate problem (P_{sl}) and we obtain its reduced formulation

$$\min_{u \in U_{ad}} J(u) := F(G(u), u) = \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \tag{4.61}$$

where the functional J denotes again the reduced cost functional. As in the linear elliptic case we call a control $\bar{u} \in U_{ad}$ optimal or solution of problem (P_{sl}) with associated state $\bar{y} = G(\bar{u})$ if

$$J(\bar{u}) \le J(u) \quad \forall u \in U_{ad}.$$

Let us remark that the reduced cost functional J is non-convex in general due to the semilinear state equation. Therefore, we cannot expect a unique solution of the optimal control problem. Moreover, we have to deal with locally optimal solutions which are defined as follows: we denote a control $\bar{u} \in U_{ad}$ locally optimal or local solution of (P_{sl}) in the sense of $L^2(\Gamma)$ if there exists an $\epsilon > 0$ such that

$$J(\bar{u}) \le J(u) \quad \forall u \in U_{ad} \cap B_\epsilon(\bar{u}),$$

where the L^2-ball $B_\epsilon(\bar{u})$ around $\bar{u}$ with radius ϵ is defined by

$$B_\epsilon(\bar{u}) := \left\{ u \in L^2(\Gamma) : \|u - \bar{u}\|_{L^2(\Gamma)} \le \epsilon \right\}.$$

Next, we would like to derive the optimality system or rather the first order necessary optimality conditions for problem (P_{sl}) as in the linear elliptic case. This requires first order Fréchet derivatives of the control-to-state operator G and the reduced cost functional J. Moreover, due the non-convex character of problem (P_{sl}), as already mentioned above, a solution of the corresponding first order optimality system represents not necessarily a local solution. Therefore, we have to deal with second order sufficient optimality conditions and consequently we require Fréchet derivatives of second order of the control-to-state mapping G and the reduced cost functional J.

Note that we only need Assumptions 4.24 (A4)–(A5) for the validity of the following lemma.

Lemma 4.26. *The mapping $G : L^2(\Gamma) \to H^1(\Omega) \cap C^0(\bar{\Omega})$, defined by (4.60) is twice continuously Fréchet differentiable. Moreover, for all $u, v \in L^2(\Gamma)$, $G'(u)v = y_v \in H^1(\Omega)$ is defined as the unique weak solution of*

$$\begin{aligned} -\Delta y_v + d_y(\cdot, y) y_v &= 0 \quad \text{in } \Omega, \\ \partial_n y_v &= v \quad \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{4.62}$$

where $y = G(u)$*. Furthermore, for every* $u, v_1, v_2 \in L^2(\Gamma)$*,* $G''(u)[v_1, v_2] := (G''(u)v_1)v_2 = y_{v_1,v_2} \in H^1(\Omega)$ *is the unique weak solution of*

$$\begin{aligned} -\Delta y_{v_1,v_2} + d_y(\cdot, y)y_{v_1,v_2} &= -d_{yy}(\cdot, y)y_{v_1}y_{v_2} && \text{in } \Omega, \\ \partial_n y_{v_1,v_2} &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{4.63}$$

where $y = G(u)$ *and* $y_{v_i} = G'(u)v_i$*,* $i = 1, 2$*.*

Proof. The proof is based on the implicit function theorem and can be found in e.g. [32, Theorem 3.1]. We also refer to [27, Theorem 2.4] and [107, Section 4.10.3 and Section 4.10.6]. □

Next, we introduce the adjoint state $p \in H^1(\Omega)$ as the unique weak solution of the adjoint equation

$$\begin{aligned} -\Delta p + d_y(\cdot, y)p &= y - y_d && \text{in } \Omega, \\ \partial_n p &= 0 && \text{on } \Gamma_j, \quad j = 1, \dots, m, \end{aligned} \tag{4.64}$$

where $y \in L^\infty(\Omega)$ is given. According to Assumptions 4.24 (A4)–(A5) we have $d_y(x, y(x)) \geq 0$ for a.a. $x \in \Omega$, $d_y(x, y(x)) \geq c_\Omega$ for a.a. $x \in E_\Omega$. Furthermore, we can conclude

$$\|d_y(\cdot, y) - d_y(\cdot, 0)\|_{L^\infty(\Omega)} \leq c\|y\|_{L^\infty(\Omega)}$$

with a constant $c = c(\|y\|_{L^\infty(\Omega)})$ and consequently

$$\begin{aligned} \|d_y(\cdot, y)\|_{L^\infty(\Omega)} &\leq \|d_y(\cdot, y) - d_y(\cdot, 0)\|_{L^\infty(\Omega)} + \|d_y(\cdot, 0)\|_{L^\infty(\Omega)} \\ &\leq c\|y\|_{L^\infty(\Omega)} + \|d_y(\cdot, 0)\|_{L^\infty(\Omega)} \leq c \end{aligned} \tag{4.65}$$

with a constant $c = c(\|y\|_{L^\infty(\Omega)})$. Thus, Lemma 3.4, the regularity of y_d according to Assumption 4.24 (A2) and the Lipschitz continuity and regularity of d_y from Assumptions 4.24 (A4)–(A5) imply the well-posedness of (4.64) for every $y \in L^\infty(\Omega)$. Consequently, we can introduce an operator $P : L^\infty(\Omega) \to H^1(\Omega)$ by $P(y) := p$ where $p \in H^1(\Omega)$ is the weak solution of (4.64) associated to $y \in L^\infty(\Omega)$. The adjoint state allows us to state expressions for the Fréchet derivatives of J in a simple manner.

Lemma 4.27. *The functional* $J : L^2(\Gamma) \to \mathbb{R}$ *from* (4.61) *is twice continuously Fréchet differentiable. Moreover, for every* $u, v, v_1, v_2 \in L^2(\Gamma)$ *there holds*

$$J'(u)v = \int_\Gamma (P(G(u)) + \nu u)\, v$$

and

$$J''(u)[v_1, v_2] := (J''(u)v_1)v_2 = \int_\Omega (1 - P(G(u))d_{yy}(\cdot, G(u)))\, G'(u)v_1 G'(u)v_2 + \nu \int_\Gamma v_1 v_2.$$

Proof. This lemma is given in [32, Theorem 3.2 and Remark 3.3]. The proof is based on the chain rule and the results of Lemma 4.26 together with

$$\int_\Omega (G(u) - y_d)G'(u)v = \int_\Gamma P(G(u))v$$

and

$$\int_\Omega (G(u) - y_d) G''(u)[v_1, v_2] = -\int_\Gamma P(G(u)) d_{yy}(\cdot, G(u)) G'(u) v_1 G'(u) v_2,$$

which can be deduced from the weak formulations of (4.62), (4.63) and (4.64). □

The next theorem is devoted to the existence of solutions of problem (P_{sl}), first order necessary optimality conditions and regularity results for local solutions.

Theorem 4.28. *The optimal control problem (P_{sl}) admits at least one solution in U_{ad}. For every local solution $\bar{u} \in U_{ad}$ of problem (P_{sl}) there exists a unique optimal state $\bar{y} = G(\bar{u})$ and optimal adjoint state $\bar{p} = P(\bar{y})$ such that*

$$(\bar{p} + \nu \bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad}, \tag{4.66}$$

which is equivalent to

$$\bar{u}(x) = \Pi_{[u_a, u_b]} \left(-\frac{1}{\nu} \bar{p}(x) \right) \quad \text{for a.a. } x \in \Gamma. \tag{4.67}$$

Moreover, let $t < 3/2$ and let β_j, γ_j, τ_j and κ_j satisfy the conditions

$$\begin{aligned} 1 > \beta_j &> \max(0, 1 - \lambda_j) \quad \text{or} \quad \beta_j = 0 \text{ and } 1 - \lambda_j < 0, \\ 2 > \gamma_j &> \max(0, 2 - \lambda_j) \quad \text{or} \quad \gamma_j = 0 \text{ and } 2 - \lambda_j < 0, \\ 1 > \tau_j &> \max(0, 1 - \lambda_j) \quad \text{or} \quad \tau_j = 0 \text{ and } 1 - \lambda_j < 0, \\ 3/2 > \kappa_j &> \max(-1/2, 3/2 - \lambda_j) \end{aligned}$$

for each $j \in \{1, \ldots, m\}$. Then $\bar{y}$ and $\bar{p}$ fulfill the a priori estimates

$$\|\bar{y}\|_{H^t(\Omega)} + \|\bar{p}\|_{H^t(\Omega)} \leq c$$

and

$$\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} + \|\bar{p}\|_{W^{1,\infty}_{\vec{\tau}}(\Gamma)} + \|\bar{p}\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \leq c.$$

Proof. First, we observe that the admissible set U_{ad} is not empty. Consequently, the convexity of the cost functional F with respect to the control u implies the existence of at least one solution of problem (P_{sl}) in U_{ad} under Assumption 4.24, see e.g. [107, Section 4.4.2]. The first order necessary optimality condition (4.66) can be derived by standard arguments based on Lemma 4.27 and the convexity of the admissible set U_{ad}, see e.g. [107, Lemma 4.18]. The equivalence between (4.66) and (4.67) follows as in the linear elliptic case, see e.g. [107, Section 4.6]. The regularity assertions can be deduced similar to the linear elliptic case as well. Let us point out the main steps. For a local solution $\bar{u} \in L^2(\Gamma)$ we obtain from Lemma 3.20 that the state $\bar{y} \in H^1(\Omega) \cap C^0(\bar{\Omega})$ fulfills

$$\|\bar{y}\|_{H^1(\Omega)} + \|\bar{y}\|_{L^\infty(\Omega)} \leq c. \tag{4.68}$$

Furthermore, Theorem 2.7 and Corollary 3.22 imply

$$\|\bar{y}\|_{C^{0,t-1}(\bar{\Omega})} \leq c\|\bar{y}\|_{H^t(\Omega)} \leq c \tag{4.69}$$

with some arbitrary $t \in (1, 3/2)$. Using Lemma 3.4 and Corollary 3.6, together with (4.65) and (4.68), we conclude that the adjoint state $\bar{p} \in H^1(\Omega)$ satisfies

$$\|\bar{p}\|_{H^1(\Omega)} + \|\bar{p}\|_{H^t(\Omega)} \leq c \tag{4.70}$$

with some $t \in (1, 3/2)$. Next, we show the regularity results in weighted Sobolev spaces. Lemma 3.11, together with (4.65) and (4.68), imply

$$\|\bar{p}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c. \tag{4.71}$$

Moreover, using Corollary 3.23, Theorem 2.12 and (4.70), we can argue as in (4.13) and (4.14) to deduce

$$\|\bar{y}\|_{W^{2,2}_{\vec{\beta}}(\Omega)} \leq c. \tag{4.72}$$

It remains to show the higher regularity and the regularity on the boundary of the adjoint state. As in (3.67) we obtain by using the Lipschitz continuity of d_y from Assumption 4.24 (A4) that

$$\|d_y(\cdot, \bar{y}) - d_y(\cdot, 0)\|_{C^{0,\min(t-1,\sigma)}(\bar{\Omega})} \leq c(1 + \|\bar{y}\|_{C^{0,\min(t-1,\sigma)}(\bar{\Omega})}) \leq c(1 + \|\bar{y}\|_{C^{0,t-1}(\bar{\Omega})}).$$

Consequently, we can deduce from (4.69)

$$\begin{aligned}\|d_y(\cdot, \bar{y})\|_{C^{0,\min(t-1,\sigma)}(\bar{\Omega})} &\leq \|d_y(\cdot, \bar{y}) - d_y(\cdot, 0)\|_{C^{0,\min(t-1,\sigma)}(\bar{\Omega})} + \|d_y(\cdot, 0)\|_{C^{0,\min(t-1,\sigma)}(\bar{\Omega})} \\ &\leq c(1 + \|\bar{y}\|_{C^{0,t-1}(\bar{\Omega})}) + \|d_y(\cdot, 0)\|_{C^{0,\sigma}(\bar{\Omega})} \leq c. \end{aligned} \tag{4.73}$$

Thus, Lemma 3.13, Corollary 3.14 and Corollary 3.15 imply as in (4.16), together with (4.69),

$$\|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Omega)} + \|\bar{p}\|_{W^{2,\infty}_{\vec{\gamma}}(\Gamma)} + \|\bar{p}\|_{W^{1,\infty}_{\vec{\tau}}(\Gamma)} + \|\bar{p}\|_{W^{2,2}_{\vec{\kappa}}(\Gamma)} \leq c. \tag{4.74}$$

Collecting the inequalities (4.68)–(4.72) and (4.74) yields the regularity assertion. □

In order to state second order sufficient optimality conditions we will rely on so-called strongly active sets. We start with the definition of the τ-critical cone associated to a control $\bar{u}$,

$$C_\tau(\bar{u}) := \{v \in L^2(\Gamma) : v \text{ satisfies } (4.76)\}, \tag{4.75}$$

where

$$v(x) \begin{cases} \geq 0, & \text{if } \bar{u}(x) = u_a, \\ \leq 0, & \text{if } \bar{u}(x) = u_b, \\ = 0, & \text{if } |\bar{p}(x) + \nu\bar{u}(x)| > \tau. \end{cases} \tag{4.76}$$

Now, we are in the position to formulate second order sufficient optimality conditions.

Theorem 4.29. *Let $\bar{u} \in U_{ad}$ be a control satisfying the first order optimality conditions given in Theorem 4.28. Further, it is assumed that there are two constants $\tau > 0$ and $\delta > 0$ such that*

$$J''(\bar{u})[v,v] \geq \delta \|v\|^2_{L^2(\Gamma)} \quad \forall v \in C_\tau(\bar{u}). \tag{4.77}$$

Then there exist constants $\beta > 0$ and $\varrho > 0$ such that

$$J(u) \geq J(\bar{u}) + \beta \|u - \bar{u}\|^2_{L^2(\Gamma)} \quad \forall u \in U_{ad} \cap B_\varrho(\bar{u}).$$

Proof. For the proof we refer to e.g. [32, Corollary 3.6]. Note that we do not have to deal with the two-norm discrepancy due to the special structure of problem (P_{sl}), cf. the general setting in [32, Section 3]. □

4.4 Discretization and error estimates for semilinear problems

This section is devoted to the discretization of problem (P_{sl}) using the concept of variational discretization and the postprocessing approach. Before we turn our attention to the numerical analysis for both discretization strategies, we start with the introduction of the discrete versions of the operators G and P and the discretization of the reduced cost functional J, denoted by J_h. Furthermore, we state first and second order derivatives of the discrete functional J_h.

For each $u \in L^2(\Gamma)$ we denote by $G_h(u) = y_h \in V_h$ the unique element that satisfies

$$a(y_h, v_h) + \int_\Omega d(\cdot, y_h) v_h = (u, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h, \tag{4.78}$$

where the bilinear form a has been defined in (3.58). The unique solvability of this equation has been discussed in Lemma 3.69 based on Assumption 4.24. Consequently, the discrete version of the functional J is defined by

$$J_h(u) := F(G_h(u), u) = \frac{1}{2}\|G_h(u) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|_{L^2(\Gamma)}. \tag{4.79}$$

Next, we define the discrete operator $P_h : L^\infty(\Omega) \to V_h$ which assigns to each $y \in L^\infty(\Omega)$ the function $P_h(y) := p_h$, where $p_h \in V_h$ is the unique element that fulfills

$$a(v_h, p_h) + \int_\Omega d_y(\cdot, y) p_h v_h = (y - y_d, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h. \tag{4.80}$$

Similar to the continuous case the well-posedness of the operator P_h can be shown by means of Lemma 3.40. Now, let us state results about the differentiability of the discrete operator G_h and the reduced cost functional J_h, which can be proven as in the continuous case, see e.g. [22].

Lemma 4.30. *The mapping $G_h : L^2(\Gamma) \to V_h$, defined by* (4.78) *is twice continuously Fréchet differentiable. Moreover, for all $u, v \in L^2(\Gamma)$, $G'_h(u)v = y^v_h \in V_h$ is defined as the unique solution of*

$$a(y^v_h, v_h) + \int_\Omega d_y(\cdot, y_h) y^v_h v_h = \int_\Gamma v v_h \quad \forall v_h \in V_h, \tag{4.81}$$

where $y_h = G_h(u)$*. Furthermore, for every* $u, v_1, v_2 \in L^2(\Gamma)$*,* $G_h''(u)[v_1, v_2] := (G_h''(u)v_1)v_2 = y_h^{v_1,v_2} \in V_h$ *is the unique solution of*

$$a(y_h^{v_1,v_2}, v_h) + \int_\Omega d_y(\cdot, y_h) y_h^{v_1,v_2} v_h = -\int_\Omega d_{yy}(\cdot, y_h) y_h^{v_1} y_h^{v_2} v_h \quad \forall v_h \in V_h, \tag{4.82}$$

where $y_h = G_h(u)$ *and* $y_h^{v_i} = G_h'(u)v_i$*,* $i = 1, 2$*.*

Lemma 4.31. *The functional* $J_h : L^2(\Gamma) \to \mathbb{R}$ *from* (4.79) *is twice continuously Fréchet differentiable. Moreover, for every* $u, v, v_1, v_2 \in L^2(\Gamma)$ *there holds*

$$J_h'(u)v = \int_\Gamma (P_h(G_h(u)) + \nu u)\, v$$

and

$$J_h''(u)[v_1, v_2] := (J_h''(u)v_1)v_2 = \int_\Omega (1 - P_h(G_h(u))d_{yy}(\cdot, G_h(u)))\, G_h'(u)v_1 G_h'(u)v_2 + \nu \int_\Gamma v_1 v_2.$$

An important ingredient of the numerical analysis in the sequel are the following two lemmas. The first one summarizes some basic estimates concerning the operators G, P and their discrete versions. It can easily be deduced by means of the results of Chapter 3. In the second one, we state comparable estimates for the reduced cost functionals and their derivatives. Similar results can be found in e.g. [14] and [27].

Lemma 4.32. *The following assertions hold:*

(i) *Let* $u \in L^2(\Gamma)$ *and* $y \in L^\infty(\Omega)$*. Then there are the estimates*

$$\begin{aligned} \|G(u)\|_{L^\infty(\Omega)} + \|G_h(u)\|_{L^\infty(\Omega)} &\le c, \\ \|d_y(\cdot, G(u))\|_{L^\infty(\Omega)} + \|d_y(\cdot, G_h(u))\|_{L^\infty(\Omega)} &\le c, \\ \|d_{yy}(\cdot, G(u))\|_{L^\infty(\Omega)} + \|d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)} &\le c \end{aligned}$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$ *and*

$$\|P(y)\|_{L^\infty(\Omega)} + \|P_h(y)\|_{L^\infty(\Omega)} \le c$$

with a constant $c = c(\|y\|_{L^\infty(\Omega)})$*.*

(ii) *Let* $u_1, u_2 \in L^2(\Gamma)$*. Then the Lipschitz estimates*

$$\|G(u_1) - G(u_2)\|_{H^1(\Omega)} + \|G_h(u_1) - G_h(u_2)\|_{H^1(\Omega)} \le c\|u_1 - u_2\|_{L^2(\Gamma)}$$

are valid.

(iii) *For every* $y_1, y_2 \in L^\infty(\Omega)$ *there is the estimate*

$$\|P(y_1) - P(y_2)\|_{H^1(\Omega)} + \|P_h(y_1) - P_h(y_2)\|_{H^1(\Omega)} \le c\|y_1 - y_2\|_{L^2(\Omega)}$$

with a constant $c = c(\|y_1\|_{L^\infty(\Omega)}, \|y_2\|_{L^\infty(\Omega)})$*.*

Proof. (i) The estimates for $G(u)$ and $G_h(u)$ follow from Lemma 3.20 and Corollary 3.75, respectively. As consequence, the estimates for the nonlinearity d_y hold according to (4.65). In the same manner, we get the estimates for d_{yy}. Finally, the estimates for $P(y)$ and $P_h(y)$ can be obtained from Corollary 3.6 and Corollary 3.47, respectively, together with (4.65) and Theorem 2.7.

(ii) This can be deduced from Lemma 3.29 and Lemma 3.79.

(iii) The Lipschitz estimates of Lemma 3.17 imply

$$\|P(y_1) - P(y_2)\|_{H^1(\Omega)} \leq c\left(\|y_1 - y_2\|_{L^2(\Omega)} + \|d_y(\cdot, y_1) - d_y(\cdot, y_2)\|_{L^2(\Omega)}\|y_1 - y_d\|_{L^2(\Omega)}\right).$$

We can conclude the first estimate by employing the Lipschitz continuity of d_y according to Assumption 4.24 (A4). The second one can be obtained analogously by means of Lemma 3.55. □

Next, let us state comparable estimates for the reduced cost functionals and their derivatives.

Lemma 4.33. *The following assertions are valid:*

(i) Let $u_1, u_2 \in L^2(\Gamma)$. Then the inequality

$$|J(u_1) - J(u_2)| \leq c\|u_1 - u_2\|_{L^2(\Gamma)}$$

is satisfied with a constant $c = c(\|u_1\|_{L^2(\Gamma)}, \|u_2\|_{L^2(\Gamma)})$.

(ii) Let $u \in L^2(\Gamma)$. Then there exists an $\epsilon > 0$ such that

$$|J(u) - J_h(u)| \leq ch^{1+\epsilon}$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$.

(iii) Let $u_1, u_2, v \in L^2(\Gamma)$. Then there holds

$$|J_h''(u_1)[v, v] - J_h''(u_2)[v, v]| \leq c\|u_1 - u_2\|_{L^2(\Gamma)}\|v\|^2_{L^2(\Gamma)}$$

with a constant $c = c(\|u_1\|_{L^2(\Gamma)}, \|u_2\|_{L^2(\Gamma)})$.

(iv) Let $u, v \in L^2(\Gamma)$. Then there exists an $\epsilon > 0$ such that

$$|J''(u)[v, v] - J_h''(u)[v, v]| \leq ch^{1+\epsilon}\|v\|^2_{L^2(\Gamma)}$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$.

(v) Let $u, v_1, v_2 \in L^2(\Gamma)$. Then there holds

$$|J_h''(u)[v_1, v_2] - J_h''(u)[v_1, v_1]| \leq c\|v_1\|_{L^2(\Gamma)}\|v_1 - v_2\|_{L^1(\Gamma)} + \nu\left|\int_\Gamma v_1(v_1 - v_2)\right|$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$.

Proof. (i) The definition of J and some straightforward calculations yield

$$\begin{aligned}
|J(u_1) - J(u_2)| &= \left|\frac{1}{2}\int_\Omega \left((G(u_1) - y_d)^2 - (G(u_2) - y_d)^2\right) + \nu \int_\Gamma \left(u_1^2 - u_2^2\right)\right| \\
&= \left|\frac{1}{2}\int_\Omega (G(u_1) + G(u_2) - 2y_d)(G(u_1) - G(u_2)) + \nu \int_\Gamma (u_1 + u_2)(u_1 - u_2)\right| \\
&\leq \frac{1}{2}\|G(u_1) + G(u_2) - 2y_d\|_{L^2(\Omega)}\|G(u_1) - G(u_2)\|_{L^2(\Omega)} + \nu\|u_1 + u_2\|_{L^2(\Gamma)}\|u_1 - u_2\|_{L^2(\Gamma)} \\
&\leq c\left(\frac{1}{2}\|G(u_1) + G(u_2) - 2y_d\|_{L^2(\Omega)} + \|u_1 + u_2\|_{L^2(\Gamma)}\right)\|u_1 - u_2\|_{L^2(\Gamma)},
\end{aligned}$$

where we applied the Lipschitz estimates of Lemma 4.32 (ii) in the last step. Using Lemma 4.32 (i) we obtain the existence of a constant $c = c(\|u_1\|_{L^2(\Gamma)}, \|u_2\|_{L^2(\Gamma)})$ such that

$$|J(u_1) - J(u_2)| \leq c\|u_1 - u_2\|_{L^2(\Gamma)},$$

which is the first assertion.

(ii) Similar to the first part we obtain

$$\begin{aligned}
|J(u) - J_h(u)| &= \frac{1}{2}\left|\int_\Omega \left((G(u) - y_d)^2 - (G_h(u) - y_d)^2\right)\right| \\
&= \frac{1}{2}\left|\int_\Omega (G(u) + G_h(u) - 2y_d)(G(u) - G_h(u))\right| \\
&\leq \frac{1}{2}\|G(u) + G_h(u) - 2y_d\|_{L^2(\Omega)}\|G(u) - G_h(u)\|_{L^2(\Omega)}.
\end{aligned}$$

The boundedness of $G(u)$ and $G_h(u)$ in $L^2(\Omega)$ for any bounded $u \in L^2(\Gamma)$ is proven in Lemma 4.32 (i). Moreover, using the finite element error estimates of Corollary 3.74, together with Remark 3.46, we get the existence of an $\epsilon > 0$ such that

$$|J(u) - J_h(u)| \leq ch^{1+\epsilon}$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$.

(iii) We begin with inserting the definitions of J_h'' and introducing some intermediate functions. This yields

$$\begin{aligned}
|J_h''(u_1)[v,v] &- J_h''(u_2)[v,v]| \\
&= \left|\int_\Omega \{[P_h(G_h(u_2)) - P_h(G_h(u_1))]\, d_{yy}(\cdot, G_h(u_2)) \right. \\
&+ P_h(G_h(u_1))\,[d_{yy}(\cdot, G_h(u_2)) - d_{yy}(\cdot, G_h(u_1))]\}\, G_h'(u_1)vG_h'(u_1)v \\
&+ \left.\int_\Omega [1 - P_h(G_h(u_2))d_{yy}(\cdot, G_h(u_2))]\,[G_h'(u_1)v + G_h'(u_2)v]\,[G_h'(u_1)v - G_h'(u_2)v]\right| \\
&\leq \Big[\|P_h(G_h(u_2)) - P_h(G_h(u_1))\|_{L^2(\Omega)}\|d_{yy}(\cdot, G_h(u_2))\|_{L^\infty(\Omega)} \\
&+ \|P_h(G_h(u_1))\|_{L^\infty(\Omega)}\|d_{yy}(\cdot, G_h(u_2)) - d_{yy}(\cdot, G_h(u_1))\|_{L^2(\Omega)}\Big]\,\|G_h'(u_1)v\|_{L^4(\Omega)}^2 \\
&+ \|1 - P_h(G_h(u_2))d_{yy}(\cdot, G_h(u_2))\|_{L^\infty(\Omega)}\Big[\|G_h'(u_1)v\|_{L^2(\Omega)} \\
&+ \|G_h'(u_2)v\|_{L^2(\Omega)}\Big]\,\|G_h'(u_1)v - G_h'(u_2)v\|_{L^2(\Omega)}, \qquad (4.83)
\end{aligned}$$

where we applied the triangle inequality and the Hölder inequality several times. Next, we estimate each term of (4.83) separately. By means of parts (i), (ii) and (iii) of Lemma 4.32 we can conclude

$$\begin{aligned}\|P_h(G_h(u_2)) - P_h(G_h(u_1))\|_{L^2(\Omega)}\|d_{yy}(\cdot, G_h(u_2))\|_{L^\infty(\Omega)} &\leq c\|G_h(u_2) - G_h(u_1)\|_{L^2(\Omega)} \\ &\leq c\|u_2 - u_1\|_{L^2(\Gamma)} \end{aligned} \tag{4.84}$$

with a constant $c = c(\|u_1\|_{L^2(\Gamma)}, \|u_2\|_{L^2(\Gamma)})$. Lemma 4.32 (i) and (ii), and the Lipschitz continuity of d_{yy} according to Assumption 4.24 (A4) yield

$$\begin{aligned}\|P_h(G_h(u_1))\|_{L^\infty(\Omega)}\|d_{yy}(\cdot, G_h(u_2)) - d_{yy}(\cdot, G_h(u_1))\|_{L^2(\Omega)} &\leq c\|G_h(u_2) - G_h(u_1)\|_{L^2(\Omega)} \\ &\leq c\|u_2 - u_1\|_{L^2(\Gamma)} \end{aligned} \tag{4.85}$$

with a constant $c = c(\|u_1\|_{L^2(\Gamma)}, \|u_2\|_{L^2(\Gamma)})$. By employing Theorem 2.7 and Lemma 3.40, together with part (i) of Lemma 4.32 to ensure $d_y(\cdot, G_h(u_1)) \in L^\infty(\Omega)$, we obtain

$$\|G_h'(u_1)v\|_{L^4(\Omega)}^2 \leq c\|G_h'(u_1)v\|_{H^1(\Omega)}^2 \leq c\|v\|_{L^2(\Gamma)}^2. \tag{4.86}$$

We conclude by means of Lemma 4.32 (i) that

$$\begin{aligned}\|1 - P_h(G_h(u_2))d_{yy}(\cdot, G_h(u_2))\|_{L^\infty(\Omega)} &\leq 1 + \|P_h(G_h(u_2))\|_{L^\infty(\Omega)}\|d_{yy}(\cdot, G_h(u_2))\|_{L^\infty(\Omega)} \\ &\leq 1 + c \end{aligned} \tag{4.87}$$

with a constant $c = c(\|u_2\|_{L^2(\Gamma)})$. Moreover, estimates as for (4.86), the Lipschitz estimates of Lemma 3.55, the Lipschitz continuity of d_y according to Assumption 4.24 (A4), together with Lemma 4.32 (i) to have the boundedness of $G_h(u_1)$, $G_h(u_2)$, $d_y(\cdot, G_h(u_1))$ and $d_y(\cdot, G_h(u_2))$ in $L^\infty(\Omega)$, and the Lipschitz estimates of Lemma 4.32 (ii) yield

$$\begin{aligned}&\Big[\|G_h'(u_1)v\|_{L^2(\Omega)} + \|G_h'(u_2)v\|_{L^2(\Omega)}\Big]\|G_h'(u_1)v - G_h'(u_2)v\|_{L^2(\Omega)} \\ &\qquad \leq c\|v\|_{L^2(\Gamma)}\|d_y(\cdot, G_h(u_1)) - d_y(\cdot, G_h(u_2))\|_{L^2(\Omega)}\|v\|_{L^2(\Gamma)} \\ &\qquad \leq c\|G_h(u_1) - G_h(u_2)\|_{L^2(\Omega)}\|v\|_{L^2(\Gamma)}^2 \\ &\qquad \leq c\|u_1 - u_2\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)}^2 \end{aligned} \tag{4.88}$$

with a constant $c = c(\|u_1\|_{L^2(\Gamma)}, \|u_2\|_{L^2(\Gamma)})$. Collecting the results (4.83)–(4.88) yields the Lipschitz estimate for J_h''.

(iv) As in (4.83) we get

$$\begin{aligned}|J''(u)[v,v] - J_h''(u)[v,v]| \leq &\Big[\|P_h(G_h(u)) - P(G(u))\|_{L^2(\Omega)}\|d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)} \\ &+ \|P(G(u))\|_{L^\infty(\Omega)}\|d_{yy}(\cdot, G_h(u)) - d_{yy}(\cdot, G(u))\|_{L^2(\Omega)}\Big]\,\|G'(u)v\|_{L^4(\Omega)}^2 \\ &+ \|1 - P_h(G_h(u))d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)}\,\Big[\|G'(u)v\|_{L^2(\Omega)} \\ &+ \|G_h'(u)v\|_{L^2(\Omega)}\Big]\,\|G'(u)v - G_h'(u)v\|_{L^2(\Omega)}. \end{aligned} \tag{4.89}$$

Again, we estimate each term separately. Using Corollary 3.56, the Lipschitz continuity of d_y, cf. Assumption 4.24 (A4), Lemma 4.32 (i) to deduce the boundedness of $G(u)$, $G_h(u)$,

$d_y(\cdot, G(u))$, $d_y(\cdot, G_h(u))$ and $d_{yy}(\cdot, G_h(u))$ in $L^\infty(\Omega)$, and the finite element error estimates of Corollary 3.74, we can show the existence of an $\epsilon > 0$ such that

$$\begin{aligned}
&\|P_h(G_h(u)) - P(G(u))\|_{L^2(\Omega)} \|d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)} \\
&\le c \left(\|G_h(u) - G(u)\|_{L^2(\Omega)} + \left(h^{1+\epsilon} + \|d_y(\cdot, G_h(u)) - d_y(\cdot, G(u))\|_{L^2(\Omega)} \right) \|G(u) - y_d\|_{L^2(\Omega)} \right) \\
&\le c \left(\|G_h(u) - G(u)\|_{L^2(\Omega)} + \left(h^{1+\epsilon} + \|G_h(u) - G(u)\|_{L^2(\Omega)} \right) \|G(u) - y_d\|_{L^2(\Omega)} \right) \\
&\le c \left(h^{1+\epsilon} + h^{1+\epsilon} \|G(u) - y_d\|_{L^2(\Omega)} \right) \le c h^{1+\epsilon} \qquad (4.90)
\end{aligned}$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$. Next, we obtain from Lemma 4.32 (i), the Lipschitz continuity of d_{yy} as stated in Assumption 4.24 (A4), and the finite element error estimates of Corollary 3.74

$$\|P(G(u))\|_{L^\infty(\Omega)} \|d_{yy}(\cdot, G_h(u)) - d_{yy}(\cdot, G(u))\|_{L^2(\Omega)} \le c \|G_h(u) - G(u)\|_{L^2(\Omega)} \le c h^{1+\epsilon} \qquad (4.91)$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$ and some $\epsilon > 0$. As in (4.86) we conclude by means of Theorem 2.7 and Lemma 3.4, together with Lemma 4.32 (i) in order to ensure $d(\cdot, G(u)) \in L^\infty(\Omega)$, that

$$\|G'(u)v\|^2_{L^4(\Omega)} \le c \|G'(u)v\|^2_{H^1(\Omega)} \le c \|v\|^2_{L^2(\Gamma)}. \qquad (4.92)$$

Analogously to (4.87) we get the existence of a constant $c = c(\|u\|_{L^2(\Gamma)})$ such that

$$\|1 - P_h(G_h(u)) d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)} \le 1 + c. \qquad (4.93)$$

Furthermore, estimates as for (4.86) and (4.92), Corollary 3.56, the Lipschitz continuity of d_y as stated in Assumption 4.24 (A4), together with the boundedness of $G(u)$, $G_h(u)$ and $d_y(\cdot, G(u))$ in $L^\infty(\Omega)$ according to Lemma 4.32 (i), and the finite element error estimates of Corollary 3.74 imply

$$\begin{aligned}
&\left[\|G'(u)v\|_{L^2(\Omega)} + \|G_h'(u)v\|_{L^2(\Omega)} \right] \|G'(u)v - G_h'(u)v\|_{L^2(\Omega)} \\
&\qquad \le c \|v\|_{L^2(\Gamma)} \left(h^{1+\epsilon} + \|d_y(\cdot, G(u)) - d_y(\cdot, G_h(u))\|_{L^2(\Omega)} \right) \|v\|_{L^2(\Gamma)} \\
&\qquad \le c \left(h^{1+\epsilon} + \|G(u) - G_h(u)\| \right) \|v\|^2_{L^2(\Gamma)} \le c h^{1+\epsilon} \|v\|^2_{L^2(\Gamma)} \qquad (4.94)
\end{aligned}$$

with some $\epsilon > 0$ and a positive constant $c = c(\|u\|_{L^2(\Gamma)})$, see also (4.90). Summarizing the inequalities (4.89)–(4.94) yields the assertion.

(v) As in the previous parts we start with inserting the definition of J_h''. We obtain

$$\begin{aligned}
&|J_h''(u)[v_1, v_2] - J_h''(u)[v_1, v_1]| \\
&= \left| \int_\Omega \left[1 - P_h(G_h(u)) d_{yy}(\cdot, G_h(u))\right] G_h'(u)v_1 \left(G_h'(u)v_2 - G_h'(u)v_1 \right) + \nu \int_\Gamma v_1 (v_2 - v_1) \right| \\
&\le \|1 - P_h(G_h(u)) d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)} \|G_h'(u)v_1\|_{L^2(\Omega)} \|G_h'(u)v_2 - G_h'(u)v_1\|_{L^2(\Omega)} \\
&+ \nu \left| \int_\Gamma v_1 (v_2 - v_1) \right|, \qquad (4.95)
\end{aligned}$$

where we applied the Hölder inequality and the Cauchy-Schwarz inequality. Next, we conclude as in (4.86) and (4.87)

$$\|1 - P_h(G_h(u)) d_{yy}(\cdot, G_h(u))\|_{L^\infty(\Omega)} \|G_h'(u)v_1\|_{L^2(\Omega)} \le c \|v_1\|_{L^2(\Gamma)} \qquad (4.96)$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$. Furthermore, the Lipschitz estimates from Lemma 3.54, together with Lemma 4.32 (i) to get $d_y(\cdot, G_h(u)) \in L^\infty(\Omega)$, imply

$$\|G_h'(u)v_2 - G_h'(u)v_1\|_{L^2(\Omega)} \leq c\|v_1 - v_2\|_{L^1(\Gamma)} \tag{4.97}$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$. One can deduce from the inequalities (4.95)–(4.97)

$$|J_h''(u)[v_1, v_2] - J_h''(u)[v_1, v_1]| \leq c\|v_1\|_{L^2(\Gamma)}\|v_1 - v_2\|_{L^1(\Gamma)} + \nu \left| \int_\Gamma v_1(v_1 - v_2) \right|$$

with a constant $c = c(\|u\|_{L^2(\Gamma)})$, which ends the proof. □

Remark 4.34. The finite element error estimates for the reduced cost functionals are worst case estimates. Apparently, taking into account the maximal size of the interior angles or using graded meshes improve the convergence rates.

4.4.1 Error estimates for the concept of variational discretization

This section is concerned with discretization error estimates for the concept of variational discretization applied to semilinear elliptic Neumann boundary control problems. As far as we know, this concept was used in [26] for the first time in the context of such problems. The underlying idea in the semilinear setting does not differ from that in the linear one, i.e., we only discretize the state but not the control. This leads to the discrete optimal control problem

$$\min_{u \in U_{ad}} J_h(u). \tag{4.98}$$

In the further course of this section we are going to show that for every local solution of the continuous problem (P_{sl}) which satisfies the second order sufficient optimality condition (4.77), there exists a local solution of (4.98) which converges to the continuous local solution with some suboptimal rate. Based on this, we are going to show that such discrete solutions possess the quasi-optimal convergence rates which we have seen for linear problems.

We start with the proof of existence of such a solution and some sub-optimal convergence results, which are needed in the sequel. For that purpose we introduce an auxiliary discrete optimal control problem according to an idea already presented in [29]. Let $\bar{u}$ be a local solution of the continuous problem (P_{sl}), which satisfies the second order sufficient optimality condition (4.77). Then the auxiliary problem reads as follows:

$$\min_{u \in U_{ad} \cap B_{\varrho'}(\bar{u})} J_h(u) \tag{4.99}$$

with some radius $\varrho' > 0$ which is specified below. Apparently, the condition $u \in B_{\varrho'}(\bar{u})$ defines an additional constraint. However, if one can show the convergence of a solution of (4.99) to the local solution $\bar{u}$ of problem (P_{sl}), then there exists a mesh size $h_0 > 0$ such that for all mesh parameters $h < h_0$ the constraint $u \in B_{\varrho'}(\bar{u})$ becomes inactive. Thus, this solution of (4.99) also represents a local solution of (4.98).

Before doing so, let us state an existence result for solutions of the auxiliary problem (4.99) and corresponding first order optimality conditions.

Lemma 4.35. *Let $\bar{u}$ be a local solution of problem (P_{sl}). For every $\varrho' > 0$ the associated discrete optimal control problem* (4.99) *has at least one solution belonging to $U_{ad} \cap B_{\varrho'}(\bar{u})$. For every local solution $\bar{u}_h \in U_{ad} \cap B_{\varrho'}(\bar{u})$ there exists a unique discrete optimal state $\bar{y}_h = G_h(\bar{u}_h)$ and unique discrete optimal adjoint state $\bar{p}_h = P_h(\bar{y}_h)$ such that the variational inequality*

$$(\bar{p}_h + \nu \bar{u}_h, u - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad} \cap B_{\varrho'}(\bar{u}) \tag{4.100}$$

is satisfied.

Proof. It is enough to notice that the set $U_{ad} \cap B_{\varrho'}(\bar{u})$ is non-empty, convex, bounded and closed. Note that we obviously have $\bar{u} \in U_{ad} \cap B_{\varrho'}(\bar{u})$ due to the variational discretization. Therefore, one can argue as in the continuous case to get the existence of at least one solution in $U_{ad} \cap B_{\varrho'}(\bar{u})$, cf. Theorem 4.28. The first order necessary optimality condition can also be deduced as in the proof of Theorem 4.28. We also refer to [107, Section 4.4.2]. □

Lemma 4.36. *Let $\bar{u}$ be a local solution of problem (P_{sl}) satisfying the second order sufficient optimality condition* (4.77)*. Furthermore, let $\varrho' > 0$ satisfy $\varrho' \leq \varrho$ with the parameter ϱ from Theorem 4.29. Then for any solution $\bar{u}_h$ of the associated discrete optimal control problem* (4.99) *there is an $\epsilon > 0$ such that the estimate*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq ch^{1/2+\epsilon} \tag{4.101}$$

is valid with a constant c independent of h. Moreover, there is a mesh size $h_0 > 0$ such that for all mesh parameters $h < h_0$ a solution $\bar{u}_h$ of (4.99) *is a local solution of* (4.98)*.*

Proof. We proceed similar to the proofs of Section 3.2 of [108], where control problems with finitely many state constraints are analyzed. Let $\bar{u}_h$ be any solution of (4.99), which exists according to Lemma 4.35. Since $\varrho' \leq \varrho$ we have $\bar{u}_h \in U_{ad} \cap B_\varrho(\bar{u})$ and we can use the quadratic growth condition of Theorem 4.29 to conclude

$$\beta \|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \leq J(\bar{u}_h) - J(\bar{u}) = J(\bar{u}_h) - J_h(\bar{u}_h) + J_h(\bar{u}_h) - J_h(\bar{u}) + J_h(\bar{u}) - J(\bar{u}).$$

Apparently, there holds $\bar{u} \in U_{ad} \cap B_\varrho(\bar{u})$. Therefore, we know $J_h(\bar{u}_h) - J_h(\bar{u}) \leq 0$ since $\bar{u}_h$ is a solution of (4.99). Thus, we obtain

$$\beta \|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \leq J(\bar{u}_h) - J_h(\bar{u}_h) + J_h(\bar{u}) - J(\bar{u}).$$

According to Lemma 4.33 there exists an $\epsilon' > 0$ such that

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)} \leq (c/\beta)^{1/2} h^{(1+\epsilon')/2} \tag{4.102}$$

with a constant c independent of h, since $\|\bar{u}_h\|_{L^2(\Gamma)}$ is uniformly bounded in h. Setting $\epsilon = \epsilon'/2$ results in the desired estimate for the control. Next, we show the estimates for the state and adjoint state. By introducing the intermediate function $G_h(\bar{u})$ we get

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq \|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)} + \|G_h(\bar{u}) - \bar{y}_h\|_{L^2(\Omega)} \leq \|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)} + c\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}, \tag{4.103}$$

where we applied the Lipschitz estimates of Lemma 4.32 (ii) in the last step. Using the finite element error estimates of Corollary 3.74, together with Remark 3.46, and (4.102) we conclude

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq c\left(h^{1+\epsilon} + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\right) \leq ch^{1/2+\epsilon}. \tag{4.104}$$

In a similar way we are able to derive an estimate for the adjoint state. First, we introduce the intermediate function $P_h(\bar{y})$. Afterwards, we use Theorem 2.8 and Theorem 2.7, respectively, and the Lipschitz estimates for P_h from Lemma 4.32 (iii), together with Lemma 4.32 (i) to deduce the uniform boundedness of $G(\bar{u})$ and $G_h(\bar{u}_h)$ in $L^\infty(\Omega)$ independent of h. This yields

$$\begin{aligned}
&\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \\
&\qquad \leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + \|P_h(\bar{y}) - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)} + \|P_h(\bar{y}) - \bar{p}_h\|_{L^2(\Omega)} \\
&\qquad \leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + \|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)} + c\|P_h(\bar{y}) - \bar{p}_h\|_{H^1(\Omega)} \\
&\qquad \leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + \|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)} + c\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}.
\end{aligned} \tag{4.105}$$

Next, we use Theorem 2.10, apply the finite element error estimates of Corollary 3.45, together with Remark 3.46, employ the boundedness of $\bar{p}$ in $H^t(\Omega)$ according to Lemma 4.28 and insert (4.104) to conclude

$$\begin{aligned}
&\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \\
&\qquad \leq c\left(\|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)}^{1/2}\|\bar{p} - P_h(\bar{y})\|_{H^1(\Omega)}^{1/2} + \|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}\right) \\
&\qquad \leq ch^{1/2+\epsilon}.
\end{aligned}$$

Finally, we show that any solution $\bar{u}_h$ of (4.99) represents a local solution of (4.98) if the mesh parameter h is small enough. Let us choose h_0 such that $(c/\beta)^{1/2}h_0^{(1+\epsilon')/2} = \varrho'$ with the constants c and β of (4.102). Then we conclude for all $h < h_0$

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)} \leq (c/\beta)^{1/2}h^{(1+\epsilon')/2} < \varrho'.$$

Therefore, the additional constraint is inactive and $\bar{u}_h$ is a local solution of (4.98). □

The following assumption is needed for the remainder of this section.

Assumption 4.37. Let $\bar{u}$ be a local solution of problem (P_{sl}), which satisfies the second order sufficient optimality condition. Furthermore, let $\bar{u}_h$ denote a local solution of (4.98) with associated state $\bar{y}_h = G_h(\bar{u}_h)$ and adjoint state $\bar{p}_h = P_h(\bar{y}_h)$, which have the approximation properties (4.101).

Remark 4.38. For a local solution $\bar{u}_h$ of (4.98) the first order necessary optimality condition can be written as

$$(\bar{p}_h + \nu\bar{u}_h, u - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \qquad \forall u \in U_{ad}, \tag{4.106}$$

since there are no additional constraints compared to (4.99). Furthermore, this condition is equivalent to

$$\bar{u}_h(x) = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h(x)\right) \quad \text{for a.a. } x \in \Gamma, \tag{4.107}$$

which can be proven as in the continuous case.

Next, we are going to show convergence in the L^∞-setting based on the results of Lemma 4.36.

Lemma 4.39. *Let Assumption 4.37 be fulfilled. Then the estimate*

$$\|\bar u - \bar u_h\|_{L^\infty(\Gamma)} + \|\bar y - \bar y_h\|_{L^\infty(\Omega)} + \|\bar p - \bar p_h\|_{L^\infty(\Gamma)} + \|\bar p - \bar p_h\|_{L^\infty(\Omega)} \le ch^{1/2-\epsilon}$$

holds for some arbitrary $\epsilon > 0$.

Proof. Let us start with the estimate for the states. We proceed similar to the proof of Lemma 4.36. According to the discrete Sobolev inequality, cf. [20, Lemma 4.9.2], and Lemma 4.32 (ii) there holds for some arbitrary $\epsilon > 0$

$$\begin{aligned}
\|\bar y - \bar y_h\|_{L^\infty(\Omega)} &\le \|\bar y - G_h(\bar u)\|_{L^\infty(\Omega)} + \|G_h(\bar u) - \bar y_h\|_{L^\infty(\Omega)} \\
&\le \|\bar y - G_h(\bar u)\|_{L^\infty(\Omega)} + c|\ln h|^{1/2}\|G_h(\bar u) - \bar y_h\|_{H^1(\Omega)} \\
&\le \|\bar y - G_h(\bar u)\|_{L^\infty(\Omega)} + c|\ln h|^{1/2}\|\bar u - \bar u_h\|_{L^2(\Gamma)} \\
&\le ch^{1/2-\epsilon}, \qquad (4.108)
\end{aligned}$$

where we applied the finite element error estimates of Corollary 3.74 and Lemma 4.36 in the last steps. In the same manner we can show the estimates for the adjoint states. Due to the continuity of $\bar p - \bar p_h$, the discrete Sobolev inequality [20, Lemma 4.9.2], and Lemma 4.32 (iii), together with Lemma 4.32 (i) to ensure that $G(\bar u)$ and $G_h(\bar u_h)$ are uniformly bounded in $L^\infty(\Omega)$ independent of h, we obtain

$$\begin{aligned}
\|\bar p - \bar p_h\|_{L^\infty(\Gamma)} + \|\bar p - \bar p_h\|_{L^\infty(\Omega)} &\le \|\bar p - P_h(\bar y)\|_{L^\infty(\Omega)} + \|P_h(\bar y) - \bar p_h\|_{L^\infty(\Omega)} \\
&\le \|\bar p - P_h(\bar y)\|_{L^\infty(\Omega)} + c|\ln h|^{1/2}\|P_h(\bar y) - \bar p_h\|_{H^1(\Omega)} \\
&\le \|\bar p - P_h(\bar y)\|_{L^\infty(\Omega)} + c|\ln h|^{1/2}\|\bar y - \bar y_h\|_{L^2(\Omega)} \\
&\le ch^{1/2-\epsilon}, \qquad (4.109)
\end{aligned}$$

where we used Corollary 3.45, together with Lemma 4.32 (i) and Theorem 4.28 for the boundedness, and Lemma 4.36 in the last step. Finally, we derive the estimate for the controls. Using (4.67) and (4.107) and the Lipschitz continuity of the projection operator $\Pi_{[u_a,u_b]}$ we conclude

$$\begin{aligned}
\|\bar u - \bar u_h\|_{L^\infty(\Gamma)} &= \left\|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar p\right) - \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar p_h\right)\right\|_{L^\infty(\Gamma)} \le c\|\bar p - \bar p_h\|_{L^\infty(\Gamma)} \\
&\le ch^{1/2-\epsilon},
\end{aligned}$$

where we inserted (4.109). □

The following lemma represents in some sense the counterpart of Lemma 4.9 of the linear setting.

Lemma 4.40. *Suppose that Assumption 4.37 is satisfied. Moreover, let* δ *be the constant of Theorem 4.29. Then there exists a mesh size* $h_1 > 0$ *such that for all mesh parameters* $h < h_1$ *the estimate*

$$\frac{\delta}{2}\|\bar u - \bar u_h\|^2_{L^2(\Gamma)} \le (J_h'(\bar u) - J_h'(\bar u_h))(\bar u - \bar u_h)$$

is valid.

Proof. For the proof we use techniques similar to those used in e.g. [27, Lemma 4.6] and [26, Lemma 5.5]. First, we show that $\bar{u}_h - \bar{u}$ belongs to the τ-critical cone $C_\tau(\bar{u})$, cf. (4.75). Afterwards, we apply the second order sufficient optimality condition (4.77) to prove the assertion.

Throughout the proof let δ and τ be the constants of Theorem 4.29. Since $\bar{u}_h \in U_{ad}$, there apparently holds $(\bar{u}_h - \bar{u})(x) \geq 0$ if $\bar{u}(x) = u_a$ and $(\bar{u}_h - \bar{u})(x) \leq 0$ if $\bar{u}(x) = u_b$. It remains to show that $(\bar{u}_h - \bar{u})(x) = 0$ if $|(\bar{p} + \nu\bar{u})(x)| > \tau$. By means of Lemma 4.39 we know that there exists a mesh size $h_\tau > 0$ such that for all $h < h_\tau$

$$\|\bar{p} + \nu\bar{u} - \bar{p}_h - \nu\bar{u}_h\|_{L^\infty(\Gamma)} \leq \|\bar{p} - \bar{p}_h\|_{L^\infty(\Gamma)} + \nu\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} < \frac{\tau}{2}.$$

Thus, we can conclude for $(\bar{p} + \nu\bar{u})(x) > \tau$

$$(\bar{p}_h + \nu\bar{u}_h)(x) \geq -\left|(\bar{p}_h + \nu\bar{u}_h)(x) - (\bar{p} + \nu\bar{u})(x)\right| + (\bar{p} + \nu\bar{u})(x) > -\frac{\tau}{2} + \tau = \frac{\tau}{2}.$$

Therefore, we get from (4.67) and (4.107)

$$(\bar{u}_h - \bar{u})(x) = u_a - u_a = 0$$

if $(\bar{p} + \nu\bar{u})(x) > \tau$. For $(\bar{p} + \nu\bar{u})(x) < -\tau$ we can show analogously

$$(\bar{p}_h + \nu\bar{u}_h)(x) \leq \left|(\bar{p}_h + \nu\bar{u}_h)(x) - (\bar{p} + \nu\bar{u})(x)\right| + (\bar{p} + \nu\bar{u})(x) < \frac{\tau}{2} - \tau = -\frac{\tau}{2}$$

and

$$(\bar{u}_h - \bar{u})(x) = u_b - u_b = 0.$$

Summarizing the previous results we have proven $(\bar{u}_h - \bar{u}) \in C_\tau(\bar{u})$. Thus, we know

$$J''(\bar{u})[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}] \geq \delta\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \quad \forall h < h_\tau, \tag{4.110}$$

since $\bar{u}$ fulfills the second order sufficient optimality condition (4.77) by assumption. Due to the mean value theorem, we conclude that there is a function $\hat{u} = \bar{u} + \theta(\bar{u}_h - \bar{u})$ with some $0 < \theta < 1$ such that

$$\begin{aligned}
(J_h'(\bar{u}_h) - J_h'(\bar{u}))(\bar{u}_h - \bar{u}) &= J_h''(\hat{u})[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}] \\
&= J''(\bar{u})[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}] + (J_h''(\hat{u}) - J_h''(\bar{u}))[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}] \\
&\quad + (J_h''(\bar{u}) - J''(\bar{u}))[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}] \\
&\geq J''(\bar{u})[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}] - \left|(J_h''(\hat{u}) - J_h''(\bar{u}))[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}]\right| \\
&\quad - \left|(J_h''(\bar{u}) - J''(\bar{u}))[\bar{u}_h - \bar{u}, \bar{u}_h - \bar{u}]\right|.
\end{aligned}$$

By means of (4.110) and the estimates of Lemma 4.33 and Lemma 4.36 we arrive at

$$\begin{aligned}
(J_h'(\bar{u}_h) - J_h'(\bar{u}))(\bar{u}_h - \bar{u}) &\geq \left(\delta - c\|\hat{u} - \bar{u}\|_{L^2(\Gamma)} - ch^{1+\varepsilon}\right)\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \\
&\geq \left(\delta - c\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)} - ch^{1+\varepsilon}\right)\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \\
&\geq \left(\delta - ch^{1/2+\varepsilon}\right)\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{4.111}$$

Thus, there exists a mesh size $0 < h_1 \leq h_\tau$ such that

$$\frac{\delta}{2}\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \leq (J_h'(\bar{u}_h) - J_h'(\bar{u}))(\bar{u}_h - \bar{u}) \quad \forall h < h_1,$$

which is the desired result. □

Based on the previous lemma we are able to prove the main results of this section, the convergence results for the concept of variational discretization.

Theorem 4.41. *Let Assumption 4.37 be satisfied. Furthermore, let h_1 be the mesh size from Lemma 4.40. Then for all mesh parameters $h < h_1$ the discretization error estimates*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq ch^2|\ln h|^{3/2}$$

are satisfied, provided that the mesh grading parameters $\vec{\mu}$ fulfill $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$.

Proof. By testing the variational inequalities (4.66) and (4.106) with $\bar{u}_h$ and $\bar{u}$, respectively, we obtain, after adding both,

$$\begin{aligned} 0 &\leq (\bar{p} - \bar{p}_h + \nu(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u})_{L^2(\Gamma)} \\ &= (P_h(G_h(\bar{u})) - \bar{p}_h + \nu(\bar{u} - \bar{u}_h), \bar{u}_h - \bar{u})_{L^2(\Gamma)} + (\bar{p} - P_h(G_h(\bar{u})), \bar{u}_h - \bar{u})_{L^2(\Gamma)} \\ &= \left(J_h'(\bar{u}) - J_h'(\bar{u}_h)\right)(\bar{u}_h - \bar{u}) + (\bar{p} - P_h(G_h(\bar{u})), \bar{u}_h - \bar{u})_{L^2(\Gamma)}, \end{aligned}$$

where we inserted the intermediate function $P_h(G_h(\bar{u}))$ and used the definition of J_h'. According to Lemma 4.40 we conclude

$$\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (\bar{p} - P_h(G_h(\bar{u})), \bar{u}_h - \bar{u})_{L^2(\Gamma)} \leq \|\bar{p} - P_h(G_h(\bar{u}))\|_{L^2(\Gamma)}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$$

for all mesh parameters $h < h_1$. Dividing by $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$, inserting the function $P_h(\bar{y})$ applying Theorem 2.8 and using Lemma 4.32 (iii), together with Lemma 4.32 (i) to get $G(\bar{u})$ and $G_h(\bar{u}_h)$ uniformly bounded in $L^\infty(\Omega)$, yields

$$\begin{aligned} \frac{\delta}{2}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} &\leq \|\bar{p} - P_h(G_h(\bar{u}))\|_{L^2(\Gamma)} \\ &\leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + \|P_h(\bar{y}) - P_h(G_h(\bar{u}))\|_{L^2(\Gamma)} \\ &\leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + c\|P_h(\bar{y}) - P_h(G_h(\bar{u}))\|_{H^1(\Omega)} \\ &\leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + c\|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)}. \end{aligned} \tag{4.112}$$

Now, the finite element error estimates on the boundary from Theorem 3.48, together with (4.73), the finite element error estimates in the domain from Corollary 3.72 and the regularity results of Theorem 4.28 imply for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2}, \tag{4.113}$$

which is the desired estimate for the control. According to (4.103) the error of the states can be estimated by

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq \|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)} + c\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2}, \tag{4.114}$$

where we used again Corollary 3.72, Theorem 4.28 and (4.113). As before, there must hold $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$. Finally, we show the estimates for the adjoint states. From (4.105) we conclude

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + \|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)} + c\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}.$$

The finite element error estimates from Theorem 3.48 and Lemma 3.41, together with (4.73), Theorem 4.28 and (4.114) imply for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le ch^2 |\ln h|^{3/2}.$$

□

Using Corollary 3.49, Corollary 3.73 and Corollary 3.42 instead of Theorem 3.48, Corollary 3.72 and Lemma 3.41, respectively, in the proof of Theorem 4.41, we obtain the following result for quasi-uniform triangulations.

Corollary 4.42. *Let Assumption 4.37 be fulfilled and let h_1 be the mesh size from Lemma 4.40. Furthermore, let $\vec{\mu} = 1$ (quasi-uniform mesh), $0 < \vec{\epsilon} < \vec{\lambda}$ and $\rho = \min(2, \min(\vec{1}/2 + \vec{\lambda} - \vec{\epsilon}))$. Then for all mesh parameters $h < h_1$ the estimate*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le ch^{\rho} |\ln h|^{3/2}$$

is valid with a constant c independent of h.

Remark 4.43. In case of distributed semilinear elliptic control problems one can show analogously a convergence order of 2 for all variables provided that the mesh grading parameters fulfill the weaker condition $\vec{\mu} < \vec{\lambda}$. This condition is sufficient in that case since no discretization error estimates on the boundary are required but only estimates in the domain.

4.4.2 Error estimates for the postprocessing approach

This section is devoted to the numerical analysis for the postprocessing approach. To the best of our knowledge there are no results available in the literature which deal with this approach in the context of semilinear elliptic partial differential equations. In the sequel, we are going to show that the superconvergence properties of this approach, which we have seen for linear problems in Section 4.2.2, extend to semilinear problems. More precisely, we going to prove that for every local solution $\bar{u}$ of the continuous problem (P_{sl}), which satisfies the second order sufficient optimality condition (4.77), there exists a local solution of the fully discretized problem

$$\min_{u_h \in U_h^{ad}} J_h(u_h) \tag{4.115}$$

which converges to $\bar{u}$ with some rate. Here, U_h^{ad} again denotes the discrete admissible set, which is given by $U_h^{ad} := U_h \cap U_{ad}$. Based on such discrete solutions, we are going to show that one can construct a new control in a postprocessing step which has the improved approximation properties as in the linear setting. However, as we will see, we have to postulate a slightly stronger structural assumption on the optimal control compared to the linear problems.

To prove the existence of a convergent solution of (4.115), we proceed as in the beginning of Section 4.4.1. Let $\bar{u}$ be a local solution of problem (P_{sl}), which satisfies the second order sufficient optimality condition (4.77) and let $\varrho' > 0$ be a fixed real number. Furthermore, we introduce the auxiliary problem

$$\min_{u_h \in U_h^{ad} \cap B_{\varrho'}(\bar{u})} J_h(u_h). \tag{4.116}$$

We are going to show that there exist a radius ϱ and a mesh size $h_0 > 0$ such that for all $\varrho' \leq \varrho$ and $h < h_0$ a solution $\bar{u}_h$ of (4.116) represents a local solution of (4.115) which converges to the considered local solution $\bar{u}$ of problem (P_{sl}). Afterwards, we will establish the superconvergence properties of the associated postprocessed control based on several auxiliary results.

Let us start with existence results for solutions of (4.116) and first order necessary optimality conditions.

Lemma 4.44. *Let $\bar{u}$ be a local solution of problem (P_{sl}). For every $\varrho' > 0$ there exists a mesh size $h_{\varrho'} > 0$ such that problem (4.116) related to $\bar{u}$ has at least one solution which belongs to $U_h^{ad} \cap B_{\varrho'}(\bar{u})$. Furthermore, for every local solution $\bar{u}_h \in U_h^{ad} \cap B_{\varrho'}(\bar{u})$ of (4.116) there is a unique optimal discrete state $\bar{y}_h = G_h(\bar{u}_h)$ and adjoint state $\bar{p}_h = P_h(\bar{y}_h)$ such that*

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \geq 0 \quad \forall u_h \in U_h^{ad} \cap B_{\varrho'}(\bar{u}). \tag{4.117}$$

Proof. Using the projection formula (4.67) and [69, Theorem A.1], see also (4.46), we can conclude

$$\|\bar{u}\|_{H^1(\Gamma)} \leq \|\bar{u}\|_{L^2(\Gamma)} + \left|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}\right)\right|_{H^1(\Gamma)} \leq \|\bar{u}\|_{L^2(\Gamma)} + c|\bar{p}|_{H^1(\Gamma)}. \tag{4.118}$$

Next, we observe that there is an $\epsilon > 0$ such that $1/2 - \epsilon > 1 - \lambda_j$ for $j = 1, \ldots, m$ since $\omega_j \in (0, 2\pi)$. Employing Theorem 2.16, Lemma 2.29 and Theorem 4.28, we obtain

$$\|\bar{p}\|_{H^1(\Gamma)} \leq c\|\bar{p}\|_{W^{2,4/3}(\Omega)} \leq c\|\bar{p}\|_{W^{2,2}_{\vec{1}/2-\vec{\epsilon}}(\Omega)} \leq c. \tag{4.119}$$

Inequalities (4.118) and (4.119) imply $\bar{u} \in H^1(\Gamma)$ and

$$\|\bar{u}\|_{H^1(\Gamma)} \leq c. \tag{4.120}$$

Therefore, there is a mesh size $h_{\varrho'} > 0$ such that $\|\bar{u} - Q_h\bar{u}\|_{L^2(\Gamma)} \leq \varrho'$ for all mesh parameters $h < h_{\varrho'}$ according to Corollary 3.37. As a consequence, the set $U_h^{ad} \cap B_{\varrho'}(\bar{u})$ is non-empty. Furthermore, it is compact since we are in finite dimensions. Thus, the existence of a solution of (4.116) is an immediate result of the continuity of J_h. The first oder necessary optimality condition (4.117) can be deduced as in the continuous case from Lemma 4.31 and the convexity of $U_h^{ad} \cap B_{\varrho'}(\bar{u})$. □

Next, we present a first convergence result for solutions of (4.116). Furthermore, we show that such a solution is actually a local solution of (4.115) if the mesh size is small enough.

Lemma 4.45. *Let $\bar{u}$ be a local solution of problem (P_{sl}) which satisfies the second order sufficient optimality condition (4.77). Moreover, let $\varrho' > 0$ and the mesh parameter h be chosen such that $\varrho' \leq \varrho$ and $h < h_{\varrho'}$, where ϱ and $h_{\varrho'}$ denote the parameters from Theorem 4.29 and Lemma 4.44, respectively. Then any solution $\bar{u}_h$ of the related discrete optimal control problem (4.116) with state $\bar{y}_h = G_h(\bar{u}_h)$ and adjoint state $\bar{p}_h = P_h(\bar{y}_h)$ fulfills*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq ch^{1/2} \tag{4.121}$$

with a constant c independent of h. Furthermore, there is a mesh size $h_0 > 0$ such that for all mesh parameters $h < h_0$ a solution $\bar{u}_h$ of (4.116) is a local solution of (4.115).

Proof. Let $\bar{u}_h$ be an arbitrary solution of (4.116). This solution belongs to $U_{ad} \cap B_\varrho(\bar{u})$, since $\varrho' \le \varrho$ and $U_h^{ad} \subset U_{ad}$. Thus, the quadratic growth condition of Theorem 4.29 holds, i.e.,

$$\begin{aligned}\beta\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} &\le J(\bar{u}_h) - J(\bar{u}) \\ &= J(\bar{u}_h) - J_h(\bar{u}_h) + J_h(\bar{u}_h) - J_h(Q_h\bar{u}) + J_h(Q_h\bar{u}) - J(Q_h\bar{u}) + J(Q_h\bar{u}) - J(\bar{u}).\end{aligned}$$

Since $\bar{u}_h$ represents a solution of (4.116) and $Q_h\bar{u} \in U_h^{ad} \cap B_{\varrho'}(\bar{u})$, as seen in the previous proof, we know $J_h(\bar{u}_h) \le J_h(Q_h\bar{u})$. Furthermore, we can use the Lipschitz estimate for the functional J from Lemma 4.33. This yields

$$\beta\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le J(\bar{u}_h) - J_h(\bar{u}_h) + J_h(Q_h\bar{u}) - J(Q_h\bar{u}) + c\|Q_h\bar{u} - \bar{u}\|_{L^2(\Gamma)}$$

with a constant independent of h, since $Q_h\bar{u}$ is uniformly bounded in $L^2(\Gamma)$ independent of h. The finite element error estimates for the reduced cost functionals of Lemma 4.33 and Corollary 3.37, together with the boundedness of $\bar{u}$ in $H^1(\Gamma)$ according to (4.120), imply

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le (c/\beta)^{1/2} h^{1/2} \tag{4.122}$$

with a constant c independent of h due to the boundedness of $\bar{u}_h$ in $L^2(\Gamma)$ independent of h. Now, the remaining part of the proof is a word by word repetition of the proof of Lemma 4.36 using (4.122) instead of (4.102). We only have to assume $h_0 \le h_{\varrho'}$ in addition. □

For the remainder of this section we require the following assumption.

Assumption 4.46. Let $\bar{u}$ be a local solution of problem (P_{sl}) which fulfills the second order sufficient optimality condition (4.77) and let $\bar{u}_h$ be a related local solution of (4.115) with associated state $\bar{y}_h = G_h(\bar{u}_h)$ and adjoint state $\bar{p}_h = P_h(\bar{y}_h)$, which satisfy (4.121).

Remark 4.47. The first order necessary optimality condition for a local solution $\bar{u}_h$ of (4.115) can be stated as

$$(\bar{p}_h + \nu\bar{u}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} \ge 0 \quad \forall u_h \in U_h^{ad}, \tag{4.123}$$

which can be proven as in the continuous case or by standard techniques known from the finite dimensional optimization theory. Moreover, this condition is equivalent to

$$(R_h\bar{p}_h + \nu\bar{u}_h)_{|E}(u_h - \bar{u}_h)_{|E} \ge 0 \quad \forall u_h \in [u_a, u_b] \text{ and } \forall E \in \mathcal{E}_h \tag{4.124}$$

and

$$\bar{u}_h(x) = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}(R_h\bar{p}_h)(x)\right) \quad \text{for a.a. } x \in \Gamma, \tag{4.125}$$

cf. [110, Lemma 4.3].

Next, let us establish convergence results in the L^∞-norm.

Lemma 4.48. *Let Assumption 4.46 be fulfilled. Furthermore, let some arbitrary $\epsilon > 0$ be given. Then the estimate*

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^\infty(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \le ch^{1/2-\epsilon}$$

is valid.

Proof. Using the results of Lemma 4.45 the estimates for the states and adjoint states can be proven analogously to (4.108) and (4.109), respectively, i.e.,

$$\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^\infty(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \leq ch^{1/2-\epsilon}. \tag{4.126}$$

To get an estimate for the controls we use the projection formulas (4.67) and (4.125). We obtain, by employing the Lipschitz continuity of the projection operator $\Pi_{[u_a,u_b]}$,

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} = \left\|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}\right) - \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}R_h\bar{p}_h\right)\right\|_{L^\infty(\Gamma)} \leq c\|\bar{p} - R_h\bar{p}_h\|_{L^\infty(\Gamma)}.$$

Next, let E_* be the element on the boundary where $\bar{p} - R_h\bar{p}_h$ admits its maximum. We can conclude

$$\begin{aligned}
\|\bar{p} - R_h\bar{p}_h\|_{L^\infty(\Gamma)} &= \|\bar{p} - R_h\bar{p}_h\|_{L^\infty(E_*)} \\
&\leq c\left(\|\bar{p} - R_h\bar{p}\|_{L^\infty(E_*)} + \|R_h\left(\bar{p} - \bar{p}_h\right)\|_{L^\infty(E_*)}\right) \\
&\leq c\left(\|\bar{p} - R_h\bar{p}\|_{L^\infty(E_*)} + \|\bar{p} - \bar{p}_h\|_{L^\infty(E_*)}\right),
\end{aligned}$$

where we used in the last step that R_h is a bounded operator from $L^\infty(E_*)$ to $L^\infty(E_*)$ independent of h. Corollary 3.35, (4.119) and (4.126) imply

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq ch^{1/2-\epsilon}.$$

□

In our error analysis we will need a discrete control u_h, which is admissible for (4.115), close to the optimal control $\bar{u}$ and the direction $\bar{u}_h - u_h$ should belong to the critical cone $C_\tau(\bar{u})$, cf. (4.75), such that the second order sufficient optimality condition can be applied. An intuitive choice seems to be $u_h = R_h\bar{u}$. Indeed, the element $R_h\bar{u}$ is admissible for (4.115) and close to $\bar{u}$ but $\bar{u}_h - R_h\bar{u}$ does not necessarily belong to the critical cone. To overcome this difficulty, we will modify the interpolator R_h. Due to the regularity of the adjoint state, see Theorem 4.28, and the fact that the optimal control is given by the projection formula (4.67), we can distinguish between active points ($\bar{u}(x) \in \{u_a, u_b\}$) and inactive points ($\bar{u}(x) \in (u_a, u_b)$). Based on this we can classify the edges $E \in \mathcal{E}_h$ in the following two sets K_1 and K_2 as in Section 2 of [100]:

$$\begin{aligned}
K_1 &:= \{E \in \mathcal{E}_h : \ E \text{ contains active and inactive points}\}, \\
K_2 &:= \{E \in \mathcal{E}_h : \ E \text{ contains only active points or only inactive points}\}.
\end{aligned}$$

The modified interpolation operator is now defined by

$$(R_h^{\bar{u}} f)(x) := \begin{cases} (R_h f)(x) & \text{for } x \in E, \ E \in K_2, \\ f(x_K) & \text{for } x \in E, \ E \in K_1 \end{cases} \tag{4.127}$$

with $x_K \in E$ such that either $\bar{u}(x_K) = u_a$ or $\bar{u}(x_K) = u_b$. We make the following assumption on the measure of the set K_1, which is quite common in the context of PDE constrained optimization, see e.g. [100, 26].

Assumption 4.49. We suppose that $|K_1| \leq ch$.

Remark 4.50. Compared to linear quadratic elliptic optimal control problems the Assumption 4.49 is slightly stronger. In the linear case the set K_1 is only the union of all elements E where the optimal control has kinks with the control constraints, whereas the present definition of the set K_1 also contains elements E where the optimal control intersects smoothly the control constraints. However, the definition of the modified interpolation operator $R_h^{\bar{u}}$ makes the stronger assumption necessary to prove the superconvergence properties of the postprocessed control in the further course of this section.

Next, we prove several auxiliary results which are needed for the proof of the superconvergence result on page 154. We start with a result analogous to that of Lemma 4.15 derived for the linear setting.

Lemma 4.51. *Suppose that Assumptions 4.46 and 4.49 hold. Then for all mesh grading parameters $\vec{\mu}$ with $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$ the estimate*

$$\|G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})\|_{L^2(\Omega)} \leq ch^2$$

holds true.

Proof. Initially, we observe that according to Corollary 3.36 and (4.120) there holds

$$\|\bar{u} - R_h^{\bar{u}}\bar{u}\|_{L^2(\Gamma)} + h^{1/2}\|\bar{u} - R_h^{\bar{u}}\bar{u}\|_{L^\infty(\Gamma)} \leq ch|\bar{u}|_{H^1(\Gamma)} \leq ch. \tag{4.128}$$

Next, we introduce a dual problem and its discrete counterpart following the ideas of [26, Appendix]. Let $w \in H^1(\Omega)$ be the unique solution of

$$a(w, v) + \int_\Omega \psi w v = \int_\Omega (G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u}))v \quad \forall v \in H^1(\Omega)$$

with

$$\psi(x) := \begin{cases} \frac{d(x, G_h(\bar{u})(x)) - d(x, G_h(R_h^{\bar{u}}\bar{u})(x))}{G_h(\bar{u})(x) - G_h(R_h^{\bar{u}}\bar{u})(x)} & \text{if } G_h(\bar{u})(x) - G_h(R_h^{\bar{u}}\bar{u})(x) \neq 0, \\ \alpha(x) & \text{otherwise,} \end{cases}$$

where α is defined at the beginning of Section 3.1.2. Due to the Assumptions 4.24 (A4)–(A5), together with Lemma 4.32 (i) and (4.128) to ensure the uniform boundedness of $G_h(\bar{u})$ and $G_h(R_h^{\bar{u}}\bar{u})$ in $L^\infty(\Omega)$ independent of h, we can conclude for the function ψ that $\psi \geq c_\Omega$ on E_Ω and $\|\psi\|_{L^\infty(\Omega)} \leq c(\|\bar{u}\|_{L^2(\Gamma)})$. Thus, the problem is well-posed according to Lemma 3.4. The corresponding discrete counterpart $w_h \in V_h$ is the unique solution of the problem

$$a(w_h, v_h) + \int_\Omega \psi w_h v_h = \int_\Omega (G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u}))v_h \quad \forall v_h \in V_h.$$

By means of the definition of G_h and ψ we derive

$$\begin{aligned} \|G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})\|_{L^2(\Omega)}^2 &= a(w_h, G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})) + \int_\Omega \psi w_h (G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})) \\ &= a(G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u}), w_h) + \int_\Omega (d(\cdot, G_h(\bar{u})) - d(\cdot, G_h(R_h^{\bar{u}}\bar{u}))w_h \\ &= \int_\Gamma (\bar{u} - R_h^{\bar{u}}\bar{u})w_h, \end{aligned} \tag{4.129}$$

see the definition of the operator G_h for the last step. From now on, the proof proceeds similar to that of Lemma 4.15. We split the last term in two by introducing the intermediate function w, i.e.,

$$(\bar u - R_h^{\bar u}\bar u, w_h)_{L^2(\Gamma)} = (\bar u - R_h^{\bar u}\bar u, w_h - w)_{L^2(\Gamma)} + (\bar u - R_h^{\bar u}\bar u, w)_{L^2(\Gamma)}. \tag{4.130}$$

For the first term, we get using the Cauchy-Schwarz inequality, (4.128), the Trace Theorem 2.8 and the finite element error estimates from Lemma 3.41

$$\begin{aligned}(\bar u - R_h^{\bar u}\bar u, w_h - w)_{L^2(\Gamma)} &\le \|\bar u - R_h^{\bar u}\bar u\|_{L^2(\Gamma)}\|w_h - w\|_{L^2(\Gamma)} \le c\|\bar u - R_h^{\bar u}\bar u\|_{L^2(\Gamma)}\|w_h - w\|_{H^1(\Omega)}\\ &\le ch^2\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}\end{aligned} \tag{4.131}$$

if $\vec\mu < \vec\lambda$. For the second term in (4.130) we introduce $Q_h\bar u$. This yields

$$(\bar u - R_h^{\bar u}\bar u, w)_{L^2(\Gamma)} = (\bar u - Q_h\bar u, w)_{L^2(\Gamma)} + (Q_h\bar u - R_h^{\bar u}\bar u, w)_{L^2(\Gamma)}. \tag{4.132}$$

According to Corollary 3.38 we estimate

$$(\bar u - Q_h\bar u, w)_{L^2(\Gamma)} \le ch^2|\bar u|_{H^1(\Gamma)}|w|_{H^1(\Gamma)}. \tag{4.133}$$

Analogously to (4.119) we can conclude with Lemma 3.11 that

$$|w|_{H^1(\Gamma)} \le c\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}. \tag{4.134}$$

From the previous estimate and the boundedness of $\bar u$ in $H^1(\Gamma)$, see (4.120), we obtain for (4.133)

$$(\bar u - Q_h\bar u, w)_{L^2(\Gamma)} \le ch^2\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}. \tag{4.135}$$

Next, we estimate the second term of (4.132). The Hölder inequality yields

$$\begin{aligned}(Q_h\bar u - R_h^{\bar u}\bar u, w)_{L^2(\Gamma)} &\le \|Q_h\bar u - R_h^{\bar u}\bar u\|_{L^1(\Gamma)}\|w\|_{L^\infty(\Gamma)} \le \|Q_h\bar u - R_h^{\bar u}\bar u\|_{L^1(\Gamma)}\|w\|_{H^1(\Gamma)}\\ &\le \|Q_h\bar u - R_h^{\bar u}\bar u\|_{L^1(\Gamma)}\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)},\end{aligned} \tag{4.136}$$

where we used Theorem 2.7 and (4.134) in the last steps. We deduce analogously to (4.43) and (4.44) for $\vec\mu > \vec 1/4$ using Corollary 3.36 instead of Corollary 3.35

$$\begin{aligned}\|Q_h\bar u - R_h^{\bar u}\bar u\|_{L^1(\Gamma)} &= \sum_{j=0}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} \left|\int_E \left(\bar u - R_h^{\bar u}\bar u\right)\right| + \sum_{j=0}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} \left|\int_E \left(\bar u - R_h^{\bar u}\bar u\right)\right|\\ &\le c\left(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_1}} h|E||\bar u|_{W^{1,\infty}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} h|E||\bar u|_{W^{1,\infty}_{1-\mu_j}(E)}\right.\\ &\quad\left. + \sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_2}} h^2|E|^{1/2}|\bar u|_{W^{2,2}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} h^2|E|^{1/2}|\bar u|_{W^{2,2}_{2(1-\mu_j)}(E)}\right)\\ &\le ch|K_1|\left(|\bar u|_{W^{1,\infty}(K_1\cap\Gamma^0)} + \sum_{j=1}^m |\bar u|_{W^{1,\infty}_{1-\mu_j}(K_1\cap\Gamma_j^\pm)}\right)\end{aligned}$$

$$+ ch^2|K_2|^{1/2} \left(|\bar u|_{W^{2,2}(K_2\cap\Gamma^0)} + \sum_{j=1}^m |\bar u|_{W^{2,2}_{2(1-\mu_j)}(K_2\cap\Gamma_j^\pm)} \right)$$

$$\le ch^2 \left(|\bar u|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1)} + |\bar u|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2)} \right), \tag{4.137}$$

where we applied Assumption 4.49 and the discrete Cauchy-Schwarz inequality. Arguing as for (4.47) we conclude by means of Theorem 4.28 for $\vec 1/4 < \vec\mu < \vec 1/4 + \vec\lambda/2$

$$|\bar u|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1)} + |\bar u|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2)} \le c \left(\|\bar p\|_{W^{1,\infty}_{\vec 1-\vec\mu}(K_1)} + \|\bar p\|_{W^{2,2}_{2(\vec 1-\vec\mu)}(K_2)} \right) \le c. \tag{4.138}$$

Finally, collecting (4.129), (4.130), (4.131), (4.132), (4.135), (4.136), (4.137) and (4.138) yields the assertion if we divide by $\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}$. □

Corollary 4.52. *Suppose that Assumptions 4.46 and 4.49 are fulfilled. Furthermore, let* $\vec\mu = 1$ *(quasi-uniform mesh) and* $\vec 0 < \vec\epsilon < \vec\lambda$. *Then the estimate*

$$\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)} \le ch^\rho$$

is valid with $\rho = \min(2, \min(\vec 1/2 + \vec\lambda - \vec\epsilon))$.

Proof. The proof is almost a word by word repetition of the proof of Lemma 4.51. Only (4.131) and (4.137) have to be adjusted. If we use Corollary 3.42 instead of Lemma 3.41 we obtain for (4.131)

$$(\bar u - R_h^{\bar u}\bar u, w_h - w)_{L^2(\Gamma)} \le ch^{1+\lambda}\|G_h(\bar u) - G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}$$

with $\lambda = \min(1, \min(\vec\lambda - \vec\epsilon))$. Next, let $\vec\kappa \in \mathbb{R}^m$ fulfill $\vec\kappa = \vec 3/2 - \vec\lambda + \vec\epsilon$ and $\vec\kappa \ge \vec 0$. Moreover, let $\tau \in \mathbb{R}^m$ satisfy $\vec\tau = \vec 1 - \vec\lambda + \vec\epsilon$ and $\vec\tau \ge \vec 0$. Then, if we use Corollary 3.36 with $\vec\mu = \vec 1$, we can conclude for (4.137) as in (4.48) that

$$\|Q_h\bar u - R_h\bar u\|_{L^1(\Gamma)} \le ch^\rho \left(|\bar u|_{W^{1,\infty}_{\vec\tau}(K_1)} + |\bar u|_{W^{2,2}_{\vec\kappa}(K_2)} \right).$$

All other arguments remain unchanged. □

Lemma 4.53. *Let* χ_1 *and* χ_2 *be the characteristic functions of the sets* K_1 *and* K_2, *respectively. Furthermore, let* δ *be the constant of Theorem 4.29 and suppose that Assumptions 4.46 and 4.49 are fulfilled. Then there exists a mesh size* $h_1 > 0$ *and a constant* $\varepsilon > 0$ *such that for all mesh parameters* $h < h_1$ *the estimate*

$$\frac{\delta}{2} \left\| \left(h^{1/2}\chi_1 + \varepsilon^{1/2}\chi_2 \right) \left(\bar u_h - R_h^{\bar u}\bar u \right) \right\|^2_{L^2(\Gamma)} \le \left(J_h'(\bar u_h) - J_h'(R_h^{\bar u}\bar u) \right) \left((h\chi_1 + \varepsilon\chi_2) \left(\bar u_h - R_h^{\bar u}\bar u \right) \right)$$

holds.

Proof. We proceed similar to the proof of Lemma 4.40. Let δ and τ be the constants of Theorem 4.29. First, we show that $\bar u_h - R_h^{\bar u}\bar u$ belongs to the critical cone $C_\tau(\bar u)$, see (4.75) for the definition. In a second step we apply the second order sufficient optimality condition to deduce the desired estimate. Since $\bar u_h(x) \in [u_a, u_b]$ we can easily conclude $(\bar u_h - R_h^{\bar u}\bar u)(x) \ge 0$

if $\bar{u}(x) = u_a$ and $(\bar{u}_h - R_h^{\bar{u}}\bar{u})(x) \leq 0$ if $\bar{u}(x) = u_b$ due to the definition of the interpolation operator $R_h^{\bar{u}}$. Thus, we only have to show $(\bar{u}_h - R_h^{\bar{u}}\bar{u})(x) = 0$ if $|(\bar{p} + \nu\bar{u})(x)| \geq \tau$. According to Lemma 4.48 there exists a mesh size $h_\tau > 0$ such that

$$\|\bar{p} + \nu\bar{u} - \bar{p}_h - \nu\bar{u}_h\|_{L^\infty(\Gamma)} \leq \|\bar{p} - \bar{p}_h\|_{L^\infty(\Gamma)} + \nu\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} < \frac{\tau}{4} \tag{4.139}$$

is satisfied for all mesh parameters $h < h_\tau$. Furthermore, we choose $0 < h_\xi \leq h_\tau$ such that

$$|(\bar{p} + \nu\bar{u})(x_1) - (\bar{p} + \nu\bar{u})(x_2)| < \frac{\tau}{4} \quad \text{if } |x_1 - x_2| < h_\xi, \tag{4.140}$$

which is possible due to the continuity of $\bar{p}+\nu\bar{u}$. Next, let $E \in \mathcal{E}_h$ and $x \in E$ an arbitrary point with $(\bar{p} + \nu\bar{u})(x) > \tau$. Due to (4.140) we can conclude for all $\xi \in E$ and all mesh parameters $h < h_\xi$

$$(\bar{p} + \nu\bar{u})(\xi) \geq -|(\bar{p} + \nu\bar{u})(\xi) - (\bar{p} + \nu\bar{u})(x)| + (\bar{p} + \nu\bar{u})(x) > -\frac{\tau}{4} + \tau = \frac{3\tau}{4}$$

and as a consequence, together with (4.139),

$$(\bar{p}_h + \nu\bar{u}_h)(\xi) \geq -|(\bar{p}_h + \nu\bar{u}_h)(\xi) - (\bar{p} + \nu\bar{u})(\xi)| + (\bar{p} + \nu\bar{u})(\xi) > -\frac{\tau}{4} + \frac{3\tau}{4} = \frac{\tau}{2}.$$

Therefore, we obtain from the projection formula (4.67), together with the definition of the operator $R_h^{\bar{u}}$, and the projection formula (4.125) that

$$(\bar{u}_h - R_h^{\bar{u}}\bar{u})(x) = u_a - u_a = 0$$

if $(\bar{p} + \nu\bar{u})(x) > \tau$. Next, let $x \in E$ with $(\bar{p} + \nu\bar{u})(x) < -\tau$. Analogously, we get for all $\xi \in E$ and all mesh parameters $h < h_\xi$

$$(\bar{p} + \nu\bar{u})(\xi) \leq |(\bar{p} + \nu\bar{u})(\xi) - (\bar{p} + \nu\bar{u})(x)| + (\bar{p} + \nu\bar{u})(x) < \frac{\tau}{4} - \tau = -\frac{3\tau}{4}$$

and consequently

$$(\bar{p}_h + \nu\bar{u}_h)(\xi) \leq |(\bar{p}_h + \nu\bar{u}_h)(\xi) - (\bar{p} + \nu\bar{u})(\xi)| + (\bar{p} + \nu\bar{u})(\xi) < \frac{\tau}{4} - \frac{3\tau}{4} = -\frac{\tau}{2}.$$

and therefore

$$(\bar{u}_h - R_h^{\bar{u}}\bar{u})(x) = u_b - u_b = 0$$

if $(\bar{p} + \nu\bar{u})(x) < -\tau$. Thus, we have proven $(\bar{u}_h - R_h^{\bar{u}}\bar{u}) \in C_\tau(\bar{u})$. Obviously, $\chi_1(\bar{u}_h - R_h^{\bar{u}}\bar{u})$ and $\chi_2(\bar{u}_h - R_h^{\bar{u}}\bar{u})$ also belong to the critical cone whenever $(\bar{u}_h - R_h^{\bar{u}}\bar{u}) \in C_\tau(\bar{u})$. Now, we can apply the second order sufficient optimality condition (4.77), i.e., there holds

$$J''(\bar{u})[\chi_1(\bar{u}_h - R_h^{\bar{u}}\bar{u}), \chi_1(\bar{u}_h - R_h^{\bar{u}}\bar{u})] \geq \delta\|\bar{u}_h - R_h^{\bar{u}}\bar{u}\|^2_{L^2(K_1)} \quad \forall h < h_\xi \tag{4.141}$$

and

$$J''(\bar{u})[\chi_2(\bar{u}_h - R_h^{\bar{u}}\bar{u}), \chi_2(\bar{u}_h - R_h^{\bar{u}}\bar{u})] \geq \delta\|\bar{u}_h - R_h^{\bar{u}}\bar{u}\|^2_{L^2(K_2)} \quad \forall h < h_\xi. \tag{4.142}$$

Furthermore, if we use the results of Lemma 4.45 and (4.128) we deduce

$$\|\bar{u}_h - R_h^{\bar{u}}\bar{u}\|_{L^2(\Gamma)} \leq \|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)} + \|\bar{u} - R_h^{\bar{u}}\bar{u}\|_{L^2(\Gamma)} \leq ch^{1/2}. \tag{4.143}$$

To shorten the notation in the sequel let us define $v = \bar{u}_h - R_h^{\bar{u}}\bar{u}$, $v_1 = \chi_1(\bar{u}_h - R_h^{\bar{u}}\bar{u})$ and $v_2 = \chi_2(\bar{u}_h - R_h^{\bar{u}}\bar{u})$. Furthermore, let ε be a positive constant exactly specified below. Due to the mean value theorem, we can conclude for $\hat{u} = R_h^{\bar{u}}\bar{u} + \theta(\bar{u}_h - R_h^{\bar{u}}\bar{u})$ with some $\theta \in (0, 1)$

$$\begin{aligned}
&(J_h'(\bar{u}_h) - J_h'(R_h^{\bar{u}}\bar{u}))\,(hv_1 + \varepsilon v_2) = J_h''(\hat{u})[v, hv_1 + \varepsilon v_2] = hJ_h''(\hat{u})[v, v_1] + \varepsilon J_h''(\hat{u})[v, v_2] \\
&= h\left(J''(\bar{u})[v_1, v_1] + J_h''(\hat{u})[v - v_1, v_1] + (J_h''(\hat{u}) - J_h''(\bar{u}))[v_1, v_1] + (J_h''(\bar{u}) - J''(\bar{u}))[v_1, v_1]\right) \\
&+ \varepsilon\left(J''(\bar{u})[v_2, v_2] + J_h''(\hat{u})[v - v_2, v_2] + (J_h''(\hat{u}) - J_h''(\bar{u}))[v_2, v_2] + (J_h''(\bar{u}) - J''(\bar{u}))[v_2, v_2]\right) \\
&\geq h\left(J''(\bar{u})[v_1, v_1] - \left|J_h''(\hat{u})[v - v_1, v_1]\right| - \left|(J_h''(\hat{u}) - J_h''(\bar{u}))[v_1, v_1]\right| - \left|(J_h''(\bar{u}) - J''(\bar{u}))[v_1, v_1]\right|\right) \\
&+ \varepsilon\left(J''(\bar{u})[v_2, v_2] - \left|J_h''(\hat{u})[v - v_2, v_2]\right| - \left|(J_h''(\hat{u}) - J_h''(\bar{u}))[v_2, v_2]\right| - \left|(J_h''(\bar{u}) - J''(\bar{u}))[v_2, v_2]\right|\right),
\end{aligned}$$

where we introduced several intermediate functions. Next, we employ (4.141), (4.142) and the results of Lemma 4.33. We obtain

$$\begin{aligned}
&(J_h'(\bar{u}_h) - J_h'(R_h^{\bar{u}}\bar{u}))\,(hv_1 + \varepsilon v_2) \geq h\left(\delta\|v_1\|^2_{L^2(\Gamma)} - c\|v_1\|_{L^2(\Gamma)}\|v - v_1\|_{L^1(\Gamma)} - \nu\left|\int_\Gamma v_1(v - v_1)\right|\right. \\
&\left.-c\|\hat{u} - \bar{u}\|_{L^2(\Gamma)}\|v_1\|^2_{L^2(\Gamma)} - ch\|v_1\|^2_{L^2(\Gamma)}\right) + \varepsilon\left(\delta\|v_2\|^2_{L^2(\Gamma)} - c\|v_2\|_{L^2(\Gamma)}\|v - v_2\|_{L^1(\Gamma)}\right. \\
&\left.-\nu\left|\int_\Gamma v_2(v - v_2)\right| - c\|\hat{u} - \bar{u}\|_{L^2(\Gamma)}\|v_2\|^2_{L^2(\Gamma)} - ch\|v_2\|^2_{L^2(\Gamma)}\right) \qquad (4.144)
\end{aligned}$$

Let us consider selected terms of (4.144). First, we observe that

$$\left|\int_\Gamma v_1(v - v_1)\right| = \left|\int_\Gamma v_2(v - v_2)\right| = \left|\int_\Gamma v_1 v_2\right| = 0 \qquad (4.145)$$

by construction. Next, we employ the definitions of the functions v, v_1 and v_2, and use the Hölder inequality and Young's inequality to deduce

$$\begin{aligned}
\|v_1\|_{L^2(\Gamma)}\|v - v_1\|_{L^1(\Gamma)} &= \|v_1\|_{L^2(\Gamma)}\|v_2\|_{L^1(\Gamma)} \leq c\|v_1\|_{L^2(\Gamma)}\|v_2\|_{L^2(\Gamma)} \\
&\leq c\left(\epsilon_1\|v_1\|^2_{L^2(\Gamma)} + \frac{1}{\epsilon_1}\|v_2\|^2_{L^2(\Gamma)}\right) \qquad (4.146)
\end{aligned}$$

with some arbitrary $\epsilon_1 > 0$. In the same manner we obtain by inserting the definitions of the functions v, v_1 and v_2 together with the Hölder inequality, Assumption 4.49 and the Young's inequality

$$\begin{aligned}
\varepsilon\|v_2\|_{L^2(\Gamma)}\|v - v_2\|_{L^1(\Gamma)} &= \varepsilon\|v_2\|_{L^2(\Gamma)}\|v_1\|_{L^1(K_1)} \leq \varepsilon|K_1|^{1/2}\|v_2\|_{L^2(\Gamma)}\|v_1\|_{L^2(K_1)} \\
&\leq c\varepsilon h^{1/2}\|v_2\|_{L^2(\Gamma)}\|v_1\|_{L^2(\Gamma)} \leq c\left(h\epsilon_2\|v_1\|^2_{L^2(\Gamma)} + \frac{\varepsilon^2}{\epsilon_2}\|v_2\|^2_{L^2(\Gamma)}\right), \qquad (4.147)
\end{aligned}$$

where $\epsilon_2 > 0$ is arbitrary. Moreover, we conclude according to (4.128) and (4.143)

$$\|\hat{u} - \bar{u}\|_{L^2(\Gamma)} \leq \|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^2(\Gamma)} + \|\bar{u}_h - R_h^{\bar{u}}\bar{u}\|_{L^2(\Gamma)} \leq ch^{1/2}. \qquad (4.148)$$

Summarizing (4.144)–(4.148) we get

$$\begin{aligned}
(J_h'(\bar{u}_h) - J_h'(R_h^{\bar{u}}\bar{u}))\,(hv_1 + \varepsilon v_2) &\geq h\left(\delta - c\epsilon_1 - ch^{1/2} - ch - c\epsilon_2\right)\|v_1\|^2_{L^2(\Gamma)} \\
&+ \varepsilon\left(\delta - c\frac{\varepsilon}{\epsilon_2} - ch^{1/2} - ch - \frac{ch}{\varepsilon\epsilon_1}\right)\|v_2\|^2_{L^2(\Gamma)}. \qquad (4.149)
\end{aligned}$$

Finally, let us choose ϵ_1 end ϵ_2 such that $c\epsilon_1 + c\epsilon_2 \leq \delta/4$ and ε such that $c\varepsilon/\epsilon_2 \leq \delta/4$. Then there exists a mesh size $0 < h_1 \leq h_\xi$ such that for all $h < h_1$

$$\max\left(ch^{1/2} + ch, ch^{1/2} + ch + \frac{ch}{\varepsilon\epsilon_1}\right) \leq \frac{\delta}{4}.$$

The desired result follows from (4.149) and the definitions of the functions v_1 and v_2. □

Based on the previous lemma we can show the following supercloseness result, i.e., the function $R_h^{\bar u}\bar u$ is closer to the discrete control $\bar u_h$ than to the continuous control $\bar u$. The result is the counterpart of Lemma 4.18 of the linear setting, but due to the definition of the operator $R_h^{\bar u}$ we cannot expect a comparable result. We will comment on this in Remark 4.56.

Lemma 4.54 (Supercloseness). *Let Assumptions 4.46 and 4.49 be satisfied. Furthermore, let χ_1 and χ_2 be the characteristic functions of the sets K_1 and K_2, respectively, and let δ and ε be the constants of Theorem 4.29 and Lemma 4.53, respectively. Then for all mesh parameters $h < h_1$ with mesh size h_1 from Lemma 4.53 there is the estimate*

$$\frac{\delta}{2}\left\|\left(h^{1/2}\chi_1 + \varepsilon^{1/2}\chi_2\right)\left(\bar u_h - R_h^{\bar u}\bar u\right)\right\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2},$$

provided that the mesh grading parameters $\vec\mu$ fulfill $\vec 1/4 < \vec\mu < \vec 1/4 + \vec\lambda/2$.

Proof. We start with the pointwise a.e. version of the variational inequality (4.66), i.e.,

$$(\bar p(x) + \nu\bar u(x))(u - \bar u(x)) \geq 0 \quad \forall u \in [u_a, u_b] \text{ and for a.a. } x \in \Gamma.$$

For every $E \in \mathcal{E}_h$ and $E \subset K_1$ we set $x = x_K$ and $u = \bar u_h(x_K)$, where $x_K \in E$ is a point satisfying either $\bar u(x_K) = u_a$ or $\bar u(x_K) = u_b$. Then we multiply this formula with h, integrate over E and sum up over all $E \subset K_1$. Using the operator $R_h^{\bar u}$, this yields

$$(R_h^{\bar u}\bar p + \nu R_h^{\bar u}\bar u, h\chi_1(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)} \geq 0. \tag{4.150}$$

For every $E \in \mathcal{E}_h$ and $E \subset K_2$ we proceed similarly. In contrast, we set $x = S_E$ and $u = \bar u_h(S_E)$ with S_E being the midpoint of E. Then multiplying the formula with ε, integrating over E and summing up over all $E \subset K_2$ yields

$$(R_h^{\bar u}\bar p + \nu R_h^{\bar u}\bar u, \varepsilon\chi_2(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)} \geq 0 \tag{4.151}$$

due to the definition of the operator $R_h^{\bar u}$. As a consequence of (4.150) and (4.151) we obtain

$$(R_h^{\bar u}\bar p + \nu R_h^{\bar u}\bar u, (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)} \geq 0. \tag{4.152}$$

From (4.123) we deduce

$$(\bar p_h + \nu\bar u_h, u_h - \bar u_h)_{L^2(E)} \geq 0 \quad \forall u_h \in [u_a, u_b] \text{ and } \forall E \in \mathcal{E}_h.$$

We choose $u_h = R_h^{\bar u}\bar u$. Furthermore, for every $E \subset K_1$ and for every $E \subset K_2$ we multiply this formula with h and ε, respectively. Then we sum up over all $E \subset K_1$ and $E \subset K_2$, respectively, and add the resulting inequalities. This yields

$$(\bar p_h + \nu\bar u_h, (h\chi_1 + \varepsilon\chi_2)(R_h^{\bar u}\bar u - \bar u_h))_{L^2(\Gamma)} \geq 0. \tag{4.153}$$

Next, we add the inequalities (4.152) and (4.153), i.e.,

$$(R_h^{\bar u}\bar p - \bar p_h + \nu(R_h^{\bar u}\bar u - \bar u_h), (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)} \geq 0.$$

We obtain by inserting some intermediate functions that

$$\begin{aligned}&(R_h^{\bar u}\bar p - \bar p + \bar p - P_h(G_h(R_h^{\bar u}\bar u)), (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)}\\ &+ (P_h(G_h(R_h^{\bar u}\bar u)) - \bar p_h + \nu(R_h^{\bar u}\bar u - \bar u_h), (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)} \geq 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(J_h'(R_h^{\bar u}\bar u) - J_h'(\bar u_h))(h\chi_1 + \varepsilon\chi_2)(R_h^{\bar u}\bar u - \bar u_h))_{L^2(\Gamma)} &\leq (R_h^{\bar u}\bar p - \bar p, (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)}\\ &+ (\bar p - P_h(G_h(R_h^{\bar u}\bar u)), (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)}\end{aligned}$$

due to the definition of J_h'. Lemma 4.53 implies

$$\begin{aligned}\frac{\delta}{2}\| \left(h^{1/2}\chi_1 + \varepsilon^{1/2}\chi_2\right)\left(\bar u_h - R_h^{\bar u}\bar u\right)\|_{L^2(\Gamma)}^2 &\leq (R_h^{\bar u}\bar p - \bar p, (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)}\\ &+ (\bar p - P_h(G_h(R_h^{\bar u}\bar u)), (h\chi_1 + \varepsilon\chi_2)(\bar u_h - R_h^{\bar u}\bar u))_{L^2(\Gamma)}.\end{aligned} \tag{4.154}$$

Now, we estimate each term on the right hand side of the previous inequality separately. To shorten the notation let us set $v_1 = \chi_1(\bar u_h - R_h^{\bar u}\bar u)$ and $v_2 = \chi_2(\bar u_h - R_h^{\bar u}\bar u)$. For the first term of (4.154) we conclude using Corollary 3.36 for $\vec\mu > \vec 1/4$

$$\begin{aligned}(R_h^{\bar u}\bar p - \bar p, hv_1 + \varepsilon v_2)_{L^2(\Gamma)} &= h \sum_{\substack{E\in\mathcal{E}_h\\ E\subset K_1}} \int_E \left(R_h^{\bar u}\bar p - \bar p\right) v_1 + \varepsilon \sum_{\substack{E\in\mathcal{E}_h\\ E\subset K_2}} \int_E \left(R_h^{\bar u}\bar p - \bar p\right) v_2\\
&= h \sum_{\substack{E\in\mathcal{E}_h\\ E\subset K_1}} v_{1|E} \int_E \left(R_h^{\bar u}\bar p - \bar p\right) + \varepsilon \sum_{\substack{E\in\mathcal{E}_h\\ E\subset K_2}} v_{2|E} \int_E \left(R_h^{\bar u}\bar p - \bar p\right)\\
&\leq ch^{3/2}\left(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_1}} h^{1/2}|E| \left|v_{1|E}\right| |\bar p|_{W^{1,\infty}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} h^{1/2}|E| \left|v_{1|E}\right| |\bar p|_{W^{1,\infty}_{1-\mu_j}(E)}\right)\\
&+ c\varepsilon^{1/2}h^2\left(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_2}} \varepsilon^{1/2}|E|^{1/2} \left|v_{2|E}\right| |\bar p|_{W^{2,2}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} \varepsilon^{1/2}|E|^{1/2} \left|v_{2|E}\right| |\bar p|_{W^{2,2}_{2(1-\mu_j)}(E)}\right)\\
&= ch^{3/2}\left(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_1}} |E|^{1/2}\|h^{1/2}v_1\|_{L^2(E)}|\bar p|_{W^{1,\infty}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_1}} |E|^{1/2}\|h^{1/2}v_1\|_{L^2(E)}|\bar p|_{W^{1,\infty}_{1-\mu_j}(E)}\right)\\
&+ c\varepsilon^{1/2}h^2\left(\sum_{\substack{E\in\mathcal{E}_{h,0}\\ E\subset K_2}} \|\varepsilon^{1/2}v_2\|_{L^2(E)}|\bar p|_{W^{2,2}(E)} + \sum_{j=1}^m \sum_{\substack{E\in\mathcal{E}_{h,j}\\ E\subset K_2}} \|\varepsilon^{1/2}v_2\|_{L^2(E)}|\bar p|_{W^{2,2}_{2(1-\mu_j)}(E)}\right)\end{aligned}$$

$$\leq ch^{3/2}|K_1|^{1/2}\left(\|h^{1/2}v_1\|_{L^2(K_1\cap\Gamma^0)}|\bar p|_{W^{1,\infty}(K_1\cap\Gamma^0)}+\sum_{j=1}^m\|h^{1/2}v_1\|_{L^2(K_1\cap\Gamma_j^\pm)}|\bar p|_{W^{1,\infty}_{1-\mu_j}(K_1\cap\Gamma_j^\pm)}\right)$$
$$+c\varepsilon^{1/2}h^2\left(\|\varepsilon^{1/2}v_2\|_{L^2(K_2\cap\Gamma^0)}|\bar p|_{W^{2,2}(K_2\cap\Gamma^0)}+\sum_{j=1}^m\|\varepsilon^{1/2}v_2\|_{L^2(K_2\cap\Gamma_j^\pm)}|\bar p|_{W^{2,2}_{2(1-\mu_j)}(K_2\cap\Gamma_j^\pm)}\right)$$
$$\leq ch^2\|h^{1/2}v_1+\varepsilon^{1/2}v_2\|_{L^2(\Gamma)}\left(|\bar p|_{W^{1,\infty}_{1-\mu_j}(\Gamma)}+|\bar p|_{W^{2,2}_{2(1-\mu_j)}(\Gamma)}\right)$$

where we used the discrete Cauchy-Schwarz inequality and Assumption 4.49 in the last steps. If we set $\vec{1}/4<\vec{\mu}<\vec{1}/4+\vec{\lambda}/2$ we deduce from Theorem 4.28

$$(R_h^{\bar u}\bar p-\bar p,(hv_1+\varepsilon v_2)_{L^2(\Gamma)}\leq ch^2\|h^{1/2}v_1+\varepsilon^{1/2}v_2\|_{L^2(\Gamma)}. \tag{4.155}$$

For the second term of (4.154) we first apply the Cauchy-Schwarz inequality and introduce the intermediate functions $P_h(\bar y)$. We obtain

$$\begin{aligned}(\bar p-P_h(G_h(R_h^{\bar u}\bar u)),hv_1+\varepsilon v_2)_{L^2(\Gamma)}&\leq c\|\bar p-P_h(G_h(R_h^{\bar u}\bar u))\|_{L^2(\Gamma)}\|hv_1+\varepsilon v_2\|_{L^2(\Gamma)}\\ &\leq c\left(\|\bar p-P_h(\bar y)\|_{L^2(\Gamma)}+\|P_h(\bar y)-P_h(G_h(R_h^{\bar u}\bar u))\|_{L^2(\Gamma)}\right)\|h^{1/2}v_1+\varepsilon^{1/2}v_2\|_{L^2(\Gamma)}.\end{aligned} \tag{4.156}$$

By means of the finite element error estimates on the boundary from Theorem 3.48, together with (4.73), and Theorem 4.28 we can conclude for $\vec{1}/4<\vec{\mu}<\vec{1}/4+\vec{\lambda}/2$

$$\|\bar p-P_h(\bar y)\|_{L^2(\Gamma)}\leq ch^2|\ln h|^{3/2}. \tag{4.157}$$

Next, we deduce by means of Lemma 4.32 (iii), together with Lemma 4.32 (i) and (4.128) to ensure the boundedness of $G_h(R_h^{\bar u}\bar u)$ in $L^\infty(\Omega)$ independent of h, that

$$\|P_h(\bar y)-P_h(G_h(R_h^{\bar u}\bar u))\|_{L^2(\Gamma)}\leq c\|\bar y-G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}. \tag{4.158}$$

The finite element error estimates in the domain from Corollary 3.72, the regularity results of Theorem 4.28, and Lemma 4.51 imply for $\vec{1}/4<\vec{\mu}<\vec{1}/4+\vec{\lambda}/2$

$$\|\bar y-G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}\leq\|\bar y-G_h(\bar u)\|_{L^2(\Omega)}+\|G_h(\bar u)-G_h(R_h^{\bar u}\bar u)\|_{L^2(\Omega)}\leq ch^2. \tag{4.159}$$

Finally, the inequalities (4.154)–(4.159) yield the assertion. □

Corollary 4.55 (Supercloseness)**.** *Suppose that Assumptions 4.46 and 4.49 are satisfied and let* $\vec{\mu}=1$ *(quasi-uniform mesh). Moreover, let* χ_1 *and* χ_2 *be the characteristic functions of the sets* K_1 *and* K_2*, respectively, and let* δ *and* ε *be the constants of Theorem 4.29 and Lemma 4.53, respectively. Then, for all mesh parameters* $h<h_1$ *with* h_1 *from Lemma 4.53, the estimate*

$$\frac{\delta}{2}\left\|\left(h^{1/2}\chi_1+\varepsilon^{1/2}\chi_2\right)\left(\bar u_h-R_h^{\bar u}\bar u\right)\right\|_{L^2(\Gamma)}\leq ch^\rho|\ln h|^{3/2}$$

is valid with $\rho=\min(2,\min(\vec{1}/2+\vec{\lambda}-\vec{\epsilon}))$ *and* $\vec{0}<\vec{\epsilon}<\vec{\lambda}$*.*

Proof. The proof is similar to that of Lemma 4.54. Let us point out the differences. For (4.155) we conclude analogously by means of Corollary 3.36 with $\vec{\mu} = 1$ and Theorem 4.28

$$(R_h^{\bar{u}}\bar{p} - \bar{p}, (hv_1 + \varepsilon v_2)_{L^2(\Gamma)} \leq ch^\rho \|h^{1/2}v_1 + \varepsilon^{1/2}v_2\|_{L^2(\Gamma)},$$

see also (4.48). Instead of (4.157), we get using the finite element error estimates on the boundary from Corollary 3.49 and the regularity results of Theorem 4.28

$$\|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} \leq ch^\rho |\ln h|^{3/2}.$$

Using the finite element error estimates in the domain from Corollary 3.73 and Corollary 4.52 we replace the estimate (4.159) by

$$\|\bar{y} - G_h(R_h^{\bar{u}}\bar{u})\|_{L^2(\Omega)} \leq \|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)} + \|G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})\|_{L^2(\Omega)} \leq c\left(h^\lambda + h^\rho\right) \leq ch^\rho$$

with $\lambda = \min(2, \min(2(\vec{\lambda} - \vec{\epsilon})))$. The remainder of the proof remains unchanged. □

Remark 4.56. In general one cannot expect an estimate better than

$$\|\bar{u}_h - R_h^{\bar{u}}\bar{u}\|_{L^2(K_1)} \leq ch^{3/2}.$$

The reason for this is that on the set K_1 the integration formula

$$\int_E (R_h^{\bar{u}} f - f) = 0, \quad E \in K_1,$$

induced by our modified interpolator $R_h^{\bar{u}}$, is only exact for constant polynomials f on the element E, since the interpolation point is not the midpoint S_E of the element E. This is different to the linear quadratic case.

Let us now present the main result. As in the linear elliptic case we define the postprocessed control $\tilde{u}$ by

$$\tilde{u}_h := \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h\right).$$

Theorem 4.57. *Suppose that Assumptions 4.46 and 4.49 are fulfilled. Then for all mesh parameters $h < h_1$ with mesh size h_1 from Lemma 4.53 the estimate*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} + \|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \leq ch^2|\ln h|^{3/2}$$

is valid, provided that the mesh grading parameters $\vec{\mu}$ fulfill $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$.

Proof. We introduce intermediate functions and apply the triangle inequality such that

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq \|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)} + \|G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})\|_{L^2(\Omega)} + \|G_h(R_h^{\bar{u}}\bar{u}) - \bar{y}_h\|_{L^2(\Omega)}. \quad (4.160)$$

For the first term we conclude according to the finite element error estimates of Corollary 3.72 and the regularity results from Theorem 4.28 that

$$\|\bar{y} - G_h(\bar{u})\|_{L^2(\Omega)} \leq ch^2 \quad (4.161)$$

if $\vec{\mu} < \vec{\lambda}$. For the second term we apply Lemma 4.51 to deduce for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|G_h(\bar{u}) - G_h(R_h^{\bar{u}}\bar{u})\|_{L^2(\Omega)} \le ch^2. \tag{4.162}$$

The third term can be estimated by means of the Lipschitz estimates of Lemma 3.79, together with (4.128) to deduce the uniform boundedness of $R_h^{\bar{u}}\bar{u}$ in $L^2(\Gamma)$, i.e.,

$$\|G_h(R_h^{\bar{u}}\bar{u}) - \bar{y}_h\|_{L^2(\Omega)} \le c\|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^1(\Gamma)} \le c\left(\|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^1(K_1)} + \|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^1(K_2)}\right).$$

Next, we apply the Hölder inequality, Assumption 4.49 and Lemma 4.54, which yields for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\begin{aligned} \|G_h(R_h^{\bar{u}}\bar{u}) - \bar{y}_h\|_{L^2(\Omega)} &\le c\left(|K_1|^{1/2}\|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^2(K_1)} + |K_2|^{1/2}\|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^2(K_2)}\right) \\ &\le c\left(h^{1/2}\|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^2(K_1)} + \|R_h^{\bar{u}}\bar{u} - \bar{u}\|_{L^2(K_2)}\right) \le ch^2|\ln h|^{3/2}. \end{aligned} \tag{4.163}$$

One obtains from (4.160)–(4.163) the estimate for the states

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2. \tag{4.164}$$

To derive an estimate for the adjoint states we first argue as in (4.105) to derive

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le \|\bar{p} - P_h(\bar{y})\|_{L^2(\Gamma)} + \|\bar{p} - P_h(\bar{y})\|_{L^2(\Omega)} + c\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}.$$

The finite element error estimates of Theorem 3.48 and Lemma 3.41, the regularity results from Theorem 4.28, and (4.164) imply for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le ch^2|\ln h|^{3/2}. \tag{4.165}$$

Finally, the Lipschitz continuity of the operator $\Pi_{[u_a,u_b]}$ and (4.165) yield for $\vec{1}/4 < \vec{\mu} < \vec{1}/4 + \vec{\lambda}/2$

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} = \left\|\Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}\right) - \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h\right)\right\|_{L^2(\Gamma)} \le c\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \le ch^2|\ln h|^{3/2}.$$

□

Using Corollary 3.73, Corollary 4.52, Corollary 4.55, Corollary 3.49 and Corollary 3.42 instead of Corollary 3.72, Lemma 4.51, Lemma 4.54, Theorem 3.48 and Lemma 3.41, respectively, we can derive the following result for quasi-uniform meshes.

Corollary 4.58. *Let Assumptions 4.46 and 4.49 be fulfilled. Furthermore, let $\vec{\mu} = \vec{1}$ (quasi-uniform mesh), $\vec{0} < \vec{\epsilon} < \vec{\lambda}$ and $\rho = \min(2, \min(\vec{1}/2 + \vec{\lambda} - \vec{\epsilon}))$. Then the estimate*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} + \|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \le ch^\rho|\ln h|^{3/2}$$

is satisfied for all mesh parameters $h < h_1$ with mesh size h_1 from Lemma 4.53.

Remark 4.59. If one would consider semilinear elliptic optimal control problems with a distributed control, one can analogously derive a convergence rate of 2 in all variables provided that the mesh grading parameters $\vec{\mu}$ satisfy $\vec{\mu} < \vec{\lambda}$.

4.4.3 Numerical example for the postprocessing approach

This section is devoted to the numerical verification of theoretical results for the postprocessing approach. To this aim we present two numerical examples. In the first one we know the exact solution, whereas in the second we do not know any local minimum and we use a reference solution on a finer mesh for the sake of comparison. In both examples we numerically solve the semilinear Neumann boundary control problem (P_{ex})

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_\omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma_\omega)} + \int_{\Gamma_\omega} g_1 y,\\
&\text{subject to} \quad u \in U_{ad} := \{u \in L^2(\Gamma_\omega) : u_a \le u \le u_b \text{ a.e. on } \Gamma_\omega\},\\
&\qquad -\Delta y + y + y^3 = f \qquad \text{in } \Omega_\omega,\\
&\qquad \partial_n y = u + g_2 \qquad \text{on } \Gamma_j, \quad j = 1, \ldots, m,
\end{aligned}$$

where the computational domain is defined by (3.142). Similar to Section 4.2.3 this control problem differs from problem (P_{sl}) in the additional term $\int_{\Gamma_\omega} g_1 y$ and the additional functions f and g_2. But this problem can be analyzed analogously. The optimality system of problem (P_{ex}) can be formulated as

$$\begin{aligned}
-\Delta y + y + y^3 &= f && \text{in } \Omega_\omega,\\
\partial_n y &= u + g_2 && \text{on } \Gamma_j, \quad j = 1, \ldots, m,\\
-\Delta p + p + 3y^2 p &= y - y_d && \text{in } \Omega_\omega,\\
\partial_n p &= g_1 && \text{on } \Gamma_j, \quad j = 1, \ldots, m,\\
u &= \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}p\right) && \text{on } \Gamma_j, \quad j = 1, \ldots, m.
\end{aligned}$$

In the first example we choose the data such that we get a solution which exactly possesses the singular behavior proven in Theorem 4.28. In the second one we set the additional terms equal to zero such that we are in the framework of the foregoing section. As described in Section 4.4.2 we discretize the state equation by linear finite elements and the control by piecewise constant functions such that we end up with the discrete optimal control problem

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{2}\Big\|\sum_{k\in I_X} y_k\phi_k - y_d\Big\|^2_{L^2(\Omega_\omega)} + \frac{\nu}{2}\sum_{k\in I_E} u_k^2\|e_k\|^2_{L^2(\Gamma_\omega)} + \sum_{k\in I_X} y_k \int_{\Gamma_\omega} g_1\phi_k,\\
&\text{subject to} \quad u_k \in [u_a, u_b],\ k \in I_E,\\
&\sum_{k\in I_X} y_k \int_{\Omega_\omega} (\nabla\phi_k\cdot\nabla\phi_i + \phi_k\phi_i) + \int_{\Omega_\omega}\left(\sum_{k\in I_X} y_k\phi_k\right)^3\phi_i\\
&\qquad\qquad = \int_{\Omega_\omega} f\phi_i + \int_{\Gamma_\omega} g_2\phi_i + \sum_{k\in I_E} u_k\int_{\Gamma_\omega} e_k\phi_i, \quad \forall i \in I_X,
\end{aligned}$$

where we used the notation introduced in Section 3.2.1. In order to solve the optimal control problem we implemented a standard SQP-method as described in [107, Section 4.11.3]. We also refer to [59], [68] and [76]. Thus, we approximate the solution by a sequence of solutions of linear quadratic sub-problems, i.e., given $\vec{y}_l$ and $\vec{p}_l$ and $\vec{u}_l$ we have to solve in step $l+1$ the following optimality system on the discrete level

$$\sum_{j\in I_X} y_{l+1,j}\left[\int_{\Omega_\omega}(\nabla\phi_j\cdot\nabla\phi_i+\phi_j\phi_i)+\int_{\Omega_\omega}3\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)^2\phi_j\phi_i\right]-\sum_{j\in I_E}u_{l+1,j}\int_{\Gamma_\omega}e_j\phi_i$$
$$=\sum_{j\in I_X}y_{l,j}\int_{\Omega_\omega}3\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)^2\phi_j\phi_i-\int_{\Omega_\omega}\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)^3\phi_i+\int_{\Omega_\omega}f\phi_i+\int_{\Gamma_\omega}g_2\phi_i,\ \forall i\in I_X,$$

$$\sum_{j\in I_X} p_{l+1,j}\left[\int_{\Omega_\omega}(\nabla\phi_j\cdot\nabla\phi_i+\phi_j\phi_i)+\int_{\Omega_\omega}3\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)^2\phi_j\phi_i\right]$$
$$+\sum_{j\in I_X}y_{l+1,j}\left[\int_{\Omega_\omega}6\left(\sum_{k\in I_X}p_{l,k}\phi_k\right)\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)\phi_j\phi_i-\int_{\Omega_\omega}\phi_j\phi_i\right]$$
$$=\sum_{j\in I_X}y_{l,j}\int_{\Omega_\omega}6\left(\sum_{k\in I_X}p_{l,k}\phi_k\right)\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)\phi_j\phi_i-\int_{\Omega_\omega}y_d\phi_i+\int_{\Gamma_\omega}g_1\phi_i,\quad\forall i\in I_X,$$

$$\left(\sum_{j\in I_X}p_{l+1,j}\int_{\Gamma_\omega}\phi_j e_i+\nu u_{l+1,i}\int_{\Gamma_\omega}e_i^2\right)(v_i-u_i)\geq 0,\quad\forall v_i\in[u_a,u_b],\ \forall i\in I_E,$$

where we set $\vec{y}_0=\vec{0}\in\mathbb{R}^N$, $\vec{p}_0=\vec{0}\in\mathbb{R}^N$ and $\vec{u}_0=0.5(u_a+u_b)\vec{1}\in R^M$ with $N=\#I_X$ and $M=\#I_E$. The implementation is realized using a finite element method as described in [2] having regard to (3.152), (3.153), (4.57) and (4.58). Furthermore, we have to extend the algorithms to be able to calculate

$$\int_{\Omega_\omega}6\left(\sum_{k\in I_X}p_{l,k}\phi_k\right)\left(\sum_{k\in I_X}y_{l,k}\phi_k\right)\phi_j\phi_i\quad\forall j,i\in I_X.$$

The optimality system of each sub-problem is solved by a primal-dual active set strategy as already used in Section 4.2.3. As stopping criterion for the SQP-method we choose

$$\frac{\|\sum_{k\in I_X}(y_{l+1,k}-y_{l,k})\phi_k\|_{L^2(\Omega_\omega)}}{\|\sum_{k\in I_X}y_{l+1,k}\phi_k\|_{L^2(\Omega_\omega)}}+\frac{\|\sum_{k\in I_X}(p_{l+1,k}-p_{l,k})\phi_k\|_{L^2(\Omega_\omega)}}{\|\sum_{k\in I_X}p_{l+1,k}\phi_k\|_{L^2(\Omega_\omega)}}$$
$$+\frac{\|\sum_{k\in I_E}(u_{l+1,k}-u_{l,k})e_k\|_{L^2(\Gamma_\omega)}}{\|\sum_{k\in I_E}u_{l+1,k}e_k\|_{L^2(\Gamma_\omega)}}<3TOL$$

with $TOL=10^{-8}$.

Next, we present the two numerical examples.

Example 4.60. Let us set $\nu=1$, $u_a=-0.5$ and $u_b=0.5$. Furthermore, we choose the data

f, y_d, g_1 and g_2 as follows

$$
\begin{aligned}
f &= r^\lambda \cos(\lambda\varphi) + \Big(r^\lambda \cos(\lambda\varphi)\Big)^3 && \text{in } \Omega_\omega,\\
y_d &= 2r^\lambda \cos(\lambda\varphi) + 3\Big(r^\lambda \cos(\lambda\varphi)\Big)^3 && \text{in } \Omega_\omega,\\
g_1 &= -\partial_n \Big(r^\lambda \cos(\lambda\varphi)\Big) && \text{on } \Gamma_j, \quad j = 1,\dots,m,\\
g_2 &= \partial_n \Big(r^\lambda \cos(\lambda\varphi)\Big) - \Pi_{[u_a,u_b]}\Big(r^\lambda \cos(\lambda\varphi)\Big) && \text{on } \Gamma_j, \quad j = 1,\dots,m,
\end{aligned}
$$

with $\lambda = \pi/\omega$. One can easily check, that

$$
\begin{aligned}
\bar{y} &= r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega,\\
\bar{p} &= -r^\lambda \cos(\lambda\varphi) && \text{in } \Omega_\omega,\\
\bar{u} &= \Pi_{[u_a,u_b]}\Big(r^\lambda \cos(\lambda\varphi)\Big) && \text{on } \Gamma_j, \quad j = 1,\dots,m,
\end{aligned}
$$

satisfy the respective first order necessary optimality conditions, possess exactly the singular behavior discussed in Theorem 4.28 and fulfill the second order sufficient optimality condition (4.77) by construction. In Figures 4.4–4.6 the corresponding discrete state $\bar{y}_h$, the discrete adjoint state $\bar{p}_h$ and the postprocessed control $\tilde{u}_h$ are illustrated for $\omega = 3\pi/2$ on a graded mesh with $\mu = 0.5$ and $R = 0.4$, which was generated by a transformation of the nodes. We calculated the discretization errors of the state and the adjoint state in the $L^2(\Omega_\omega)$-norm and of the postprocessed control in $L^2(\Gamma_\omega)$-norm for $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes h and different mesh grading parameters μ, where we produced the graded meshes by a transformation of the nodes as described in Section 3.2.5. Moreover, we determined the experimental orders of convergence as for Example 3.66. The results are given in Tables 4.11–4.15. The observations are equal to those for Example 4.22. But let us repeat them for the convenience of the reader. For $\omega = 2\pi/3$ the approximation rates for all three variables are equal to 2 or almost 2 on quasi-uniform meshes as we haven proven in Theorem 4.57. In case of an interior angle of $3\pi/4$ we observe on quasi-uniform meshes a convergence order of about 1.82 for the postprocessed control which fits to the theoretical results of Corollary 4.58. However, the state and adjoint state show better approximation properties which we have already seen in the linear elliptic case in Example 4.22. For a discussion on this effect we refer to that example. Next, if we use graded meshes with mesh grading parameter $\mu = 0.83 < 0.92 \approx 1/4 + \lambda/2$ we gain for the postprocessed control an approximation rate of almost 2 which confirms the estimate of Theorem 4.57. For the domain $\Omega_{3\pi/2}$ we observe on quasi-uniform meshes an order of convergence of about 1.15 for the postprocessed control which we have proven in Corollary 4.58. The state and adjoint state are approximated with a rate of about 1.35 which is again better than expected. Finally, if we choose $\mu = 0.5 < 0.58 \approx 1/4 + \lambda/2$ we retain the full order of convergence in all three variables as we have proven in Theorem 4.57.

Example 4.61. We set $\nu = 1$, $u_a = -0.15$, $u_b = 0.15$ and $\omega > \pi/4$. We define

$$
b(x) := \left(\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2\right)^{1/2}, \quad x = (x_1, x_2) \in \Omega_\omega.
$$

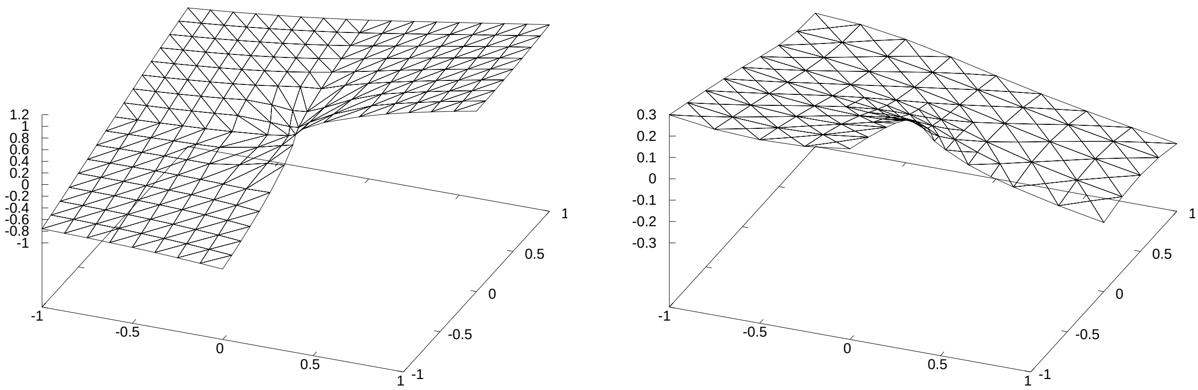

Figure 4.4: Solution $\bar{y}_h$ of Example 4.60 (left) and solution $\bar{y}_h$ of Example 4.61 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

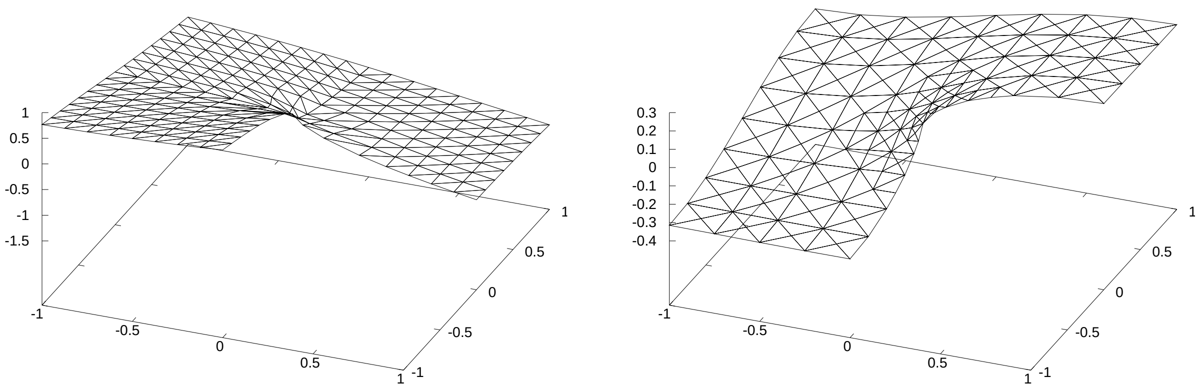

Figure 4.5: Solution $\bar{p}_h$ of Example 4.60 (left) and solution $\bar{p}_h$ of Example 4.61 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

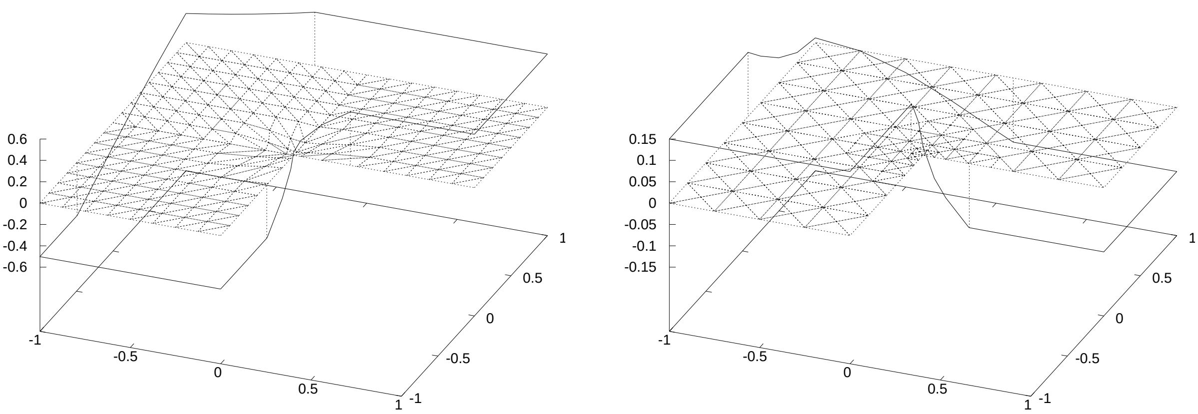

Figure 4.6: Solution $\tilde{u}_h$ of Example 4.60 (left) and solution $\tilde{u}_h$ of Example 4.61 (right) on $\Omega_{3\pi/2}$ with graded mesh ($\mu = 0.5,\ R = 0.4$)

Moreover, we choose the data f, y_d, g_1 and g_2 in the following way

$$\begin{aligned} f &= 0 && \text{in } \Omega_\omega, \\ y_d &= -b^{1/10}\cos(\lambda\varphi) && \text{in } \Omega_\omega, \\ g_1 &= 0 && \text{on } \Gamma_j, \quad j = 1,\dots,m, \\ g_2 &= 0 && \text{on } \Gamma_j, \quad j = 1,\dots,m, \end{aligned}$$

with $\lambda = \pi/\omega$. We solved this problem for the angles $\omega \in \{2\pi/3, 3\pi/4, 3\pi/2\}$, different mesh sizes h and different grading parameters μ, where we realized the graded meshes by a newest vertex bisection algorithm as described in Section 3.2.5. One can find exemplarily for $\omega = 3\pi/2$ the discrete state $\bar{y}_h$, adjoint state $\bar{p}_h$ and postprocessed control $\tilde{u}_h$ in Figures 4.4–4.6 on a graded mesh with $\mu = 0.5$ and $R = 0.4$. Furthermore, we calculated the discretization errors for the state and adjoint state in $L^2(\Omega_\omega)$ and for the postprocessed control in $L^2(\Gamma_\omega)$. The results are collected in Tables 4.16–4.20. Since we do not know the solution of this problem, we compared each solution with a reference solution, which was computed on a mesh with mesh size h_{ref} and grading parameter μ_{ref} as indicated in the different tables. We also determined the approximate experimental orders of convergence as in Section 3.2.5. The observations do not differ fundamentally from those of the previous example. However, the reference solution does no reproduce the approximation rates as well as a known singular solution. But the significant effects of the corner singularities are obvious.

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.577350	1.41e-02		8.44e-03		4.41e-02	
0.288675	3.63e-03	1.95	2.68e-03	1.65	1.32e-02	1.74
0.144338	9.55e-04	1.93	6.69e-04	2.00	3.67e-03	1.84
0.072169	2.43e-04	1.98	1.67e-04	2.00	1.00e-03	1.87
0.036084	6.08e-05	2.00	4.23e-05	1.98	2.69e-04	1.90
0.018042	1.53e-05	1.99	1.05e-05	2.01	7.16e-05	1.91
0.009021	3.83e-06	2.00	2.62e-06	2.00	1.89e-05	1.92
0.004511	9.59e-07	2.00	6.54e-07	2.00	4.97e-06	1.93
0.002255	2.40e-07	2.00	1.63e-07	2.00	1.30e-06	1.93

Table 4.11: Discretization errors for Example 4.60 with $\omega = 2\pi/3$ and $\mu = 1$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	1.96e-02		1.07e-02		5.52e-02	
0.353553	5.55e-03	1.82	3.52e-03	1.61	1.87e-02	1.57
0.176777	1.36e-03	2.03	9.09e-04	1.95	5.74e-03	1.70
0.088388	3.42e-04	1.99	2.23e-04	2.03	1.70e-03	1.76
0.044194	8.42e-05	2.02	5.44e-05	2.04	4.91e-04	1.79
0.022097	2.09e-05	2.01	1.33e-05	2.03	1.41e-04	1.80
0.011049	5.19e-06	2.01	3.26e-06	2.03	4.03e-05	1.81
0.005524	1.29e-06	2.01	8.05e-07	2.02	1.15e-05	1.81
0.002762	3.22e-07	2.00	1.99e-07	2.01	3.25e-06	1.82

Table 4.12: Discretization errors for Example 4.60 with $\omega = 3\pi/4$ and $\mu = 1$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	1.96e-02		1.07e-02		5.52e-02	
0.370133	5.90e-03	1.86	3.80e-03	1.60	1.83e-02	1.70
0.195646	1.55e-03	2.10	1.05e-03	2.01	5.27e-03	1.95
0.103664	4.03e-04	2.12	2.70e-04	2.14	1.41e-03	2.07
0.052560	1.01e-04	2.03	6.79e-05	2.03	3.67e-04	1.98
0.026439	2.54e-05	2.01	1.70e-05	2.02	9.43e-05	1.98
0.013258	6.36e-06	2.01	4.24e-06	2.01	2.40e-05	1.98
0.006639	1.59e-06	2.00	1.06e-06	2.00	6.08e-06	1.99
0.003324	3.98e-07	2.00	2.65e-07	2.00	1.53e-06	1.99

Table 4.13: Discretization errors Example 4.60 with $\omega = 3\pi/4$ and $\mu = 0.83$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	6.65e-02		4.56e-02		9.06e-02	
0.353553	2.54e-02	1.39	1.78e-02	1.36	4.55e-02	1.00
0.176777	1.02e-02	1.31	6.98e-03	1.35	2.39e-02	0.93
0.088388	3.98e-03	1.36	2.72e-03	1.36	1.15e-02	1.05
0.044194	1.53e-03	1.38	1.06e-03	1.37	5.41e-03	1.09
0.022097	5.96e-04	1.36	4.09e-04	1.37	2.49e-03	1.12
0.011049	2.32e-04	1.36	1.59e-04	1.36	1.14e-03	1.13
0.005524	9.07e-05	1.35	6.21e-05	1.36	5.16e-04	1.14
0.002762	3.56e-05	1.35	2.43e-05	1.35	2.33e-04	1.15

Table 4.14: Discretization errors for Example 4.60 with $\omega = 3\pi/2$ and $\mu = 1$

mesh size h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.707107	6.65e-02		4.56e-02		9.06e-02	
0.425046	2.93e-02	1.61	1.90e-02	1.72	4.26e-02	1.48
0.258029	1.09e-02	1.99	7.45e-03	1.88	1.66e-02	1.89
0.156360	3.40e-03	2.32	2.33e-03	2.32	5.33e-03	2.26
0.083008	9.41e-04	2.03	6.44e-04	2.03	1.55e-03	1.95
0.042742	2.49e-04	2.01	1.70e-04	2.01	4.28e-04	1.94
0.021687	6.41e-05	2.00	4.39e-05	2.00	1.14e-04	1.94
0.010923	1.63e-05	1.99	1.12e-05	1.99	3.00e-05	1.95
0.005496	4.13e-06	2.00	2.82e-06	2.00	7.75e-06	1.97

Table 4.15: Discretization errors for Example 4.60 with $\omega = 3\pi/2$ and $\mu = 0.5$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	6.24e-03		5.21e-03		6.02e-03	
0.250000	1.42e-03	2.13	1.55e-03	1.75	1.21e-03	2.31
0.125000	3.48e-04	2.03	4.11e-04	1.91	3.58e-04	1.76
0.062500	8.87e-05	1.97	1.05e-04	1.97	9.65e-05	1.89
0.031250	2.17e-05	2.03	2.66e-05	1.98	2.71e-05	1.83
0.015625	5.29e-06	2.04	6.64e-06	2.00	7.42e-06	1.87
0.007812	1.23e-06	2.10	1.62e-06	2.03	1.99e-06	1.90

Table 4.16: Discretization errors for Example 4.61 with $\omega = 2\pi/3$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	6.19e-03		8.06e-03		6.81e-03	
0.250000	1.62e-03	1.93	2.16e-03	1.90	1.56e-03	2.12
0.125000	3.25e-04	2.32	6.11e-04	1.82	4.59e-04	1.77
0.062500	1.11e-04	1.55	1.41e-04	2.11	7.86e-05	2.55
0.031250	2.52e-05	2.13	3.74e-05	1.92	2.63e-05	1.58
0.015625	6.26e-06	2.01	9.24e-06	2.01	6.51e-06	2.02
0.007812	1.33e-06	2.24	2.36e-06	1.97	1.94e-06	1.74

Table 4.17: Discretization errors for Example 4.61 with $\omega = 3\pi/4$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.07e-02		8.21e-03		1.16e-02	
0.250000	2.42e-03	2.14	2.13e-03	1.94	2.82e-03	2.04
0.125000	7.56e-04	1.68	6.57e-04	1.70	9.56e-04	1.56
0.062500	2.12e-04	1.84	1.34e-04	2.29	1.93e-04	2.30
0.031250	4.82e-05	2.14	3.70e-05	1.85	5.44e-05	1.83
0.015625	1.41e-05	1.77	9.42e-06	1.98	1.48e-05	1.88
0.007812	3.05e-06	2.21	2.46e-06	1.94	3.82e-06	1.95

Table 4.18: Discretization errors for Example 4.61 with $\omega = 3\pi/4$, $\mu = 0.83$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.97e-02		2.17e-02		7.46e-03	
0.250000	7.84e-03	1.33	7.01e-03	1.63	2.32e-03	1.69
0.125000	2.90e-03	1.43	2.56e-03	1.46	1.62e-03	0.52
0.062500	1.07e-03	1.44	9.66e-04	1.40	1.17e-03	0.46
0.031250	4.14e-04	1.37	3.67e-04	1.40	6.73e-04	0.80
0.015625	1.56e-04	1.40	1.44e-04	1.35	3.40e-04	0.99
0.007812	6.14e-05	1.35	5.61e-05	1.36	1.65e-04	1.04

Table 4.19: Discretization errors for Example 4.61 with $\omega = 3\pi/2$, $\mu = 1$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

mesh size h	$\|\bar{y}_{ref} - \bar{y}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\bar{p}_{ref} - \bar{p}_h\|_{L^2(\Omega_\omega)}$	eoc	$\|\tilde{u}_{ref} - \tilde{u}_h\|_{L^2(\Gamma_\omega)}$	eoc
0.500000	1.20e-02		1.49e-02		8.32e-03	
0.250000	2.73e-03	2.13	4.00e-03	1.90	2.31e-03	1.85
0.125000	6.83e-04	2.00	1.08e-03	1.89	8.34e-04	1.47
0.062500	1.89e-04	1.86	2.46e-04	2.14	2.22e-04	1.91
0.031250	4.29e-05	2.14	6.97e-05	1.82	6.22e-05	1.84
0.015625	1.25e-05	1.78	1.53e-05	2.19	1.53e-05	2.02
0.007812	2.85e-06	2.13	3.68e-06	2.06	3.84e-06	2.00

Table 4.20: Discretization errors for Example 4.61 with $\omega = 3\pi/2$, $\mu = 0.5$, $h_{ref} = 0.001953$ and $\mu_{ref} = 0.4$

CHAPTER 5

Conclusion and perspectives

In this work we discussed discretization error estimates for Neumann boundary control problems governed by linear and semilinear elliptic partial differential equations in general polygonal domains with pointwise inequality constraints on the control. We focused on two discretization strategies, the concept of variational discretization as well as the postprocessing approach. For both, each applied to linear and semilinear problems, we derived quasi-optimal finite element error estimates on quasi-uniform as well as on gradually refined meshes. As most challenging step we had to derive discretization error estimates in the $L^2(\Gamma)$-norm for linear elliptic boundary value problems. Finally, for the purpose of a numerical verification of our theoretical results we calculated experimental orders of convergence for different numerical examples based on a Matlab implementation.

Let us briefly discuss some possible extensions. So far we have only considered two dimensional polygonal domains. It might be interesting to derive the estimates in three dimensional polyhedral domains. Then one has to extend the finite element error estimates on the boundary to such domains. For quasi-uniform triangulations this might be done in an analogous manner. Whereas, in case of gradually refined triangulations one has to do decide first whether isotropic or anisotropic refinement should be used. For distributed control problems we can refer to [13] and [12]. In the former one isotropic refinement is used, whereas in the latter one anisotropic meshes are considered. Let us also remark, that another difficulty arises when deriving error estimates in $L^2(\Gamma)$ for anisotropic meshes. To the best of our knowledge, the local finite element error estimates, which we have used within the proofs, are only proven for locally quasi-uniform meshes. Since this local quasi-uniformity is no longer fulfilled for anisotropic meshes, one has to extend these results first.

Besides the error estimates in $L^2(\Gamma)$, which we have considered in this work, one might be interested in pointwise error estimates. Such estimates have already been derived for Neumann boundary control problems, where the problems are discretized by a full discretization on quasi-uniform triangulations. In particular, in [26] a convergence order of $o(h^{1/2})$ in $L^\infty(\Gamma)$ is proven for semilinear problems using a discretization of the control with piecewise linear functions. Furthermore, the authors of that paper got a convergence rate of one in $L^\infty(\Gamma)$

assuming a structural assumption on the control, which we have already seen in this work for the postprocessing approach. In [24] discretization error estimates on quasi-uniform meshes are derived for a full discretization of quasilinear problems, where the control is discretized by piecewise constant functions. There, a convergence order of almost one is proven for convex domains and a convergence rate of almost 1/2 for non-convex domains. Error estimates in $L^\infty(\Gamma)$ for the variational discretization concept applied to linear Neumann boundary control problems can be found in [61]. There, an error bound of $ch^{2-2/p}|\ln h|$ is proven for quasi-uniform meshes, where the parameter p depends on the largest interior angle of the domain, see Chapter 1 for details. If we would like to extend our results to the L^∞-setting, we have to derive pointwise error estimates for the Neumann boundary value problem. In [106] such estimates are proven for quasi-uniform triangulations, but to the best of our knowledge there is no reference, where this is done for gradually refined meshes. Here, one might transfer the results of [8] for Dirichlet boundary value problems to Neumann boundary value problems.

In this work we have presented results for linear and semilinear elliptic problems. A next logical step might be to transfer the results to more general problems, such as quasilinear elliptic optimal control problems. First results in this direction can be found in [31, 30, 24]. In [31, 30] quasilinear elliptic optimal control problems are considered in convex domains with distributed control. The boundary of the domain is assumed to be of class $C^{1,1}$ in two and three space dimensions. Furthermore, convex polygonal domains are allowed in two space dimensions. The authors of these papers derived discretization error estimates for the variational discretization concept as well as for the full discretization approach using quasi-uniform triangulations. More precisely, they could prove a convergence order of one in $L^2(\Omega)$ for the full discretization with piecewise constant functions and a superlinear convergence if the control is discretized by piecewise linear functions. Assuming a structural assumption on the control, which is comparable to that used in the present work for the postprocessing approach, they even got a convergence order of 3/2 in $L^2(\Omega)$. For the variational discretization concept they obtained a convergence order of two in $L^2(\Omega)$. The extension to Neumann boundary control problems is treated in [24]. In this reference, error estimates in $L^2(\Gamma)$ are derived for a discretization of the control by piecewise constant functions using the finite element error estimates of [23]. In particular, a convergence order of one is proven for convex polygonal domains and an approximation rate of 1/2 for non-convex polygonal domains. For both results a quasi-uniform triangulation is assumed.

A further interesting topic could be the consideration of different regularization terms. In this work the control costs have been measured in $L^2(\Gamma)$. But one might replace the $L^2(\Gamma)$-norm by the $H^{-1/2}(\Gamma)$-norm or an equivalent norm because a control in the dual space of $H^{1/2}(\Gamma)$ suffices to get the well-posedness of the state equation in $H^1(\Omega)$. A comparable regularization has already been used for Dirichlet boundary control problems in [92].

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